11-2022

NUMERICAL METHODS FOR LOCATING ZEROS OF POLYNOMIAL SYSTEMS USING RESULTANT

Ayade Salah Abdelmalk

United Arab Emirates University

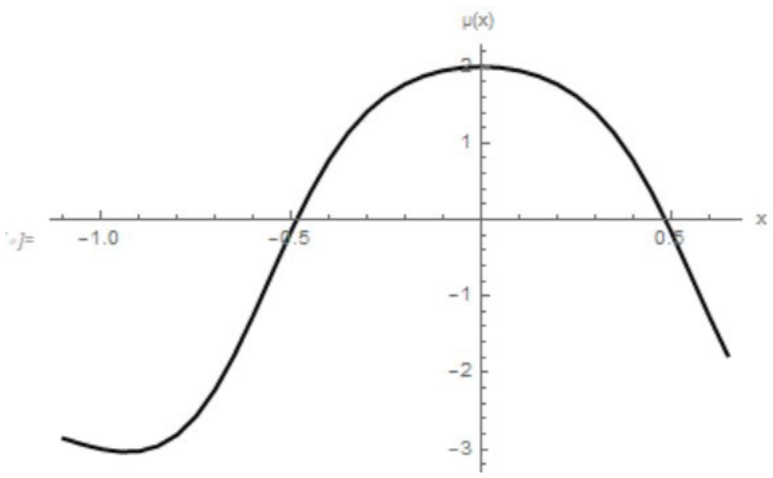
MASTER THESIS NO. 2022: 90

College of Science

Department of Mathematics

NUMERICAL METHODS FOR LOCATING ZEROS OF POLYNOMIAL SYSTEMS USING RESULTANT

Ayade Salah Ayade Abdelmalk



November 2022

United Arab Emirates University

College of Science

Department of Mathematical Sciences

NUMERICAL METHODS FOR LOCATING ZEROS OF POLYNOMIAL SYSTEMS USING RESULTANT METRICS

Ayade Salah Ayade Abdelmalk

This thesis is submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

Under the Supervision of Prof. Muhammed I. Syam

November 2022

Declaration of Original Work

I, Ayade Salah Ayade Abdelmalk, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled "*Numerical Methods for Locating Zeros of Polynomial Systems Using Resultant Metrics*", hereby, solemnly declare that this thesis is my own original research work that has been done and prepared by me under the supervision of Professor Muhammed I. Syam, in the College of Science at UAEU. This work has not previously been presented or published, or formed the basis for the award of any academic degree, diploma or a similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my thesis have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this thesis.

Student's Signature ______________ Date 11/ 11/ 2022

Approval of the Master Thesis

This Master Thesis is approved by the following Examining Committee Members:

1) Advisor (Committee Chair): Muhammed I. Syam

Title: Professor

Department of Mathematical Sciences

College of Science

Signature ______________________ Date 11/11/2022

2) Member: Victor Bodi

Title: Professor

Department of Mathematical Sciences

College of Science

Signature ______________________ Date 11/11/2022

3) Member (External Examiner): Shaher Momani

Title: Professor

Dean, College of Humanities and Sciences

Institution: Ajman University, Ajman, UAE

Signature ______________________ Date Nov. 11, 2022

This Master Thesis is accepted by:

Dean of the College of Science: Professor Maamar Benkraouda

Signature Maamar Benkraouda Date Jan. 18, 2023

Dean of the College of Graduate Studies: Professor Ali Al-Marzouqi

Signature Ali Hassan Date 18/01/2023

Abstract

In this thesis, we modify two methods for locating zeros of polynomial systems which are one dimensional path following and Lancrzos method. Both approaches are based on calculating the resultant matrix corresponding to each variable in the system. These methods are stable and preserving the spareness of these matrices. These methods are developed to avoid using the zeros of the multiresultant of each variable which are condition problems. Theoretical and numerical results will be given to show the efficiency of these methods. Finally, algorithms for the Mathematica codes are given.

Keywords: Resultant matrix, Lanczos method, One dimensional path following method, multiresultant.

Title and Abstract (in Arabic)

طرق عدديه لتحديد اصفار انظمه متغيرات الحدود باستخدام مصفوفه متعدده النتائج

الملخص

في هذه الاطروحه، تم تطوير طريقتين لتحديد اصفار أنظمه كثيرات الحدود وهما طريقه تتبع ذات البعد الواحد وطريقه لنشص كلا الطريقتان يعتمدان على حاسب المصفوف الناتجة المقابله لكل متغير في النظام.

هذه الطرق مستقره وتحافظ على الاصفار الموجودة في المصفونات. هذه الطرق طورت لتجنب ايجاد اصفار محددة المصفوفات الناتجة عن كل متغير والتي تكون غير مستقره. العديد من النتائج النظرية والعدديه سوف يتم عرضها لاثبات فاعليه هذه الطرق.

واخيرا سوف يتم عرض خوارزميات البرامج المكتوبه بلغه ماثيماتيكا.

مفاهيم البحث الرئيسية: المصفوفة الناتجة، مصفوفه شبه صفرية، طريقة لنشص، طريقة التتبع ذات البعد الواحد، متعدده النتائج.

Acknowledgements

Throughout the writing of this thesis, I have received a great deal of support and assistance. This journey would not have been possible without the support of my professors, family, and friends. I would like to take this opportunity to thank them.

I would first like to thank my supervisor, Professor Muhammed I Syam, who was an inexhaustible source of support. Thank you for your continuous help, hard-work, and the motivational words that encouraged and supported me to complete this research work with love and sincerity. Working with you was a great experience for me; your insightful feedback pushed me to sharpen my thinking and brought my work to a higher level. Besides, I am grateful for your guidance which is delivered for me to uphold my experience and research skills, and open my horizon to pave my way for better understanding. I can never thank you enough for all that you have done for me. It's my pleasure and pride to be your student. I would like to thank the Faculty Members of Mathematics Department, especially the head of the department, Dr.Adama Diene, the master's coordinator, Dr. Ahmad Al Rwashdeh, Dr.Mohamed Hajji, Prof.Fathalla Ali Rihan, Prof.Humberto Rafeiro and all my teachers for their encouragement, even those who supported me with just one word in this journey. Thank you all for your assistance and input in getting me acquainted with the research skills and my master's thesis.

My special thanks go to my parents Salah Ayade, Mona Rasmi, my brother Joseph Salah and my wife Dina Raafat, who are the main reason for reaching this achievement. They never felt tired of encouraging me in all of my pursuits and inspiring me to follow my dreams. Thank you for supporting me from all aspects; emotionally and financially. Thank you for the strength and faith that you have given me, to walk on the path of success. I can never thank you enough for everything you have done for me. Also, many thanks to my wife, who always encouraged me to complete this journey in peace.

Last but not least, I would like to thank all my friends, particularly, Mark, Sarah Elfahil, Sondos Syam, and Qamar Dallashi, who were my second family.

Dedication

To my beloved parents and teachers

Table of Contents

List of Tables

List of Figures

Chapter 1: Introduction

In this thesis, we will introduce a new practicable method for approximating all real zeros of polynomial systems using the multiresultant method. Multiresultant method is used to solve systems of polynomial equations to determine whether or not solutions exist, or to reduce a given system to one with fewer variables and/or fewer equations. Historically, a number of authors have considered the task of numerically determining all of the zero points of polynomial systems of equations. In [3], Morozov et al. discussed hidden-variable multiresultant method is a popular class of algorithms for global multidimensional root finding. They study how to compute all the solutions of polynomial systems of the form

$$G(x) = \begin{pmatrix} G_1\,(x_1,\ldots\ldots\ldots x_n) \\ \vdots \\ G_n\,(x_1,\ldots\ldots\ldots x_n) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (x_1,\ldots\ldots\ldots x_n) \in \mathbb{C}^n$$

where $n > 0$ and $G_1 \ldots G_n$ are polynomials in $x_1,\ldots,x_n$ with real coefficients. Mathematically, they are based on these methods appear to be a practitioner's dream as a difficult root finding problem is solved by the robust QR or QZ algorithm, which exploits the semiseparable matrix structure to approximate the eigenvalues in a fast and robust way and gives access to intermediate results in the computation of generalized eigenvalues [17]. Desirably, these methods have received considerable research attention from the scientific computing community. However, in higher dimensions they are known to miss zeros, calculate roots to low precision, and introduce spurious solutions. Noferini & Townsend [14] show that the hidden variable multiresultant method based on the Cayley (Dixon or Bezout) matrix is inherently and spectacularly numerically unstable by a factor that grows exponentially with the dimension. They also show that the Sylvester matrix for solving bivariate polynomial systems can square the condition number of the problem. In other words, two popular hidden variable multiresultant method is numerically unstable, and this mathematically explains the difficulties that are frequently reported by practitioners. Along the way, they prove that the Cayley resultant is a generalization of Cramer's rule for solving linear systems and generalize Clenshaw's algorithm to an evaluation scheme for polynomials expressed in a degree-graded polynomial basis. In recent years, a number of authors have considered the task of numerically determining all of the zeros of polynomial systems of equations. In particular, we mention the resultant method of Collins [1] and the homotopy methods [4]. Since the calculation of the determinant of the resultant is an unstable problem, Collins' method has heretofore been confined to systems involving integer coefficients, and the use of exact integer arithmetic plays a crucial role. In the homotopy approach, one calculates all of the complex zero points by numerical continuation. The homotopy method is generally stable but its computational cost is high. Most of the applications arising in science concerning polynomial systems are of this nature. Allgower et al. [2] gave preliminary work for computing real zeros of polynomial systems using aspects of both the multiresultant method and the conjugate gradient method. The two major tasks which they had been dealt with the construction of the multiresultant matrix $M(x_i)$ and the instability of the equation

$$det(M(x_i) = 0.$$

Since typically $G(x)$ is a polynomial of very high degree in the unknown x, they handle the latter problem by replacing the condition $G(x) = 0$ with the equivalent condition

$$\min_{||u||=1} ||M_i(x_i)u||^2 = 0.$$

However, they used the conjugate gradient method to calculate the smallest eigenvalue of the matrix $M_i(x_i)^t M_i(x_i)$ and testing whether it is zero. Here and in the following, we denote transposition by t. Their work was preliminary. They explained how to construct the Multiresultant matrix but they did not concentrate too much on the numerical techniques for solving these kind of problems. Syam [6] discussed the same problem and he solved examples using Lanczos method. Also, he wrote some algorithms to construct the multiresultant matrix. Both techniques in [2] and [6] have the following two problems.

- Their work is preliminary to present the idea of the multiresultant. So, the complexity of their techniques is high which means that their techniques are not practicable.
- They did not discuss the case of singular situation arising in the resultant matrix application.

To explain the research question of the thesis, we present the idea of multiresultant matrix. We want to describe how to construct the multiresultant matrix for both homogenous and inhomogeneous systems. First, we will study the homogeneous case.
Let

$$G(x) = \begin{pmatrix} G_1(x_1, x_2, \ldots, x_n) \\ G_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ G_n(x_1, x_2, \ldots, x_n) \end{pmatrix}$$

be n polynomials with real coefficients in n variables. Let r_i be the degree of $\mathrm{G_i}(x)$ for $i = 1, 2, \ldots\ldots, n$ and let Υ_n be the vector space that is spanned by the set

$$\beta_n = \left\{ x_1^{i_1} x_2^{i_2} \ldots\ldots\ldots x_n^{i_n} : 0 \leq i_i, i_2, \ldots\ldots, i_n, \text{ and } i_1 + i_2 + \ldots\ldots + i_n = d \right\}$$

where $\mathrm{d} = 1 - n + \sum_{i=1}^{n} r_i$. Then β_n is a basis for Υ_n. It is easy to see that the dimension of Υ_n is the binomial coefficient.

$$S = \frac{\gamma!}{(n-1)!(\gamma - n + 1)!}$$

where $\gamma = \sum_i^n r_i$. Write the basis vectors in β_n in the "reverse lexicographical" order with $\left(x_n^d\right)$ first, $\left(x_n^{d-1} x_{n-1)}\right.$, etc. Then, partition the basis β_n of Υ_n into n disjoint sets

$\lambda_i, i = 1,\ldots,n$, as follows:

$$\lambda_i = \left\{g \in \beta n : \text{ is divisible by } x_1^{r_i} \text{ but not divisible by any of } x_1^{r_1},\ldots,x_{i-1}^{r_{i-1}}\right\}.$$

Let d_i be the number of elements in the set $\lambda_i, i = 1,\ldots,n$. It is easy to see that $s = \sum_{i=1}^{n} d_i$. Now, we are ready to define the multiresultant matrix of the system $\mathrm{G}(x) = 0$. It is a square matrix of order s and it is denoted by M. For any $1 \le i \le n$, there exists an integer $1 \le j_i \le n$ such that $\sum_{l=1}^{j_i-1} d_l \le i \le \sum_{l=1}^{j_i} d_l$. Let $K_i = i - \sum_{l=1}^{j_i-1} d_l$ and q_{j_i} be the K_i^{th} element of the set λ_{j_i}. We should note that q_{j_i} is a monomial of degree d and it is divisible by $x_{j_i}^{r_{j_i}}$. Now, we describe how to homogenize inhomogenize polynomial system of the form

$$G(x) = \begin{pmatrix} \mathrm{G1}\left(x_1,x_2,\ldots,x_n\right) \\ \mathrm{G2}\left(x_1,x_2,\ldots,x_n\right) \\ \vdots \\ \mathrm{Gn}\left(x_1,x_2,\ldots,x_n\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

First, we describe how to construct the multiresultant matrix $\mathrm{M}\left(x_i\right)$ and the multiresultant $\mathrm{G}\left(x_i\right)$ for each $i = 1,2,3,\ldots,n$. Choose any $j \in \{1,2,\ldots\ldots,n\}$ and fix the value of x_j. Thus, the system becomes an inhomogeneous system in n equations and $(n-1)$ variables $x_1,\ldots x_{j-1},x_{j+1},\ldots\ldots x_n$. To homogenize $G_1\left(x_1,x_2,\ldots x_n\right) = 0$, we introduce a new variable x_0. Then, multiply each term in each polynomial by x_0^{μ} to make the system homogeneous. The variable x_0 is called an auxiliary variable and the new polynomial is called the homogenization of $G(x)$ and it is denoted by $\mathrm{G}_0^{(j)}$. Thus, the system becomes

$$G^{(j)}(x) = \begin{pmatrix} \mathrm{G}_0^{(j)}\left(\mathrm{x}_0,\mathrm{x}_1;\ldots,\mathrm{x}_{j+1};,\ldots;\mathrm{x}_\mathrm{n}\right) \\ \mathrm{G}_0^{(j)}\left(\mathrm{x}_0,\mathrm{x}_1;\ldots,\mathrm{x}_{j+1};,\ldots;\mathrm{x}_\mathrm{n}\right) \\ \vdots \\ \mathrm{G}_{n-1}^{(j)}\left(\mathrm{x}_0,\mathrm{x}_1;\ldots,\mathrm{x}_{j+1};,\ldots;\mathrm{x}_\mathrm{n}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Now, we see that the coefficients of the homogeneous for the new system are polynomial expressions in the coefficients of the old system and the chosen variable $\mathrm{x_i}$. Let $\mathrm{M}(x_i)$ be the multiresultant matrix of $G^{(j)}(x)$. Then $\mathrm{M}(x_i)$ is called the multiresultant matrix of $\mathrm{G}(x)$ with respect to the variable x_i. For more details, see [6]. Let α_j be the set of all real roots of the equation $\det(M(x_i)) = 0$ for all $\mathrm{j} = 1, 2, \ldots .\mathrm{n}$. Then, the set of all real solution of $G(x) = 0$ is a subset of the Cartesian product $\prod_{j=1}^{n} \alpha_j$. We should test all the points of $\prod_{j=1}^{n} \alpha_j$ numerically to find all real solutions of $G(x) = 0$. For more details, see [7]-[10]. DUff et al. [11] studied different methods for finding the root set of a generic system in a family of polynomial systems with parametric coefficients. Although, he presented a framework for characterizing monodromy-based solvers in terms of decorated graphs. Under the theoretical that monodromy actions are produced uniformly, they show that the estimated number of homotopy paths followed by an algorithm following this framework is linear in the number of roots.

Loisel and Maxwell [12] used Path Following Method to determine the field of values of a matrix with high accuracy. Additionally, characterizing a unique and efficient algorithm for evaluating the field of values boundary, $\partial W(\cdot)$, of an arbitrary complex square matrix. The boundary is designed by a system of ordinary differential equations which are solved using Runge–Kutta (Dormand-Prince) numerical integration to achieve control points with derivatives, then finally Hermite interpolation is applied to provide a dense output. The algorithm computes $\partial W(\cdot)$ both efficiently and with low error. Formal error bounds are proven for specific classes of matrix. Furthermore, they summarize the prevailing state of the art and make comparisons with the new algorithm. Finally, numerical experiments are performed to quantify the cost-error trade-off between the new algorithm and existing algorithms.

Musco et al. [13] presented the stability of the Lanczos Method for Matrix Function Approximation as he illustrated theoretically elegant and ubiquitous in practice, the Lanczos method can approximate $f(A)x$ for any symmetric matrix $A \in \mathbb{R}^{n\times n}$, vector $x \in \mathbb{R}^n$, and function f. By using analysis bounds, the power of stable estimating polynomials and raises the question if they fully characterize the behavior of finite precision Lanczos in solving linear systems.

Chapter 2: The Multiresultant of Polynomial Systems

In this chapter, the resultant matrix of homogeneous and inhomogeneous polynomial systems will be presented. The relation between the resultant matrix and the zeros of polynomial systems will be investigated. This technique will produce a large sparse matrix. This chapter will be divided into three sections. Section one is devoted to the homogeneous polynomial systems while the Second section devoted for inhomogeneous polynomials systems. Several numerical examples will be presented. In the last section, we present an important theorem which gives us some stable alternatives to the determinant of resultant matrix.

2.1 The Multiresultant of Homogeneous Polynomial Systems

Consider the following polynomial system

$$G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

where

$$G(x_1, x_2, ..., x_n) = \begin{pmatrix} G_1\left(x_1, x_2, \ldots, x_n\right) \\ G_2\left(x_1, x_2, \ldots, x_n\right) \\ \vdots \\ G_n\left(x_1, x_2, \ldots, x_n\right) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.1}$$

Here, we assume that each G_i is a polynomial in term of $x_1, x_2, ..., x_n$. The degree of the term

$$a x_1^{\xi_1} x_2^{\xi_2} \ldots x_n^{\xi_n}$$

is $\xi_1 + \xi_2 + \ldots + \xi_n$ where $\xi_1, \xi_2, \ldots, \xi_n$ are nonnegative integers. The degree r_i of $G_i(x)$ is the maximum of the degrees of its terms. The Polynomial $G_i\left(x_1, x_2, \cdots, x_x\right)$ is called homogeneous if its terms has same degrees.

For example,

$$G(x_1,x_2,x_3) = \begin{pmatrix} x_1^2 - x_2^2 - x_3^2 \\ x_2^2 + x_1x_2 \\ x_2 - x_1 + x_3 \end{pmatrix}$$

is homogeneous since

$$G_1(x_1,x_2,x_3) = x_1^2 - x_2^2 - x_3^2,$$

$$G_2(x_1,x_2,x_3) = x_2^2 + x_1x_2,$$

$$G_3(x_1,x_2,x_3) = x_2 - x_1 + x_3$$

are homogeneous while

$$G(x_1,x_2,x_3) = \begin{pmatrix} x_1^2 - x_2^2 - x_3^2 \\ x_2^2 + x_1 \\ x_2 - x_1 + x_3 \end{pmatrix}$$

is inhomogeneous since, $G_2(x_1,x_2,x_3) = x_2^2 + x_1$ is inhomogeneous.

Let us assume that $G(x_1,x_2,\ldots,x_n)$ in equation (2.1) be homogeneous and r_i be the degree of $G_i(x_1,x_2,\ldots,x_n)$ for $i = 1,2,\ldots,n$.

$$\text{Let } d = 1 - n + \sum_{i=1}^{n} r_i. \tag{2.2}$$

Note that any monomial of degree d in $x_i's$ must be divisible by $x_j^{r_j}$ for some j. Let $\gamma_{n,d}$ be the vector space of homogeneous polynomials in $x_1,x_2,\ldots,x_n$ of degree exactly d. The basis for $\gamma_{n,d}$ is given by the set of monomials in $(x_1,x_2,\ldots,x_n)$ of degree exactly d. The dimension of $\gamma_{n,d}$ is the binomial coefficient

$$S = \begin{pmatrix} d+n-1 \\ n-1 \end{pmatrix} = \frac{\lambda!}{(n-1)!(\lambda-n+1)!}, \lambda = \sum_{i=1}^{n} r_i. \tag{2.3}$$

Write the basis elements of $\gamma_{n,d}$ in "reverse Iexicographical" order, with x_n^d first, next $x_n^{d-1}x_{n-1},\cdots$, etc. Then, partition the basis β_n into $\lambda_i, i=1,2,\ldots,n$ as follows:

$$\lambda_i = \left\{ g \in \beta_n : x_i^{r_i} \mid g \text{ but } x_j^{r_j} x_g \text{ for } j = 1,2,\cdots i-1 \right\}. \tag{2.4}$$

The resultant matrix M is $s \times s$ matrix, and it is describe as follows. Choose an index i and a monomial $f = x_1^{e_1}\ldots x_n^{e_n}$ of λ_i. Then, $e_1 < r_i, \ldots, e_{i-1} < r_{i-1}$, and $e_i \geq r_i$. Let $g = f/x_i^{ri}$ be the corresponding element of λ_i/x_i^{ri}. Then, $gG_i(x_1,x_2,\ldots,x_n)$ is a polynomial of degree S. Then, write gG_i in terms of the basis and the row vector of the coefficients is a row in the matrix M. The matrix M is called the resultant matrix of G. The multiresultant of the System (2.1) is

$$R = \det(M). \tag{2.5}$$

Example 2.1.1 Consider the following homogeneous system

$$G(x_1,x_2,x_3) = \begin{pmatrix} G_1(x_1,x_2,x_3) \\ G_2(x_1,x_2,x_3) \\ G_3(x_1,x_2,x_3) \end{pmatrix} = \begin{pmatrix} x_1^2 - x_2^2 - x_3^2 \\ x_2^2 + x_1x_2 \\ x_2 - x_1 + x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then, the degrees of G_1, G_2, and G_3 are

$$r_1 = 2, r_2 = 2, r_3 = 1,$$

respectively. Then,

$$d = 1 - n + \sum_{i=1}^{3} r_i = 1 - 3 + 2 + 2 + 1 = 3.$$

Let $\mathbb{V}_3$ be the vector space that be spanned by

$$\beta_3 = \left\{x_3^3, x_2x_3^2, x_2^2x_3, x_2^3, x_1x_3^2, x_1x_2x_3, x_1x_2^2, x_1^2x_3, x_1^2x_2, x_1^3\right\}$$

with dimension 10. Let

$$\Lambda_1 = \{\eta \in \beta_3 : x_1^2 \mid \eta\} = \{x_1^2 x_3, x_1^2 x_2, x_1^3\},$$

$$\Lambda_2 = \{\eta \in \beta_3 : x_2^2 \mid \eta \text{ but } x_1^2 \nmid \eta\} = \{x_2^2 x_3, x_2^3, x_1 x_2^2\},$$

$$\Lambda_3 = \{\eta \in \beta_3 : x_3 \mid \eta \text{ bus } x_1^2 \times \eta \text{ and } x_2^2 \nmid \eta\} = \{x_3^3, x_2 x_3^2, x_1 x_3^2, x_1 x_2 x_3\}.$$

The resultant matrix M is formed by dividing the elements of Λ_1 by x_1^2 to get $\{x_3, x_2, x_1\}$. Then, multiply each element in $\{x_3, x_2, x_1\}$ by G_1, and write the coefficients out in "reverse Iexicographical" order to generate the first three rows of M. To explain the idea, we do the following calculations as follows.

$$x_3 G_1(x_1, x_2, x_3) = x_1^2 x_3 - x_2^2 x_3 - x_3^3,$$

$$x_2 G_1(x_1, x_2, x_3) = x_1^2 x_2 - x_2^3 - x_2 x_3^2,$$

$$x_1 G_1(x_1, x_2, x_3) = x_1^3 - x_1 x_2^2 - x_1 x_3^2.$$

Then, the first three rows of M are

$$\begin{pmatrix} -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

Divide Λ_2 by x_2^2 to get $\{x_3, x_2, x_1\}$, then multiply each element in $\{x_3, x_2, x_1\}$ by $G_2(x_1, x_2, x_3)$ to get the following

$$x_3 G_2(x_1, x_2, x_3) = x_2^2 x_3 + x_1 x_2 x_3,$$

$$x_2 G_2(x_1, x_2, x_3) = x_2^3 + x_1 x_2^2,$$

$$x_1 G_2(x_2, x_2, x_3) = x_1 x_2^2 + x_1^2 x_2^2.$$

Then the fourth, fifith and sixth rows of M are

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

In order to complete all rows of M, same steps will be processed like before. Divide Λ_3 by x_3 to get $\left\{x_3^2, x_2x_3, x_1x_3, x_1x_2\right\}$, then multiply each element in $\{x_3, x_2, x_1\}$ by $G_3\left(x_1, x_2, x_3\right)$ to get

$$\begin{aligned} x_3^2G_3\left(x_1,x_2,x_3\right) &= x_2x_3^2 - x_1x_3^2 + x_3^3, \\ x_2x_3G_3\left(x_1,x_2,x_3\right) &= x_2^2x_3 - x_1x_2x_3 + x_2x_3^2, \\ x_1x_3G_3\left(x_1,x_2,x_3\right) &= x_1x_2x_3 - x_1^2x_3 + x_1x_3^2, \\ x_1x_2G_3\left(x_1,x_2,x_3\right) &= x_1x_2^2 - x_1^2x_2 + x_1x_2x_3. \end{aligned}$$

Then, the last four rows of M are

$$\begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{pmatrix}.$$

Therefore, the resultant matrix is

$$M = \begin{pmatrix} -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{pmatrix}.$$

Then, the multiresultant R is

$$R = det(M) = 0.$$

One can see that if we change the order of the polynomials in $G(x_1, x_2, x_3)$, then the matrix M is also changed. However, its multiresultant will stay zero.

Example 2.1.2 Consider the following homogeneous system

$$G(x_1, x_2, x_3) = \begin{pmatrix} x_1x_2 - x_3^2 \\ x_1x_2 - x_1x_3 + 2x_2x_3 \\ x_1 + 2x_2 - x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then, the degrees of G_1, G_2 and G_3 are

$$r_1 = 2, r_2 = 2, r_3 = 1,$$

respectively. Then,

$$d = 1 - n + \sum_{i=1}^{3} r_i = 1 - 3 + 2 + 2 + 1 = 3.$$

Let $\mathbb{V}_3$ be the vector space that be spanned by

$$\beta_3 = \left\{x_3^3, x_2x_3^2, x_2^2x_3, x_2^3, x_1x_3^2, x_1x_2x_3, x_1x_2^2, x_1^2x_3, x_1^2x_2, x_1^3\right\}$$

with dimension 10. Let

$$\Lambda_1 = \left\{\eta \in \beta_3 : x_1^2 \mid \eta\right\} = \left\{x_1^2x_3, x_1^2x_2, x_1^3\right\}$$
$$\Lambda_2 = \left\{\eta \in \beta_3 : x_2^2|\eta \text{ but } x_1^2 \nmid \eta\right\} = \left\{x_2^2x_3, x_2^3, x_1x_2^2\right\},$$
$$\Lambda_3 = \left\{\eta \in \beta_3 : x_3 \mid \eta \text{ but } x_1^2 \nmid \eta \text{ and } x_2^2 \nmid \eta\right\} = \left\{x_3^3, x_2x_3^2, x_1x_3^2, x_1x_2x_3\right\}.$$

The resultant matrix M is formed by dividing the elements of Λ_1 by x_1^2 to get $\{x_3, x_2, x_1\}$, then multiply each element by G_1 and writing the coefficients out in "reverse Iexicographical" order to generate the first three rows of M which are

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Divide Λ_2 by x_2^2 to get $\{x_3, x_2, x_1\}$ then multiply each term by $G_2(x_1, x_2, x_3)$ to get fourth, fifith and sixth rows of M as ,

$$\begin{pmatrix} 0 & 2 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 & 1 & 0 \end{pmatrix}.$$

Divide Λ_3 by x_3 to get $\left\{x_3^2, x_2x_3, x_1x_3, x_1x_2\right\}$, then multiply each element by $G_3\left(x_1, x_2, x_3\right)$ to get the last four rows of M as

$$\begin{pmatrix} -1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore, the matrix M is given by

$$M = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & -1 & 1 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 1 & 0 \end{pmatrix}.$$

Then, the multiresultant R is

$$R = det(M) = 0.$$

Remark. 1 Since $\Lambda_1, \ldots, \Lambda_n$ is a partition of β_n, then

$a)\Lambda_i \neq \Phi$ for $i = 1, 2, \ldots, n$,

$b)\Lambda_i \cap \Lambda_j = \phi$ for $i, j \in \{1, 2, \cdots, n\}, i \neq j$,

$c)\bigcup_{i=1}^{n} \Lambda_i = \beta_n.$

Remark. 2 The degrees of the elements of β_3 in the previous two examples can be written in the matrix form as following:

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ 0 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 0 \end{bmatrix}.$$

Remark. 3 If we change the orders of the polynomials of example 2.1.2 as

$$G(x_1, x_2, x_3) = \begin{pmatrix} x_1x_2 - x_3^2 \\ 2x_2 - x_3 + x_1 \\ x_1x_2 - x_1x_3 + 2x_2x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

then the matrix M will be

$$M = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 & 1 & 0 \end{pmatrix}.$$

with multiresultant

$$R = det(M) = 0.$$

Remark. 4 The resultant matrix is always sparse matrix and the number of nonzeros in each row is the number of terms in the corresponding polynomial $G_i(x_1, x_2, \ldots, x_n)$.

2.2 The Multiresultant for Inhomogenius Systems

Consider the inhomogeneous polynomial systems in n variables

$$G(x_1, x_2, \ldots, x_n) = \begin{pmatrix} G_1(x_1, x_2, \ldots, x_n) \\ G_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ G_n(x_1, x_2, \ldots, x_n) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.6}$$

with real coefficients, which has a finite number of solutions. Choose x_i and fix a value for this x_i. Then, System (2.2) becomes a system of n inhomogeneous polynomials in the other $n-1$ variables. This new system can be homogenized by adding an auxiliary variable x_0 to obtain a system of n homogeneous polynomials in the n variables consisting of the other $n-1$, variables and the new x_0 such as

$$F_{x_i}\left(x_0,\cdots,x_{i-1},x_{i+1},\ldots,x_n\right)=\begin{pmatrix} F_1\left(x_0,\cdots,x_{i-1},x_{i+1},\ldots,x_n\right) \\ F_2\left(x_0,\cdots,x_{i-1},x_{i+1},\ldots,x_n\right) \\ \vdots \\ F_n\left(x_0,\ldots,x_{i-1},x_{i+1}\ldots,x_n\right) \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.7}$$

Note that the coefficients of System (2.7) are polynomials expressions in the coefficients of System (2.6) and x_i. Hence, the coefficients of System (2.7) are polynomials in x_i. Let R_i be the multiresultant of System (2.7) which is a polynomial of x_i for simplification can be written

$$R_i = R_i(x_i). \tag{2.8}$$

Theorem 2.2.1 *If the system* (2.6) *has a solution* $(\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_n)\in\mathbb{C}^n$*, then, for each* $i\in\{1,2,\cdots,n\}$*,*

$$R_i\left(\widetilde{x_i}\right)=0.$$

Proof. If system (2.6) has a solution $(\tilde{x}_1,\tilde{x}_2,\ldots,\vec{x}_n)\in\mathbb{C}^n$, then System (2.7) obtained by fixing $x_i=\tilde{x}_1$ has the solution $(\tilde{x}_i,\ldots,\tilde{x}_{i-1},\tilde{x}_{i+1}\ldots\tilde{x}_n)$. Hence, the homogenized system (2.7) has corresponding solution by setting $x_0=1$, for this value $\tilde{x}_i$ of x_i. Therefore, $R_i(\tilde{x}_i)$ must be zero □

Remark. 5 The converse of Theorem (2.2.1) is not always true.

One can write the real version of Theorem (2.2.1) as follows:

Theorem 2.2.2 *If System (2.6) has a real solution* $(\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_n)\in\mathbb{R}^n$*, then for each* $i,\tilde{x}$ *is a real root of* $R_i(x_i)$*.*

Thus, If A_i is the set of all real solutions of $R_i\ (x_i)$, then the set of solutions of System (2.6) is a subset of the Cartesian product $A_1 \times A_2 \times \cdots \times A_n$. To explain the idea, the following examples are investigated.

Example 2.2.1 Consider the following system

$$G(x_1,x_2) = \begin{pmatrix} x_2^2 + 2x_1x_2 + x_2 + 3 \\ \\ x_2 + 2x_1x_2 + x_1^3 + 3 \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}. \tag{2.9}$$

Using Mathematica 12.1, the real solutions are $x_1 = 4.62672, x_2 = -9.952$ and $x_1 = 1.26547, x_2 = -1.42357$.

Now, fix x_1 to get the following system

$$F_{x_1}(x_0,x_2) = \begin{pmatrix} x_2^2 + (2x_1+1)x_2x_0 + 3x_0^2 \\ \\ (1+2x_1)x_2 + (x_1^3+3)x_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \\ \\ 0 \end{pmatrix}. \tag{2.10}$$

Then, the orders of $F_{x_{1,1}}$ and $F_{x_{1,2}}$ are $r_1 = 2, r_2 = 1$ which implies that $d = 2$. Then,

$$S = \frac{3!}{(2-1)!(3-2+1)!} = 3.$$

Thus,

$$\beta_2 = \left\{x_2^2, x_0x_2, x_0^2\right\}. \quad \text{Hence}$$

$$\lambda_1 = \{\eta \in \beta_2 : x_0^2 \mid \eta\} = \left\{x_0^2\right\},$$

$$\lambda_2 = \left\{\eta \in \beta_2 : x_2 \mid \eta \text{ but } x_0^2 \nmid \eta\right\} = \left\{x_2^2, x_0x_2\right\}.$$

Then

$$\frac{\lambda_1}{x_0^2} = \{1\},$$
$$\frac{\lambda_2}{x_2} = \{x_2, x_0\}.$$

Thus,

$$1(x_2^2 + (2x_1+1)x_2x_0 + 3x_0^2) = x_2^2 + (2x_1+1)x_2x_0 + 3x_0^2,$$

$$x_2(1+2x_1)x_2 + (x_1^3+3)x_0 = (1+2x_1)x_2^2 + \left(x_1^3+3\right)x_0x_2,$$

$$x_0(1+2x_1)x_2 + (x_1^3+3)x_0 = (1+2x_1)\,x_0x_2 + \left(x_1^3+3\right)x_0^2.$$

Then, the resultant Matrix is

$$M(x_1) = \begin{pmatrix} 1 & 2x_1+1 & 3 \\ 1+2x_1 & x_1^3+3 & 0 \\ 0 & 1+2x_1 & x_1^3+3 \end{pmatrix}.$$

Hence,

$$R_1(x_1) = det(M(x_1)) = 9 + 5x_1^3 - 4x_1^4 - 4x_1^5 + x_1^6 = 0$$

which implies that the real roots are $A_1 = \{1.26547, 4.62672\}$.

Now, fix x_2 to get the following system

$$F_{x_2}(x_0,x_1) = \begin{pmatrix} (x_2^2 + x_2 + 3)x_0 + 2x_1x_2 \\ (x_2+3)x_0^3 + (2x_2)x_1x_0^2 + x_1^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.11}$$

Then, the orders of $F_{x_{2,1}}$ and $F_{x_{2,2}}$ are $r_1 = 1, r_2 = 3$ which implies that $d = 3$. Then,

$$S = \frac{4!}{(2-1)!(4-2+1)!} = 4.$$

Thus,

$$\beta_3 = \left\{x_1^3, x_0 x_1^2, x_0^2 x_1, x_0^3\right\}$$

and

$$\lambda_1 = \{\eta \in \beta_3 : x_0/\eta\} = \left\{x_0 x_1^2, x_0^2 x_1, x_0^3\right\},$$

$$\lambda_2 = \left\{\eta \in \beta_3 : x_1^3/\eta,\ \ x_0^2 x_\eta\right\} = \left\{x_1^3\right\}.$$

Thus,

$$\frac{\lambda_1}{x_0} = \left\{x_1^2, x_0 x_1, x_0^2\right\},$$
$$\frac{\lambda_2}{x_1^3} = \{1\}.$$

Hence

$$x_1^2((x_2^2 + x_2 + 3)x_0 + 2x_1 x_2) = (x_2^2 + x_2 + 3)x_0 x_1^2 + 2x_1^3 x_2,$$

$$x_0 x_1((x_2^2 + x_2 + 3)x_0 + 2x_1 x_2) = (x_2^2 + x_2 + 3)x_0^2 x_1 + 2x_0 x_1^3 x_2,$$

$$x_0^2((x_2^2 + x_2 + 3)x_0 + 2x_1 x_2) = (x_2^2 + x_2 + 3)x_0^3 + 2x_0^2 x_1 x_2,$$

$$1((x_2 + 3)x_0^3 + (2x_2)x_1 x_0^2 + x_1^3) = (x_2 + 3)x_0^3 + (2x_2)x_1 x_0^2 + x_1^3.$$

Then the resultant matrix is

$$M(x_2) = \begin{pmatrix} 2x_2 & x_2^2+x_2+3 & 0 & 0 \\ 0 & 2x_2 & x_2^2+x_2+3 & 0 \\ 0 & 0 & 2x_2 & x_2^2+x_2+3 \\ 1 & 0 & 2x_2 & x_2+3 \end{pmatrix}.$$

Hence,

$$R_2(x_2) = -27 - 27x_2 - 36x_2^2 - 19x_2^3 - 12x_2^4 - 11x_2^5 - x_2^6 = 0$$

which implies that the real roots are $A_2 = \{-9.952, -1.423572\}$. Thus, the roots of system (2.11) belongs to the following set

$$A_1 \times A_2 = \{1.2654734077068486, 4.626724494907232\} \times \{-9.952, 1.42357\}$$
$$= \{(1.26547, -9.952), (1.265473, -1.42357232), (4.62672, -9.952), (4.62672, -1.42357)\}.$$

Then, we have four order pairs and we want to verify which one of them is a solution to system (2.11). Then,

$$G(1.26547, -9.952) = \begin{pmatrix} 66.9024 \\ -30.1134 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$G(1.26547, -1.42357) = \begin{pmatrix} 0 \\ 8.88178 \times 10^{-16} \end{pmatrix} \simeq \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$G(4.62672, -1.42357) = \begin{pmatrix} -9.56997 \\ -87.4498 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$G(4.62672, -9.952) = \begin{pmatrix} -1.24345 \times 10^{-14} \\ 1.42109 \times 10^{-14} \end{pmatrix} \simeq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, the roots of system (2.11) are

$$x_1 = 4.62672, x_2 = -9.952,$$

$$x_1 = 1.26547, x_2 = -1.42357.$$

Example 2.2.2 Consider the following system

$$G(x,y,z) = \begin{pmatrix} x^2 + y^2 + z^2 - 1 \\ z - x^2 - y^2 \\ y - x^2 - z^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.12}$$

Then, this system describes the intersection of a sphere in $\mathbb{R}^3$ with two parabolas. Using Mathematica 12.1, the solutions are $x \cong \pm 0.485868, y = z \cong \pm 0.618034$

Now, fix x to get the following systems

$$F_x(x_0,y,z) = \begin{pmatrix} \left(x^2 - 1\right) x_0^2 + y^2 + z^2 \\ -x^2 x_0^2 - y^2 + z x_0 \\ -x^2 x_0^2 + y x_0 - z^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then,

$$r_1 = r_2 = r_3 = 2$$

which implies that

$$d = 1 - 3 + 2 + 2 + 2 = 4.$$

Than, the dimension of $\gamma_{3,4}$ is

$$S = \frac{6!}{(3-1)!(6-3+1)!} = \frac{6!}{2!4!} = 15.$$

Then

$$\beta_4 = \left\{z^4, yz^3, y^2z^2, y^3z, y^4, x_0z^3, x_0yz^2, x_0y^2z, x_0y^3, x_0^2z^2, x_0^2yz, x_0^2y^2, x_0^3z, x^3y, x_0^4\right\}$$

and

$$\Lambda_1 = \left\{\eta \in \beta_4 | x_0^2 | \eta\right\} = \left\{x_0^2z^2, x_0^2yz, x_0^2y^2, x_0^3z, x_0^3y, x_0^4\right\},$$

$$\Lambda_2 = \left\{\eta \in \beta_4 \left| y^2 \right| \eta \text{ but } x_0^2 \nmid \eta\right\} = \left\{y^2z^2, y^3z, y^4, x_0y^2z, x_0y^3\right\},$$

$$\Lambda_3 = \left\{\eta \in \beta_4 \mid z^2 \mid \eta \text{ but } x_0^2 \nmid \eta \text{ and } y^2 \mid \eta\right\} = \left\{z^4, yz^3, x_0z^3, x_0yz^2\right\}.$$

Using similar argument as in Example 2.2.2, the matrix $M(x)$ is given by

$$\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & x^2-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & x^2-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & x^2-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & x^2-1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & x^2-1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & x^2-1 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -x^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -x^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -x^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -x^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -x^2 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -x^2 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -x^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -x^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -x^2 & 0
\end{pmatrix}$$

Hence, the real roots of $R(x) = det(M(x)) = 0$ are $A_1 = \{-1, -0.485868, 0.485868, 1\}$

Now, fix y to get the following system

$$F_y(x_0,x,z)=\begin{pmatrix}\left(y^2-1\right)x_0^2+x^2+z^2\\ -y^2x_0^2-x^2+zx_0\\ -x^2+yx_0^2-z^2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Then,

$$r_1=r_2=r_3=2$$

which implies that

$$d=1-3+2+2+2=4$$

Thus, the dimension of $\gamma_{3,4}$ is

$$S=\frac{6!}{(3-1)!(6-3+1)!}=\frac{6!}{2!4!}=15.$$

Then,

$$\beta_4=\left\{z^4,xz^3,x^2z^2,x^3z,x^4,x_0z^3,x_0xz^2,x_0x^2z,x_0x^3,x_0^2z^2,x_0^2xz,x_0^2x^2,x_0^3z,x^3x,x_0^4\right\}$$

and

$$\Lambda_1=\left\{\eta\in\beta_4\left|x_0^2\right|\eta\right\}=\left\{x_0^2z^2,x_0^2xz,x_0^2x^2,x_0^3z,x_0^3x,x_0^4\right\}$$

$$\Lambda_2=\left\{\eta\in\beta_4\left|x^2\right|\eta \text{ but } x_0^2\nmid\eta\right\}=\left\{x^2z^2,x^3z,x^4,x_0x^2z,x_0x^3\right\}$$

$$\Lambda_3=\left\{\eta\in\beta_4\left|z^2\right|\eta \text{ but } x_0^2\nmid\eta \text{ and } x^2\mid\eta\right\}=\left\{z^4,xz^3,x_0z^3,x_0xz^2\right\}$$

Using similar argument as in Example 2.2.2, the matrix $M(y)$ is given by

$$
\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & y^2-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & y^2-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & y^2-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & y^2-1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & y^2-1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & y^2-1 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -y^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -y^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -y^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -y^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -y^2 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & y & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & y & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & y & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & y & 0
\end{pmatrix}
$$

Hence, the real roots of $R_M = det(M(y)) = 0$ are

$$A_2 = \{-1.61803, -1, 0.618034, 1\}.$$

Now, fix z to get the following system

$$
F(z) = \begin{pmatrix} \left(z^2-1\right)x_0^2 + x^2 + y^2 \\ -y^2 - x^2 + zx_0^2 \\ -x^2 + yx_0 - z^2x_0^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
$$

Then,

$$r_1 = r_2 = r_3 = 2$$

which implies that

$$d = 1 - 3 + 2 + 2 + 2 = 4.$$

Then, the dimension of $\gamma_{3,4}$ is

$$S = \frac{6!}{(3-1)!(6-3+1)!} = \frac{6!}{2!4!} = 15.$$

Then

$$\beta_4 = \left\{y^4, xy^3, x^2y^2, x^3y, x^4, x_0y^3, x_0xy^2, x_0x^2y, x_0x^3, x_0^2y^2, x_0^2xy, x_0^2x^2, x_0^3y, x^3x, x_0^4\right\}$$

and

$$\Lambda_1 = \left\{\eta \in \beta_4 \left| x_0^2 \right| \eta\right\} = \left\{x_0^2y^2, x_0^2xy, x_0^2x^2, x_0^3y, x_0^3x, x_0^4\right\},$$

$$\Lambda_2 = \left\{\eta \in \beta_4 \left| x^2 \right| \eta \text{ but } x_0^2 \nmid \eta\right\} = \left\{x^2y^2, x^3y, x^4, x_0x^2y, x_0x^3\right\},$$

$$\Lambda_3 = \left\{\eta \in \beta_4 \left| y^2 \right| \eta \text{ but } x_0^2 \nmid \eta \text{ and } x^2 \mid \eta\right\} = \left\{y^4, xy^3, x_0y^3, x_0xy^2\right\}.$$

Using similar argument as in Example 2.2.2, the matrix $M(z)$ is given by

$$\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & z^2-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & z^2-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & z^2-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & z^2-1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & z^2-1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & z^2-1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & z & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & z & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -z^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -z^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -z^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -z^2 & 0
\end{pmatrix}$$

Hence, the real roots of $R(z) = det(M(z)) = 0$ are

$$A_3 = \{-1.61803, -1, 0.618034, 1\}$$

Thus, the roots of system (2.12) are subset of $A_1 \times A_2 \times A_3$ where

$$A_1 = \{-1, -0.485868, 0.485868, 1\},$$

$$A_2 = \{-1.61803, -1, 0.618034, 1\},$$

$$A_3 = \{-1.61803, -1, 0.618034, 1\}.$$

Direct substitution of the elements of $A_1 \times A_2 \times A_3$ in the system (2.12) implies that $x = \pm 0.485868, y = 0.618034,$ and $z = 0.618034$.

Chapter 3: Path Following Method

Let $G : \mathbb{R}^n \to \mathbb{R}^n$ be n inhomogeneous polynomial with real coefficients in n variables defined by

$$G(x) = \begin{pmatrix} G_1(x_1, x_2, \ldots, x_n) \\ G_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ G_n(x_1, x_2, \ldots, x_n) \end{pmatrix}. \tag{3.1}$$

In Chapter 2, we discussed the resultant method for solving

$$G(x) = 0. \tag{3.2}$$

For each $i \in \{1, 2, \cdots, n\}$, we generate the multiresultant matrix $M(x_i)$ and the multiresultant

$$R(x_i) = det(M(x_i)). \tag{3.3}$$

Then, we proved that the roots of System (3.2) is subset of $A_1 \times A_2 \times \cdots \times A_n$ where A_i is the set of all roots of equation (3.3). We noticed that $R(x_i)$ is a polynomial of too high degree and to find its roots are numerically unstable.

In this chapter, we present a new method to deal with this problem which is numerically stable and preserve the sparseness of the matrix $M(x_i)$. This approach is called one-dimensional path following method.

3.1 Method of Solution

Let $M(x_i)$ be $m \times m$ matrix for each $i \in \{1, 2, \cdots, m\}$. Let a and b be two random vectors in $\mathbb{R}^n$ with entries from the interval $[-\alpha, \alpha]$ when $\alpha > 0$. It is worth mentioning that once a and b have been chosen, they have to be kept fixed.

Define $(m+1)\times(m+1)$ matrix $A(x_i)$ by

$$A(x_i) = \begin{bmatrix} M(x_i) & a \\ b^\top & 0 \end{bmatrix}. \tag{3.4}$$

Consider the linear system

$$A(x_i) \begin{bmatrix} z(x_i) \\ \mu(x_i) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3.5}$$

when $z : \mathbb{R} \longrightarrow \mathbb{R}^m$ and $\mu : \mathbb{R} \longrightarrow \mathbb{R}$.

Our goal is to show that $R(x_i) = 0$ has a root when $\mu(x_i) = 0$ has a root. In this case, we transform the problem of finding the roots of $det(M(x_i))$ into One-dimensional problem. To start our process, we need the following definition

Definition 3.1.1 Two subspaces U and V of $\mathbb{R}^k$ are called acute if

$$U \cap V^\perp = \{0\} \tag{3.6}$$

where $V^\perp$ is the orthogonal complement of V.

Example 3.1.1 Let

$$U = \{(x,0,0) : x \in \mathbb{R}\} = \text{span}\{(1,0,0)\},$$

$$V = \{(0,y,z) : \quad y,z \in \mathbb{R}\} = \text{span}\{(0,1,0),(0,0,1)\},$$

$$W = \{(x,y,0) : x,y \in \mathbb{R}\} = \text{span}\{(1,0,0),(0,1,0)\}.$$

Then,

$$V^{\perp} = \{(x,y,z) \in \mathbb{R}^3 : (x,y,z)\cdot(0,1,0) = (0,0,0), (x,y,z)\cdot(0,0,1) = (0,0,0)\}$$

$$= \left\{(x,y,z) \in \mathbb{R}^3 = y = 0, z = 0\right\} = \text{span}\{(1,0,0)\}.$$

and

$$W^{\perp} = \{(x,y,z) \in \mathbb{R}^3 : (x,y,z)\cdot(1,0,0) = (0,0,0), (x,y,z)\cdot(0,1,0) = (0,0,0)\},$$

$$= \left\{(x,y,z) \in \mathbb{R}^3 ; x = 0, y = 0\right\} = \text{span}\{(0,0,1)\}$$

Then,

$$U \cap V^{\perp} = v \neq \{0\},$$

$$U \cap W^{\perp} = \{0\}.$$

Thus, V and W are acute while U an v are not acute.

Definition 3.1.2 Let A be $k \times k$ real matrix. Then, kernel of A is defined by

$$\ker(A) = \left\{x \in \mathbb{R}^k : Ax = 0\right\}.$$

Example 3.1.2 Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Let $Ax = 0$. Then, the augmented matrix of $Ax = 0$ is

$$U_1 = \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right).$$

Subtract row one from row three to get

$$U_2 = \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right).$$

Multiply row two of U_2 by $\frac{1}{3}$ to get

$$U_3 = \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right).$$

Multiply row two by -2 and add it to row one and subtract row two to row three to get

$$U_4 = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus, $x_1 = x_2 = 0$ an $x_3 \in \mathbb{R}$. Thus,

$$\ker(A) = \left\{(0,0,x_3) : x_3 \in \mathbb{R}^2\right\} = \operatorname{span}\{(0,0,1)\}.$$

In the next theorem, we investigate the singularity of the matrix $A(x_i)$ in Equation (3.4)

Theorem 3.1.1 *Let a and b be two nonzero vectors such that*

1) $ker\mu(x_i)\cap(\text{span}\{b\})^{\perp}=\{0\}$,

2) $span\{a\}\cap(\text{Ker}(\mu(x_i))^{\top})^{\perp}=\{0\}$.

Then $A(x_i)$ is nonsingular matrix.

Proof. Let $x_i\in\mathbb{R}$ and $M(x_i)=M$. Consider the following system

$$A(x_i)\begin{bmatrix} z \\ \mu \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.7}$$

Then,

$$\begin{bmatrix} M & a \\ b^T & 0 \end{bmatrix}\begin{bmatrix} z \\ \mu \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

which implies that

$$Mz+\mu a=0, \tag{3.8}$$

$$b^{\top}z \quad =0. \tag{3.9}$$

We have two cases to consider.

i- Let $\mu=0$. Then, from Equation (3.8), we have

$$MZ=0. \tag{3.10}$$

Then, $z \in \text{Kar}(M)$. From Equation (3.9), we get

$$b \cdot z = 0$$

which yields to

$$z \in \text{span}\{b\}^{\perp}$$

with respect to the Euclidean inner product. Thus, by condition (1) of the theorem

$$z \in \text{Kar}(\mu) \cap (\text{span}\{b\})^{\perp} = \{0\}.$$

Hence, $z = 0$. Thus the solution of System (3.7) is

$$\begin{bmatrix} z \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

ii- Let $\mu \neq 0$. Then, from Equation (3.8)

$$a = \frac{-1}{\mu} Mz. \tag{3.11}$$

From Condition (2) of the theorem, we have

$$span\{a\} \cap \text{Ker}\left(M^{\top}\right)^{\perp} = \{0\}.$$

Then, $a \notin \ker\left(M^{\top}\right)^{\perp}$.

Thus, there exists a nonzero vector $u \in \ker\left(M^{\top}\right)$ such that

$$a^{\top} u \neq 0. \tag{3.12}$$

Since $u \in \ker\left(M^{\top}\right)$, then

$$\mu^{\top} u = 0. \tag{3.13}$$

Since $a = \frac{-1}{\mu} Mz$, then

$$a^{\top} = \frac{-1}{\mu} z^{\top} M^{T}. \tag{3.14}$$

Hence, from Equation (3.12), (3.13), and (3.14) we have

$$0 \neq a^{t} u = \frac{-1}{\mu} z^{\top} \mu^{\top} u = \frac{-1}{\mu} z^{\top} 0 = 0$$

which is a contradiction. Therefore, the only solution to System (3.7) is the trivial solution.

Thus, $A(x_i)$ is nonsingular matrix. □

In the next theorem, we discuss the sign of $det(A(x_i))$.

Theorem 3.1.2 *Under the conditions of Theorem* (3.1.1)*,* $det(A(y_i))$ *does not change its sign.*

Proof. By Theorem 3.1.1, $A(x_i)$ is nonsingular matrix matrix. It is easy to see that $\det(A(x_i))$ is a polynomial of degree $m+1$. Thus,

$$\det(A(x_i)) \neq 0. \tag{3.15}$$

If $\det(A(x_i))$ changes its sign, then by intermediate value theorem, there is a root to $\det(A(x_i))$ which contradicts Equation (3.1.5). Hence, $\det(A(x_i))$ does not change its sign. □

Thus, In the next theorem, we want to prove that $\mu(x_i)$ is well defined function and belongs to $C^{\infty}(\mathbb{R})$.

Theorem 3.1.3 *Under the conditions of Theorem* 3.1.1*,* $\mu(x_i)$ *is well defined and belongs to* $C^{\infty}(\mathbb{R})$.

Proof. Since $A(x_i)$ is nonsingular by Cramer's rule one gets

$$\mu(x_i) = \frac{\det(M(x_i))}{\det(A(x_i))}. \tag{3.16}$$

Since the denominator and the numerator are polynomials of x_i and $\det(A(x_i) \neq 0$, then From Equation (3.16), $\mu(x_i)$ is well defined and $\mu \in C^\infty(\mathbb{R})$. □

In the next theorem, we study the relation between sign of $\mu(x_i)$ and $\det(M(x_i))$.

Theorem 3.1.4 *Let a and b be two nonzero vectors in R^n such that:*

1) $Ker(M(x_i)) \cap span\{b\})^{\perp} = \{0\}$,

2) $Span\{a\} \cap Ker((M(x_i))^{\top})^{\perp} = \{0\}$.

Let

$$A(x_i) = \begin{pmatrix} M(x_i) & a \\ b^{\top} & 0 \end{pmatrix}.$$

and

$$A(x_i) \begin{pmatrix} z(x_i) \\ \mu(x_i) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then, $\mu(x_i)$ changes its sign when $\det(M(x_i))$ changes its sign.

Proof. By Cramer's rule, one gets

$$\mu(x_i) = \frac{\det(M(x_i))}{\det(A(x_i))}.$$

By Theorems (3.1.1) – (3.1.2), $\mu(x_i)$ changes its sign when $\det(M(x_i))$ changes its sign. It is worth mentioning that since $\mu(x_i)$ changes its sign, a secant method will approximate the root fast. □

One should note the following points

- $\mu(x_i)$ is smooth function.
- We handle a one-dimensional path following method to find the roots of the function $\mu(x_i)$.
- It is a bad practice to expand the determinants symbolically when the size of the matrix is large
- The secant method is fast in our approach and it is easy to use.

We end this section by summarizing our method in the following algorithm:

- Input: The vectors a and b.
- Step 1: Compute $M(x_i)$ using Chapter 2.
- Step 2: Compute $A(x_i)$ using Equation (3.4).
- Step 3: Solve System (3.5) to get $\mu(x_i)$.
- Step 4: If $\mu(\bar{x}_i)\cdot\mu(\bar{x}_i) < 0$, then do step 5.
- Step 5: Use secant method to approximate the roots of $R(x_i) = 0$.
- Step 6: Stop.

Example 3.1.3 Consider the following system

$$G(x,y) = \begin{pmatrix} y^2+2xy+y+3 \\ \\ y+2xy+x^3+3 \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}.$$

Using Mathematica 12.1, the real solutions are $x = 4.62672,\ y = -9.952$ and $x = 1.26547,\ y = -1.42357$. Let

$$a^\top = \{1,0,0,0\},$$

$$b = \{1,0,1,0\}.$$

Table 3.1: x^* and $\mu(x^*)$

$\mu(x^*)$	x^*
-0.798558	0.25
-0.673099	0.5
-0.505173	0.75
-0.28	1 .
-0.0165201	1.25
0.235862	1.5
0.428809	1.75
0.541096	2 .
0.579238	2.25
0.56311	2.5
0.513321	2.75
0.445732	3 .
0.370829	3.25
0.294895	3.5
0.221401	3.75
0.152079	4.
0.0876429	4.25
0.0282364	4.5
-0.0263015	4.75

Fix x. Then, the new system will be

$$\begin{pmatrix} 1 & 2x+1 & 3 & 1 \\ 1+2x & x^3+3 & 0 & 0 \\ 0 & 1+2x & x^3+3 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} Z(x) \\ \mu(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, for the step size 0.25, the values of $\mu(x^*)$ is given in Table (3.1)

From Table (3.1) we see the $\mu(x^*)$ changes it's sign in the intervals, $[1.25, 1.5]$ and $[4.5, 4.75]$.

Now, we implement the secant method with

$$x_0 = 1.25; x_1 = 1.5,$$

$$\mu_0 = -0.01652009424883017; \mu_1 = 0.23586206896551723.$$

Let $\mu(x_i) = t_i$. Then

$$x_2 = x_1 - \frac{t_1(x_1 - x_0)}{t_1 - t_0} = 1.26636,$$

$$t_2 = 0.000949409,$$

$$x_3 = x_2 - \frac{t_2(x_2 - x_1)}{t_2 - t_1} = 1.26542,$$

$$t_3 = -0.0000570186,$$

$$x_4 = x_3 - \frac{t_3(x_3 - x_2)}{t_3 - t_2} = 1.2654734,$$

$$t_4 = 5.697774372563688 \times 10^{-9}.$$

Also, from Table (3.1) we see the $\mu(x_i)$ changes it's sign in the interval $[4.5, 4.75]$. Now, we implement the secant method with

$$x_0 = 4.5; x_1 = 4.75,$$

$$\mu_0 = 0.028236389732285275; \mu_1 = -0.026301469746214605.$$

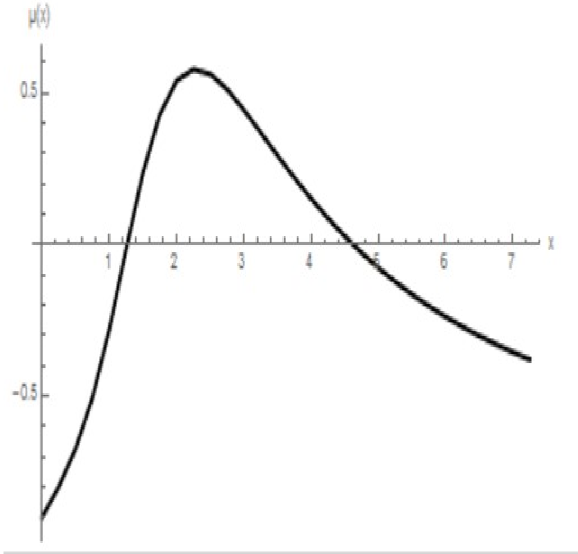


Figure 3.1: x^{*} and $\mu(x^{*})$

Let $\mu(x_i) = t_i$. Then

$$x_2 = x_1 - \frac{t_1(x_1 - x_0)}{t_1 - t_0} = 4.62943,$$

$$t_2 = -0.000590483,$$

$$x_3 = x_2 - \frac{t_2(x_2 - x_1)}{t_2 - t_1} = 4.62667,$$

$$t_3 = 0.0000127746,$$

$$x_4 = x_3 - \frac{t_3(x_3 - x_2)}{t_3 - t_2} = 4.626724,$$

$$t_4 = 6.013758251406966 \times 10^{-9}.$$

Thus, the approximate value of x^{*} are 1.2654734 and 4.626724. The graph of $\mu(x^{*})$ is presented in Figure (3.1). Secondly, for y values, let

$$a^{\top} = \{1,0,0,0,0\},$$

$$b = \{1,0,0,1,0\}.$$

The new system will be 5×5 matrix:

$$\begin{pmatrix} 2y & y^2+y+3 & 0 & 0 & 1 \\ 0 & 2y & y^2+y+3 & 0 & 0 \\ 0 & 0 & 2y & y^2+y+3 & 0 \\ 1 & 0 & 2y & y+3 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} Z(y) \\ \mu(y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Then, using the step size 0.3, the values of $\mu(y)$ are given by Table (3.2).

From Table (3.2) we see the $\mu(y)$ changes it's sign in the intervals $[-10.2, -9.9]$

Now, we implement the secant method with

$$y_0 = -10.2; y_1 = -9.9,$$

$$\mu_0 = -0.569606951512295; \mu_1 = 0.11554562830423577.$$

Let $\mu(y_i) = t_i$. Then,

$$y_2 = y_1 - \frac{t_1(y_1 - y_0)}{t_1 - t_0} = -9.95059,$$

$$t_2 = 0.00314952,$$

$$y_3 = y_2 - \frac{t_2(y_2 - y_1)}{t_2 - t_1} = -9.95201,$$

$$t_3 = -0.0000184043,$$

$$y_4 = y_3 - \frac{t_3(y_3 - y_2)}{t_3 - t_2} = -9.952,$$

Also, from Table (3.2) we see the $\mu(y^*)$ changes it's sign in the intervals $[-1.5, -1.2]$

Table 3.2: $y^{\star}$ and $\mu(y^{\star})$

$\mu(y^{\star})$	$y^{\star}$
−5.62403	−12.
−4.6693	−11.7
−3.7595	−11.4
−2.89463	−11.1
−2.0747	−10.8
−1.29969	−10.5
−0.569607	−10.2
0.115546	−9.9
0.755769	−9.6
1.35106	−9.3
1.90142	−9.
2.40684	−8.7
2.86732	−8.4
3.28284	−8.1
3.6534	−7.8
3.97896	−7.5
4.25952	−7.2
4.49503	−6.9
4.68544	−6.6
4.83071	−6.3
4.93074	−6.
4.98543	−5.7
4.99463	−5.4
4.95814	−5.1
4.87566	−4.8
4.74681	−4.5
4.57103	−4.2
4.34751	−3.9
4.07505	−3.6
3.75184	−3.3
3.375	−3.
2.93984	−2.7
2.4382	−2.4
1.85479	−2.1
1.15808	−1.8
0.274038	−1.5
−1.00365	−1.2
−3.39647	−0.9

Now, we implement the secant method with

$$y_0 = -1.5; y_1 = -1.200000000000001,$$

$$\mu_0 = 0.2740384615384618; \mu_1 = -1.0036475409836005.$$

Let $\mu(y_i) = t_i$.

$$y_2 = y_1 - \frac{t_1(y_1 - y_0)}{t_1 - t_0} = -1.43566,$$

$$t_2 = 0.0451181,$$

$$y_3 = y_2 - \frac{t_2(y_2 - y_1)}{t_2 - t_1} = -1.42552,$$

$$t_3 = 0.00731354,$$

$$y_4 = y_3 - \frac{t_3(y_3 - y_2)}{t_3 - t_2} = -1.42356,$$

$$t_4 = -0.0000587725,$$

$$y_5 = y_4 - \frac{t_4(y_4 - y_3)}{t_4 - t_3} = -1.42357,$$

$$t_5 = 7.6264 \times 10^{-8}.$$

Thus, the approximate value of y are -9.952 and -1.42357. The graph of $\mu(y^*)$ is presented by Figure (3.2). Now, we test the order pairs $(1.2654734, 9.952)$, $(1.2654734, 1.42357)$, $(4.626724, 9.952)$, $(4.626724, 1.42357)$ to check which root will satisfy the system. Thus, the roots are $(4.626724, 9.952)$ and $(1.2654734, 1.42357)$.

Example 3.1.4 Consider the following system

$$G(x,y,z) = \begin{pmatrix} x^2 + y^2 + z^2 - 1 \\ z - x^2 - y^2 \\ y - x^2 - z^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.17}$$

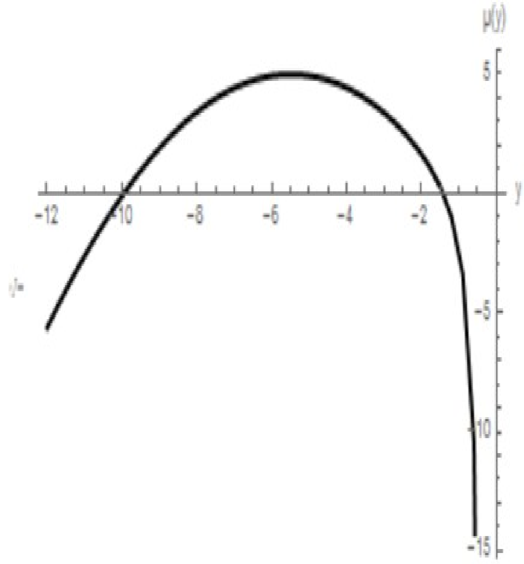


Figure 3.2: y^{*} and $\mu(y^{*})$

Then, this system describes the intersection of a sphere in $\mathbb{R}^3$ with two parabolas. Using Mathematica 12.1 the solutions are $x \cong \pm 0.485868, y = z \cong \pm 0.618034$. Let

$$a^{\top} = \{1, 0, ..., 0\},$$

$$b = \{1, 0, ..., 0\}.$$

Fix x. Then the matrix $A(x)$ is 16×16 and given by

$$A(x)=\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1+x^2 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1+x^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1+x^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1+x^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1+x^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1+x^2 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -x^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -x^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -x^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -x^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -x^2 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -x^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -x^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -x^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -x^2 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}.$$

Using step size 0.05, the values of $\mu(x)$ is given in Table (3.3).

From Table (3.3) we see the $\mu(x)$ changes it's sign in the intervals $[-0.5,-0.45]$. Now, we implement the secant method as

$$x_0 = -0.45000000000000007; x_1 = -0.5,$$

$$\mu_0 = 0.3533061794418093; \mu_1 = -0.14754098360655818.$$

Let $\mu(x_i) = t_i$. Then,

$$x_2 = x_1 - \frac{t_1(x_1 - x_0)}{t_1 - t_0} = -0.485271,$$

$$t_2 = 0.00614393,$$

$$x_3 = x_2 - \frac{t_2(x_2 - x_1)}{t_2 - t_1} = -0.48586,$$

$$t_3 = 0.000088305,$$

$$x_4 = x_3 - \frac{t_3(x_3 - x_2)}{t_3 - t_2} = -0.485868,$$

$$t_4 = -5.714600953287278 \times 10^{-8}.$$

Also, from Table(3.3) we see the $\mu(x)$ changes it's sign in the intervals $[0.45, 0.5]$ Now, we implement the secant method as

$$x_0 = 0.45; x_1 = 0.5,$$

$$\mu_0 = 0.35330617944181153; \mu_1 = -0.14754098360655818.$$

Table 3.3: x and $\mu(x)$

$\mu(x)$	x
−2.85968	−1.1
−2.93823	−1.05
−3.	−1.
−3.03397	−0.95
−3.02559	−0.9
−2.95708	−0.85
−2.80924	−0.8
−2.5653	−0.75
−2.21689	−0.7
−1.77035	−0.65
−1.24994	−0.6
−0.694427	−0.55
−0.147541	−0.5
0.353306	−0.45
0.784349	−0.4
1.1366	−0.35
1.41244	−0.3
1.62073	−0.25
1.77262	−0.2
1.87885	−0.15
1.9483	−0.1
1.98739	−0.05
2.	0.
1.98739	0.05
1.9483	0.1
1.87885	0.15
1.77262	0.2
1.62073	0.25
1.41244	0.3
1.1366	0.35
0.784349	0.4
0.353306	0.45
−0.147541	0.5
−0.694427	0.55
−1.24994	0.6
−1.77035	0.65
−2.21689	0.7
−2.5653	0.75
−2.80924	0.8
−2.95708	0.85
−3.02559	0.9
−3.03397	0.95
−3.	1.
−2.93823	1.05
−2.85968	1.1

Let $\mu(x_i) = t_i$. Then,

$$x_2 = x_1 - \frac{t_1(x_1 - x_0)}{t_1 - t_0} = 0.485271,$$

$$t_2 = 0.00614393,$$

$$x_3 = x_2 - \frac{t_2(x_2 - x_1)}{t_2 - t_1} = 0.48586,$$

$$t_3 = 0.000088305,$$

$$x_4 = x_3 - \frac{t_3(x_3 - x_2)}{t_3 - t_2} = 0.485868,$$

$$t_4 = -5.714600953287278 \times 10^{-8}.$$

Thus, the approximate values of x are 0.485868 and -0.485868. The graph of $\mu(x)$ is given by the Figure 3.3.

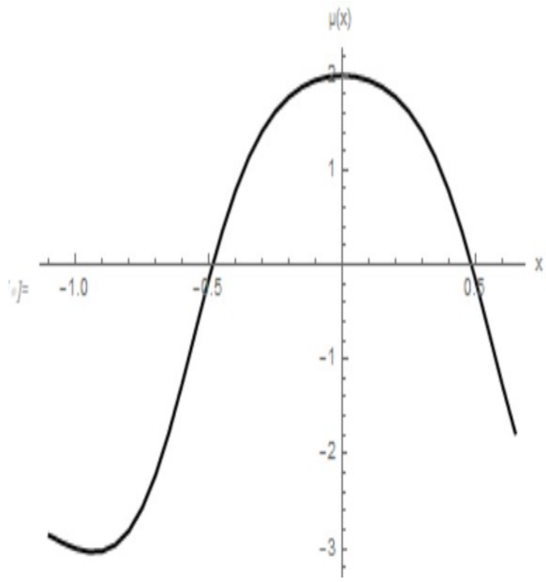


Figure 3.3: x and $\mu(x)$

Fix y. Then choose $a^\top$ and b. Then the matrix $A(y)$ is 16×16 and it's given by

$$a^\top = (1, 0, ..., 0),$$

$$b = (1, 0, ..., 0, 1, 0).$$

$$
A(y)=\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1+y^2 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1+y^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -10+y^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1+y^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1+y^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1+y^2 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -y^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -y^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -y^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -y^2 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & y & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & y & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & y & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.
$$

Using the step size 0.05, the values of $\mu(y)$ is given by in Table (3.4). From Table (3.4). We see the $\mu(y)$ changes it's sign in the intervals $[0.6, 0.65]$. Now, we implement the secant method as

$$y_0 = 0.6; y_1 = 0.65,$$

$$\mu_0 = 0.011539848539487864; \mu_1 = -0.019432092104461694.$$

Let $\mu(y_i) = t_i$. Then,

$$y_2 = y_1 - \frac{t_1(y_1 - y_0)}{t_1 - t_0} = 0.61863,$$

$$t_2 = -0.000374233,$$

$$y_3 = y_2 - \frac{t_2(y_2 - y_1)}{t_2 - t_1} = 0.618014,$$

$$t_3 = 0.0000128767,$$

$$y_4 = y_3 - \frac{t_3(y_3 - y_2)}{t_3 - t_2} = 0.618034,$$

$$t_4 = -7.681868690869802 \times 10^{-9}.$$

Thus, the approximate value of y is 0.618034, The graph of $\mu(y)$ is given in Figure (3.4). Fix z. Let

$$a^\top = (1, 0, ..., 0)$$

$$b = (1, 0, ..., 0, 1).$$

Then, the matrix $A(z)$ is 16×16 and it's given by

Table 3.4: y and $\mu(y)$

$\mu(y)$	y
0.455794	−0.7
0.504948	−0.65
0.547159	−0.6
0.582176	−0.55
0.609756	−0.5
0.629724	−0.45
0.642012	−0.4
0.646685	−0.35
0.643956	−0.3
0.634179	−0.25
0.617843	−0.2
0.595547	−0.15
0.567977	−0.1
0.535873	−0.05
0.5	0
0.461125	0.05
0.419991	0.1
0.3773	0.15
0.333699	0.2
0.289779	0.25
0.246066	0.3
0.20303	0.35
0.161088	0.4
0.120617	0.45
0.0819672	0.5
0.0454851	0.55
0.0115398	0.6
−0.0194321	0.65
−0.0468591	0.7
−0.0699161	0.75
−0.0873016	0.8
−0.0967757	0.85
−0.0940791	0.9

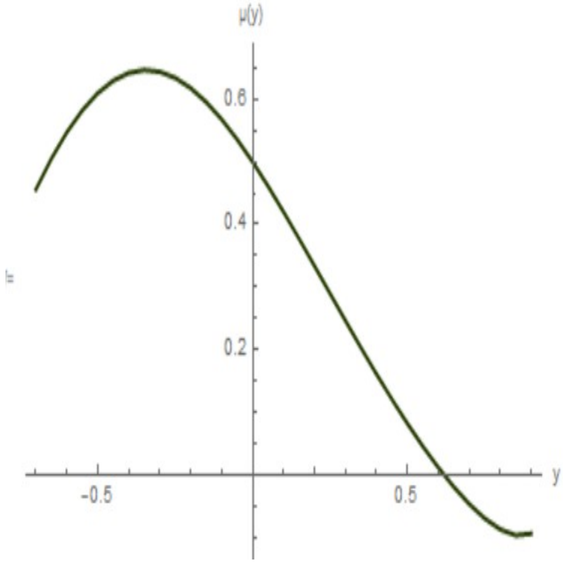


Figure 3.4: y and $\mu(y)$

$$
A(z)=\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & z^2-1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & z^2-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & z^2-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & z^2-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & z^2-1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & z^2-1 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & z & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -z^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & z^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & z^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & z^2 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

Using step size 0.05, the values of $\mu(z)$ are given by Table (3.5).

Table 3.5: z and $\mu(z)$

$\mu(z)$	z
0.605	-0.7
0.61375	-0.65
0.62	-0.6
0.62375	-0.55
0.625	-0.5
0.62375	-0.45
0.62	-0.4
0.61375	-0.35
0.605	-0.3
0.59375	-0.25
0.58	-0.2
0.56375	-0.15
0.545	-0.1
0.52375	-0.05
0.5	0
0.47375	0.05
0.445	0.1
0.41375	0.15
0.38	0.2
0.34375	0.25
0.305	0.3
0.26375	0.35
0.22	0.4
0.17375	0.45
0.125	0.5
0.07375	0.55
0.02	0.6
-0.03625	0.65
-0.095	0.7
-0.15625	0.75
-0.22	0.8
-0.28625	0.85
-0.355	0.9

From Table (3.5) we see the $\mu(z)$ changes it's sign in the intervals $[0.6, 0.65]$. Now, we implement the secant method with

$$z_0 = 0.6; z_1 = 0.65,$$

$$\mu_0 = 0.020408163265306024, \quad \mu_1 = -0.03498190591073593.$$

Let $\mu(z_i) = t_i$. Then,

$$z_2 = z_1 - \frac{t_1(z_1 - z_0)}{t_1 - t_0} = 0.618422,$$

$$t_2 = -0.000434134,$$

$$z_3 = z_2 - \frac{t_2(z_2 - z_1)}{t_2 - t_1} = 0.618025,$$

$$t_3 = 9.591148678045336 \times 10^{-6},$$

$$z_4 = z_3 - \frac{t_3(z_3 - z_2)}{t_3 - t_2} = 0.618034,$$

$$t_4 = 1.664970439296809` \times 10^{-9}.$$

Thus, the approximate value of z is 0.618034. The graph of $\mu(z)$ is given in Figure 3.5.

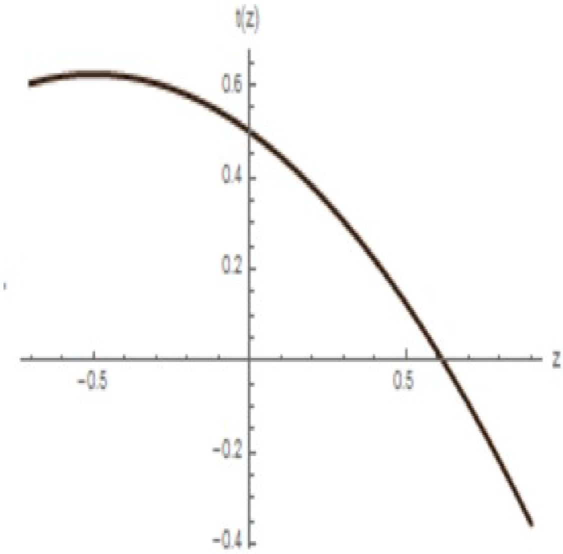


Figure 3.5: z and $\mu(z)$

Chapter 4: Lanczos Method

4.1 Equivalent Conditions to Multiresultant

As we noticed in Chapter 2, computing zeros of n inhomogeneous polynomials system

$$G(x) = 0 \tag{4.1}$$

with real coefficients in n variables is equivalent to solve

$$R(x_i) = \det(M(x_i)) = 0 \tag{4.2}$$

for $i = 1, 2, \ldots, n$. However, computing the determinant of the resultant matrix $M(x_i)$ is unstable problem. To overcome this instability, we replace problem (4.2) by the following stable problem.

$$\begin{aligned} \lambda_{\min}(x_i) &= \min\left\{ \|M(x_i)u\|^2 : u \in \mathbb{R}^s, \|u\| = 1 \right\} \\ &= \min\left\{ \frac{\|M(x_i)u\|^2}{\|u\|^2} : u \in \mathbb{R}^s, u \neq 0 \right\} \end{aligned} \tag{4.3}$$

where $\|.\|$ denotes the Euclidean norm. Theorem $(4.1.1)$ gives some equivalent conditions to Equation (4.2).

Theorem 4.1.1 *The following are equivalent.*

$a)\quad R(x_i) = \det(M(x_i)) = 0.$

$b)\quad \min\left\{ \|M(x_i)u\|^2 : u \in \mathbb{R}^s \text{ and } \|u\| = 1\right\} = 0.$

$c)\quad \min\left\{ \frac{\|M(x_i)u\|^2}{\|u\|^2} : u \in \mathbb{R}^s \text{ and } u \neq 0\right\} = 0$

$d)$ *The smallest eigenvalue of* $M(x_i)^* M(x_i)$ *is zero, when* * *weans transpose of the matrix.*

Proof. (a) $\Rightarrow$ (b): If $R(x_i) = 0$, then $M(x_i)$ is singular matrix. Thus, there exists $u_0 \in \mathbb{R}^s$ with $u_0 \neq 0$ such that

$$M(x_i)u_0 = 0.$$

Let $\nu = \frac{u_0}{\|u_0\|}$. Then, $\|\nu\| = 1$ such that $M(x_i)\nu = 0$. Thus,

$$\|M(x_i)\nu\| = \|0\| = 0.$$

Since $\|M(x_i)u\| \geqslant 0$ for all $u \in \mathbb{R}^s$ and $\|u\| = 1$, then

$$\min\left\{ \|M(x_i)u\|^2 : u \in \mathbb{R}^s \text{ and } \|u\| = 1\right\} = 0.$$

$(b) \Rightarrow (c)$: For any $u \in \mathbb{R}^s$ with $u \neq 0$, we define

$$\nu_u = \frac{u}{\|u\|}.$$

Then, $\nu_u \in \mathbb{R}^s$ with $\|\nu_u\| = 1$. Thus,

$$\frac{\|M(x_i)u\|^2}{\|u\|^2} = \|M(x_i)\nu_u\|^2 \leq \min\left\{ \|M(x_i)\nu\|^2 : \nu \in \mathbb{R}^s \text{ and } \|\nu\| = 1\right\}$$

which implies that

$$0 \leqslant \min\left\{\frac{\|M(x_i)u\|^2}{\|u\|^2} : u \in \mathbb{R}^s \text{ and } u \neq 0\right\}$$

$$\leqslant \min\left\{\|M(x_i)v\|^2 : v \in \mathbb{R}^s \text{ and } \|v\| = 1\right\} = 0.$$

Hence,

$$\min\left\{\frac{\|M(x_i)u\|^2}{\|u\|^2} : u \in \mathbb{R}^s \text{ and } u \neq 0\right\} = 0.$$

(c) $\Rightarrow$ (d) Let

$$\min\left\{\frac{\|M(x_i)u\|^2}{\|u\|^2} : u \in \mathbb{R}^s \text{ and } u \neq 0\right\} = 0.$$

Then, there exists $u \in \mathbb{R}^s$ with $u \neq 0$ such that

$$\frac{\|M(x_i)u\|^2}{\|u\|^2} = 0$$

which implies that

$$(M(x_i)u) = 0.$$

Thus,

$$M^*(x_i)M(x_i)u = 0$$

Then, zero is an eigenvalue of $M^*(x_i)M(x_i)$.

Also, all eigenvalues of $M^*(x_i)M(x_i)$ are nonnegative real numbers. Hence, the smallest eigenvalue of $M^*(x_i)M(x_i)$ is zero.

(d) $\Rightarrow (a)$: If the smallest eigenvalue of $M^*(x_i)M(x_i)$ is zero, then

$$\det(M^*(x_i)M(x_i)) = \det^2(M(x_i)) = 0$$

which implies that

$$R(x_i) = \det(M(x_i)) = 0.$$

$\square$

Therefore, we will look for x_i such that the smallest eigenvalue of $M^*(x_i)M(x_i)$ is zero.

Let us assume that

$$\mu(x_i) = M^*(x_i)M(x_i). \tag{4.4}$$

One should note that $\mu(x_i)$ is a large sparse square symmetric matrix. In some cases, $\mu(x_i)$ is singular. Therefore, we should use suitable method for such matrices which is the Lanczos method.

4.2 Lanczos Method

Let us assume that

$$\mu(x_i) = M^*(x_i)M(x_i) \tag{4.5}$$

for $i = 1,2,\ldots,n$. Then, μ is large, square, symmetric matrix of order s. Also, μ is singular matrix in sometimes. For this reason, Lanczos method is one of the most suitable methods to use in this case. In this section, we describe it.

Let us define the Rayleigh quotient as follows

$$R(u) = \frac{u^*\mu(x_i)u}{u^*u}, \quad u \neq 0. \tag{4.6}$$

Then, using Theorem (4.1.1), the minimum of $R(u)$ is the smallest eigenvalue of $\mu(x_i)$. Let us fix x_i and for simplicity write $\mu(x_i)$ by μ. Let $\{q_1,\ldots,q_s\} \subseteq \mathbb{R}^s$ be the Lanczos orthonormal vectors and define

$$Q_n = \begin{bmatrix} q_1, & q_2, & \cdots & q_n \end{bmatrix}, n = 1, 2, \cdots, s. \tag{4.7}$$

Then,

$$Q_n \mu Q_n^* = T_n = \begin{bmatrix} \alpha_1 & \beta_2 & 0 & 0 \\ \beta_2 & \alpha_2 & \ddots & 0 \\ 0 & \ddots & \ddots & \beta_n \\ 0 & 0 & \beta_n & \alpha_n \end{bmatrix}.$$

We can generate $q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n$, and $\beta_1, \ldots, \beta_{n-1}$ using the following algorithm.

Algorithm 1:

- Input: The matrix A of order n.
- Output: The matrices Q_n and T_n.

Step 1: Let $q_1 \in \mathbb{R}^s$ with $\|q_1\| = 1$ using the Euclidean norm.

Step 2: Let

$$v_1' = \mu q_1$$

$$\alpha_1 = v_1'^* q_1$$

$$v_1 = v_1' - \alpha_1 q_1.$$

Step 3: For $i = 2 : n$, do steps $4 - 8$.

Step 4: Let $\beta_i = \|v_{i-1}\|$.

Step 5: If $\beta_i \neq 0$, then $q_i = \frac{v_{i-1}}{\beta_i}$, else choose q_i of norm one and orthogonal to $q_j, j = 1 : i - 1$.

Step 6: Let $v_i' = Aq_i$.

Step 7: Let $\alpha_i = v_i'^* q_i$.

Step 8: Let $v_i = v_i' - \alpha_i q_i - \beta_i q_{i-1}$.

Step 9: Let

$$Q_n = [q_1 \dots q_n] .$$

Step 10: Let

$$T_n = \begin{bmatrix} \alpha_1 & \beta_2 & 0 & 0 \\ \beta_2 & \alpha_2 & \ddots & 0 \\ 0 & \ddots & \ddots & \beta_n \\ 0 & 0 & \beta_n & \alpha_n \end{bmatrix} .$$

Step 11: Stop.

One can see that

$$m_n = min_{u \neq 0} R(Q_n u) \geqslant \lambda_{\min}(x_i), \tag{4.8}$$

$$\text{and } m_1 \geqslant m_2 \geqslant \dots \geqslant m_s = \lambda_{min}(x_i). \tag{4.9}$$

Example 4.2.1 Let

$$\mu = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} .$$

Let

$$q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} .$$

Then

$$v_1' = \mu q_1 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}.$$

Where $\alpha_1 = v_1'^* q_1 = 1$. Thus

$$v_1 = v_1' - \alpha_1 q_1 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}.$$

Then, $\beta_2 = 5$ and

$$q_2 = \begin{bmatrix} 0 \\ 4/5 \\ 3/5 \end{bmatrix}.$$

Thus, $q_1 \cdot q_2 = 0$ and

$$v_2' = \mu q_2 = \frac{1}{5} \begin{bmatrix} 1 & 4 & 3 \\ 4 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 25 \\ 31 \\ 38 \end{bmatrix}.$$

Hence, $\alpha_2 = v_2'^* q_2 = \frac{1}{25}[0 + 124 + 114] = \frac{238}{25}$ and,

$$v_2 = v_2' - \alpha_2 q_2 - \beta_2 q_1 = \frac{1}{5} \begin{bmatrix} 25 \\ 31 \\ 38 \end{bmatrix} - \frac{238}{125} \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} - 5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{125} \begin{bmatrix} 0 \\ -177 \\ 236 \end{bmatrix}.$$

Thus, $\beta_3 = \|v_2\| = \frac{59}{25}$ and

$$q_3 = \frac{v_2}{\beta_3} = \begin{bmatrix} 0 \\ \frac{-3}{5} \\ \frac{4}{5} \end{bmatrix},$$

Where $q_1 \cdot q_2 = 0, \quad q_2 \cdot q_3 = 0, \quad q_1 \cdot q_3 = 0.$

Then,

$$v_3' = \mu q_s = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{-3}{5} \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{8}{5} \\ \frac{9}{5} \end{bmatrix},$$

and

$$\alpha_3 = v_3'^* q_3 = \begin{bmatrix} 0 & \frac{8}{5} & \frac{9}{5} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{-3}{5} \\ \frac{4}{5} \end{bmatrix} = \frac{12}{25}.$$

Thus, T_n and Q_n have been generated as

$$T_n = \begin{bmatrix} 1 & 5 & 0 \\ 5 & \frac{238}{25} & \frac{59}{25} \\ 0 & \frac{59}{25} & \frac{12}{25} \end{bmatrix}$$

$$Q_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix}.$$

It is worth mentioning that there is no need to calculate all m_i for $i = 1 : s$. since we will get an excellent approximation to $\lambda_{min}(x_i)$ by m_n for n is smaller than s. Another advantage is that T_n is tridiagonal matrix. Thus, we can write it as $n \times 3$ matrix to save storage in the computer and to reduce the computational cost.

We can write Algorithm 1

in the following form to make it more suitable for programming.

Algorithm 2:

- Input: $\varepsilon > 0$ % tolerance, $\omega \in \mathbb{R}^s$ with $\|\omega\|_2 = 1$, a matrix μ.
- output: $\alpha_i, i = 1:n$ and $\beta_j, j = 2:n$.

Step 1: Let $i = 1, \nu = 0, \beta_1 = 1$.

Step 2: While $\beta_i \geqslant \varepsilon$ and $i \leqslant n$, do steps 3-7.

Step 3: If $i \neq 1$, do steps $4-5$.

Step 4: For $k = 1:s$, do steps 5.

Step 5: Let $t = w_k, w_k = \frac{\nu_k}{\beta_i}, \nu_k = -\beta_i \cdot t$.

Step 6: Let $\nu = \nu + \mu\omega$.

Step 7: For

$$i = i+1, \alpha_{i-1} = \omega^* \nu, \nu = \nu - \alpha_{i-1}\omega$$

$$\beta_i = \|\nu_2\|.$$

Step 8: Stop.

Remark. 6 One can see the following.

1. In each step, we need one evaluation of $\mu\omega$. Thus, T_n can be generated by n evaluations of $\mu\omega$.

2. In our code, to compute $\mu\omega$, we need

a) Compute $r = \mu(x_i)\omega$,

b) Compute $M^*(x_i)r$.

3. If $M(x_i)$ has γ nonzero elements in average in each row, then the single Lanczos step need $(3\gamma+8)s$ flops.

4. The vectors ω has size $s \times 1$.

5. The main disadvantage of Algorithm 2, we loose the orthogonality of the Lanczos vectors $\omega's$ due to the cancellation.

To overcome the difficulty in point (5), We can use either complete reorthogonalization or selective orthogonalization. Unfortunately, the complete reorthogonalization is complicated to use and very expensive in terms of computational cost. Therefore, in this section, we use the selective orthogonelization. Since T_n is triadiagonal symmetric matrix, we can apply the symmetric QR methad on it. Let $\theta_1, \theta_2, \ldots, \theta_n$ be the computed Ritz values and S_n is nearly orthogonal matrix of eigenvectors. Let

$$Y_n = [y_1 \ldots y_n] = Q_n S_n. \tag{4.10}$$

Then, it can be shown that

$$|q_{n+1} y_i| \approx \frac{\varepsilon \|\mu\|_2}{\beta_n |s_{n_i}|}$$

and

$$\|\mu y_i - \theta_i y_i\| \approx |\beta_n| \, |s_{n_i}| = \beta_{n_i}$$

where ε is the machine precision. We say the computed Ritz pair (θ, y) is "good" if

$$\|\mu y - \theta y\| \approx \sqrt{\varepsilon} \|\mu\|_2.$$

One can measure the loss of orthogonality of Q_i by

$$k_i = \|I_i - Q_i^* Q_i\| \text{ and } k_1 = \|1 - q_1^* q_1\|.$$

Then,

$$k_1 \leq k_2 \leq \cdots \leq k_n.$$

The relation between k_{i+1} and k_i is given by the following theorem.

Theorem 4.2.1 *if $k_i \leqslant \eta$, then*

$$k_{i+1} \leq \frac{1}{2}\left(\eta+\varepsilon+\sqrt{(\eta-\varepsilon)^2+4\left\|Q_i^* \cdot q_{i+1}\right\|^2}\right)$$

Now, Let us fix η, say $\eta = 10^{-2}$. If $k_i \leqslant \eta$, then q_{i+1} is orthogonal on all columns of Q_i . In this case, no need to do any reorthogonalization. If $k_i > \eta$, then we orthogonalize q_{i+1} against each "good" Ritz vectors. It is easy to see that the selective orthogonlization is much less costly than the complete reorthogonalization since there are fewer "good" Ritz vectors than Lanczos vectors. Another advantage in using the selective orthogonadization is that we implement the symmetric QR method on T_n which has small size comparing with the size of μ. The following algorithms shows how can we apply the Rayleigh quotient iteration with selective orthogonatization to find the smallest eigenvalue of the matix T_n. It is easy to see that T_s and μ are similar and they have the same eigenvalues.

Algorithm 3:

- Input : $x^{(0)}$ such that $\left\|x^{(0)}\right\| = 1$.
- Output: Approximate value for smallest eigenvalue of T_n.
- Step 1: For $k = 0, 1, \ldots,$ do steps 2-5.
- Step 2: Compute $m_k = \frac{x^{(k)^*} T_n x^{(k)}}{x^{(k)^*} x^{(k)}}$.
- Step 3: Set I_n to be the identity matrix of order n.
- Step 4: Solve $(T_n - m_k I_n)\, z^{(k+1)} = x^{(k)}$ for $z^{(k+1)}$.
- Step 5: Set $x^{(k+1)} = \frac{z^{(k+1)}}{\|z^{(k+1)}\|}$.

- Step 6: Stop.

For more details about selective of orthogonalization and Lanczos method, see [16]

Example 4.2.2 Consider the following T_3

$$T_3 = \begin{bmatrix} 1 & 5 & 0 \\ 5 & \frac{238}{25} & \frac{59}{25} \\ 0 & \frac{59}{25} & \frac{12}{25} \end{bmatrix}.$$

We want to find the smallest eigenvalues. Let

$$x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then

$$\mu_0 = x_0^\top T_3 x_0 = 1.$$

Then, z_1 is the solution of

$$(T_3 - \mu_0 I_3) z_1 = x_0.$$

Which implies that,

$$z_1 = \begin{bmatrix} \frac{-10}{13} \\ \frac{1}{5} \\ \frac{59}{65} \end{bmatrix}.$$

Hint,

$$x_1 = \frac{z_1}{\|z_1\|} = \begin{bmatrix} -0.637577 \\ 0.16577 \\ 0.7527341 \end{bmatrix}.$$

Similarly, we will get

$$\mu_1 = 0.471545, \quad x_2 = \begin{bmatrix} -0.474403 \\ 0.034914 \\ 0.879615 \end{bmatrix}.$$

$$\mu_2 = 0.587372, \quad x_3 = \begin{bmatrix} -0.48391 \\ 0.0399022 \\ 0.874208 \end{bmatrix}.$$

$$\mu_3 = 0.587717$$

Thus,

$$\mu_3 = \min\{\|\lambda\| : \lambda \text{ is an eigenvalue of } T_3\}.$$

Note that the eigenvalues of T_3 are $\{12.2221, -1.8098, 0.587717\}$.

4.3 Numerical Results

In this section, we present two examples. The first example is taken from [2.2.2] to make a comparison with their results.

Example 4.3.1 Consider the following system of polynomials

$$G_1(x_1,x_2,x_3) = x_1^2 + x_2^2 + x_3^2 - 1,$$

$$G_2(x_1,x_2,x_3) = x_1^2 + x_2^2 - x_3,$$

$$G_3(x_1,x_2,x_3) = x_1^2 + x_3^2 - x_2.$$

Using Mathematica, it is easy to see that the solution to the system

$$G(x) = 0 \tag{4.11}$$

are

$$x_2 = x_3 = \frac{\sqrt{5}-1}{2} \approx 0.618 \text{ and } x_1 = \pm\sqrt{x_3 - x_2^2} \approx \pm 0.486.$$

We scan for a solution of the x_1 parameter in the interval $[-0.7, 0.7]$ and of x_2 parameter in the interval $[-0.9, 0.9]$. The parameter x_3 will give the same result as x_2. In all cases, the increment is 0.05. Table 1 and 2 show the minimal eigenvalues λ and the number of evaluations of μw which were necessary to obtain λ, say ν. We should note for Tables (4.1)-(4.3) that, the x_1 compute of the roots belongs to $[-0.5, -0.45]$ and $[0.45, 0.5]$. Also x_2 and x_3 compute of the roots belong to $[0.6, 0.65]$.

Then the matrix $M(y)$ 15×15 and it's given by

$$\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & y^2-1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & y^2-1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & y^2-1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & y^2-1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & y^2-1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & y^2-1\\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -y^2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -y^2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -y^2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -y^2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -y^2 & 0\\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & y & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & y & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & y & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & y & 0
\end{pmatrix}$$

Then the matrix $M(x)$ 15×15 and it's given by

$$\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & x^2-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & x^2-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & x^2-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & x^2-1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & x^2-1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & x^2-1 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -x^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -x^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -x^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -x^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -x^2 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -x^2 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -x^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -x^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -x^2 & 0
\end{pmatrix}$$

Then the matrix $M(z)$ 15×15 and it's given by

$$\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & z^2-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & z^2-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & z^2-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & z^2-1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & z^2-1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & z^2-1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & z & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & z & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -z^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -z^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -z^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -z^2 & 0
\end{pmatrix}$$

Table 4.1: Minimal eigenvalues for x_1

x_1	λ	ν	x_1	λ	ν
-0.70	$1.27191124e^{-02}$	8	0.05	$2.22173682e^{-02}$	5
-0.65	$8.85884225e^{-02}$	8	0.10	$2.08564155e^{-02}$	5
-0.60	$4.67641301e^{-03}$	7	0.15	$1.86667841e^{-02}$	6
-0.55	$1.51264545e^{-03}$	7	0.20	$1.57759820e^{-02}$	7
-0.50	$7.18964033e^{-05}$	7	0.25	$1.23793046e^{-02}$	7
-0.45	$4.38330123e^{-04}$	6	0.30	$8.75111892e^{-03}$	7
-0.40	$2.31806868e^{-03}$	6	0.35	$5.25114642e^{-03}$	7
-0.35	$5.25114642e^{-03}$	6	0.40	$2.31806868e^{-03}$	7
-0.30	$8.75111893e^{-03}$	6	0.45	$4.38330123e^{-04}$	6
-0.25	$1.23793046e^{-02}$	7	0.50	$7.18964033e^{-05}$	6
-0.20	$1.57759820e^{-02}$	7	0.55	$1.51264545e^{-03}$	6
-0.15	$1.86667841e^{-02}$	7	0.60	$4.67413007e^{-03}$	6
-0.10	$2.08564155e^{-02}$	7	0.65	$8.85884225e^{-03}$	7
-0.05	$2.22173682e^{-02}$	6	0.70	$1.27191124e^{-02}$	8
0.00	$2.26785889e^{-02}$	5			

Table 4.2: Minimal eigenvalues for x_2

x_2	λ	ν	x_2	λ	ν
-0.90	$7.37871009e^{-02}$	6	0.05	$3.98609845e^{-02}$	6
-0.85	$1.54541246e^{-02}$	6	0.10	$3.56493115e^{-02}$	6
-0.80	$2.52100806e^{-02}$	6	0.15	$3.11355792e^{-02}$	6
-0.75	$3.56222937e^{-02}$	6	0.20	$2.64209258e^{-02}$	6
-0.70	$4.47391043e^{-02}$	6	0.25	$2.16350960e^{-02}$	6
-0.65	$4.92109204e^{-02}$	6	0.30	$1.69348432e^{-03}$	6
-0.60	$5.19030914e^{-02}$	6	0.35	$1.24973056e^{-03}$	6
-0.55	$5.39712898e^{-02}$	5	0.40	$8.50791843e^{-03}$	6
-0.50	$5.55339702e^{-02}$	5	0.45	$5.14372620e^{-03}$	6
-0.45	$5.66056506e^{-02}$	5	0.50	$2.55436512e^{-03}$	5
-0.40	$5.71834514e^{-02}$	5	0.55	$8.44063522e^{-04}$	5
-0.35	$5.72614010e^{-02}$	5	0.60	$5.82790608e^{-05}$	5
-0.30	$5.68340255e^{-02}$	5	0.65	$1.77750934e^{-04}$	5
-0.25	$5.58976649e^{-02}$	5	0.70	$1.12056512e^{-03}$	5
-0.20	$5.44512936e^{-02}$	5	0.75	$2.74818083e^{-03}$	5
-0.15	$5.24973687e^{-02}$	5	0.80	$4.85268510e^{-03}$	5
-0.10	$5.00429927e^{-02}$	5	0.85	$6.88978374e^{-03}$	5
-0.05	$4.71016689e^{-02}$	5	0.90	$5.59574311e^{-03}$	6
0.00	$4.36959277e^{-02}$	5			

Example 4.3.2 Consider the following system

$$G_1(x,y,z,w,r) = x^2+y^2+z^2+w^2+r^2-1,$$

$$G_2(x,y,z,w,r) = x^2+y^2+z^2+w^2-r,$$

$$G_3(x,y,z,w,r) = x^2+y^2+z^2+r^2-w,$$

$$G_4(x,y,z,w,r) = x^4-z^2,$$

$$G_5(x,y,z,w,r) = z^2-y^2.$$

Using Mathematica, the solution of the system

$$G(x) = 0 \tag{4.12}$$

are

$$x = \pm 0.418202002, \quad y = \pm 0.174892914$$

$$z = \pm 0.174892914, \quad w = r = 0.618033189.$$

One can see that the size of the resultant matrix for each variable is 445×495. We make an entirely analogous analysis to that of Example (4.3.1). We Scan for a solution of the x-parameter in the interval $[-0.7, 0.7]$, y-parameter, and z-parameter in the interval $[-0.4, 0.4]$, and w-parameter and r-parameter in the interval $[-0.9, 0.9]$. In all the cases, the increment is 0.05. Tables $(4.3-4.7)$, show the minimal eigenvalues λ and the number of evaluations of μw which were necessary to obtain λ, say ν.

Table 4.3: Minimal eigenvalues for x

x	λ	ν	x	λ	ν
-0.70	$8.3714e^{-02}$	10	0.05	$3.2421e^{-02}$	11
-0.65	$4.4281e^{-02}$	10	0.10	$2.6543e^{-02}$	11
-0.60	$1.7562e^{-02}$	10	0.15	$1.6870e^{-02}$	10
-0.55	$7.2162e^{-03}$	11	0.20	$1.0057e^{-02}$	10
-0.50	$1.3869e^{-03}$	11	0.25	$5.7392e^{-03}$	10
-0.45	$3.2863e^{-04}$	10	0.30	$1.5155e^{-03}$	10
-0.40	$1.1111e^{-05}$	10	0.35	$6.5101e^{-04}$	10
-0.35	$7.5412e^{-04}$	10	0.40	$1.1081e^{-05}$	11
-0.30	$2.5410e^{-03}$	10	0.45	$3.3832e^{-04}$	10
-0.25	$7.8320e^{-03}$	10	0.50	$2.9126e^{-03}$	10
-0.20	$1.7774e^{-02}$	11	0.55	$7.2106e^{-03}$	10
-0.15	$2.7656e^{-02}$	11	0.60	$2.4111e^{-02}$	11
-0.10	$3.1826e^{-02}$	11	0.65	$5.8532e^{-02}$	11
-0.05	$4.2010e^{-02}$	11	0.70	$6.7210e^{-02}$	11
0.00	$6.2341e^{-02}$	12			

Table 4.4: Minimal eigenvalues for y

y	λ	ν	y	λ	ν
−0.40	$3.2145e^{-02}$	11	0.05	$2.8085e^{-03}$	11
−0.35	$1.2360e^{-02}$	11	0.10	$5.6910e^{-04}$	11
−0.30	$5.7681e^{-03}$	12	0.15	$3.1232e^{-05}$	10
−0.25	$4.4441e^{-04}$	11	0.20	$3.6295e^{-05}$	10
−0.20	$3.2142e^{-05}$	10	0.25	$5.6563e^{-04}$	11
−0.15	$3.0210e^{-05}$	10	0.30	$4.4222e^{-03}$	11
−0.10	$5.4980e^{-04}$	11	0.35	$8.9066e^{-03}$	12
−0.05	$3.0289e^{-03}$	12	0.40	$2.9400e^{-02}$	11
0.00	$2.1256e^{-02}$	11			

Table 4.5: Minimal eigenvalues for z

z	λ	ν	z	λ	ν
−0.40	$3.5412e^{-02}$	12	0.05	$1.9326e^{-03}$	12
−0.35	$1.8720e^{-02}$	11	0.10	$2.1211e^{-04}$	11
−0.30	$4.9998e^{-03}$	12	0.15	$3.0022e^{-05}$	10
−0.25	$3.3321e^{-04}$	11	0.20	$3.1240e^{-05}$	10
−0.20	$3.3908e^{-05}$	10	0.25	$4.4781e^{-04}$	11
−0.15	$3.2085e^{-05}$	10	0.30	$3.4061e^{-03}$	11
−0.10	$4.2106e^{-04}$	11	0.35	$7.0169e^{-03}$	12
−0.05	$5.2376e^{-03}$	12	0.40	$3.9223e^{-02}$	12
0.00	$1.8720e^{-02}$	12			

Table 4.6: Minimal eigenvalues for w

w	λ	ν	w	λ	ν
-0.90	$7.9911e^{-02}$	12	0.05	$5.3245e^{-02}$	11
-0.85	$2.6712e^{-02}$	12	0.10	$4.3876e^{-02}$	11
-0.80	$6.7802e^{-02}$	11	0.15	$3.2345e^{-02}$	11
-0.75	$8.9112e^{-02}$	11	0.20	$2.9879e^{-02}$	10
-0.70	$1.1114e^{-02}$	12	0.25	$1.0221e^{-02}$	10
-0.65	$8.9262e^{-02}$	12	0.30	$5.0211e^{-03}$	10
-0.60	$8.0925e^{-02}$	12	0.35	$4.2333e^{-03}$	10
-0.55	$7.0254e^{-02}$	12	0.40	$2.0011e^{-03}$	11
-0.50	$5.0282e^{-02}$	12	0.45	$1.0098e^{-03}$	11
-0.45	$5.9845e^{-02}$	12	0.50	$1.0001e^{-03}$	11
-0.40	$2.0186e^{-02}$	12	0.55	$4.9888e^{-04}$	10
-0.35	$7.0982e^{-02}$	11	0.60	$2.0110e^{-05}$	10
-0.30	$7.0981e^{-02}$	11	0.65	$1.2299e^{-04}$	10
-0.25	$1.1652e^{-02}$	11	0.70	$3.0098e^{-03}$	11
-0.20	$2.0931e^{-02}$	11	0.75	$4.6721e^{-03}$	10
-0.15	$1.8733e^{-02}$	11	0.80	$7.8882e^{-03}$	11
-0.10	$6.0245e^{-02}$	10	0.85	$8.9901e^{-03}$	11
-0.05	$7.9867e^{-02}$	10	0.90	$9.9913e^{-03}$	12
0.00	$6.0001e^{-02}$	10			

Table 4.7: Minimal eigenvalues for r

r	λ	ν	r	λ	ν
-0.90	$8.2234e^{-02}$	13	0.05	$7.2397e^{-02}$	12
-0.85	$4.8972e^{-02}$	12	0.10	$5.5551e^{-02}$	12
-0.80	$7.3003e^{-02}$	12	0.15	$3.2458e^{-02}$	11
-0.75	$7.5412e^{-02}$	12	0.20	$2.3145e^{-02}$	11
-0.70	$2.3341e^{-02}$	12	0.25	$1.9110e^{-02}$	10
-0.65	$7.1112e^{-02}$	12	0.30	$9.9994e^{-03}$	10
-0.60	$9.9989e^{-02}$	12	0.35	$7.8234e^{-03}$	11
-0.55	$1.2312e^{-02}$	12	0.40	$5.3572e^{-03}$	11
-0.50	$4.0026e^{-02}$	12	0.45	$2.2299e^{-03}$	11
-0.45	$4.1209e^{-02}$	12	0.50	$1.1009e^{-03}$	11
-0.40	$7.2124e^{-02}$	11	0.55	$4.4422e^{-04}$	10
-0.35	$3.9920e^{-02}$	11	0.60	$2.1180e^{-05}$	10
-0.30	$1.1191e^{-02}$	11	0.65	$1.7521e^{-04}$	10
-0.25	$2.1367e^{-02}$	11	0.70	$3.6001e^{-03}$	11
-0.20	$6.3456e^{-02}$	11	0.75	$5.9236e^{-03}$	10
-0.15	$1.9867e^{-02}$	11	0.80	$6.9221e^{-03}$	11
-0.10	$3.3332e^{-02}$	10	0.85	$9.9966e^{-03}$	11
-0.05	$1.0009e^{-02}$	10	0.90	$5.4470e^{-03}$	12
0.00	$4.5321e^{-02}$	10			

From Tables $(4.3-4.7)$, we see that this approach works nicely and efficiently. Comparing the number of evaluations ν in our approach with Allgower [2], we see that their approach is more expensive than ours.

4.4 Conclusions

In this thesis, the location of the zeros of polynomial systems using multiresultant metrics demonstrated in different methods such as "one-dimensional path following method" and the "Lancrzos method". It started with preliminaries about the multiresultant of homogenous polynomial systems and how to homogenize the inhomogenous polynomial system, although several numerical examples were presented and illustrated the technique dealing with large sparse matrices which has a finite number of solutions for homogenous as well as inhomogenous polynomial systems. Chapter 1 presented the literature review. Furthermore, Chapter 2 investigated the relationship between the resultant matrix and the zeros of polynomial systems and it is devoted to homogeneous and inhomogeneous polynomial systems. Several numerical examples were illustrated with theoretical results which prove that the multiresultant matrix has at least one zero eigenvalues. In Chapter 3, we presented a new method to deal with an unstable method which has been used in Chapter 2 to find the roots of high degree multiresultant. However, the new method is numerically stable and preserves the sparseness of the multiresultant matrix, this new method is called the one-dimensional path following method. Also, the numerical results of the singular matrix showed the efficiency and sufficiency of the proposed method. The approach of theorems (3.1.1)-(3.1.2) and Cramer's rule shows the approximation of the zeros' location when the sign of the function changes. Moreover, secant method has been used to approximate the root fastly and it is easy to use. Furthermore, a one-dimensional path following method is to find the roots of the function, and it is a bad practice to expand the determinants symbolically when the size of the matrix is large. In addition, a numerical example described the intersection of a sphere in $\mathbb{R}^3$ with two parabolas by using Mathematica-(12.1) and path following method and secant method to identify the accurate solutions of the System 3.1.4. In addition, Chapter 4 presented equivalent conditions to multiresultant, as we justify in Chapter 2 computing the determinant of the resultant matrix is unstable and costly to overcome these issues we proceed to equivalent conditions to multi-resultant by using Lanczos method which is one of the most suitable to use for large sparse square symmetric matrix. Finally, some conclusions were drawn in Chapter 4.

References

[1] Collins, G. E. (1971). The calculation of multivariate polynomial resultants. Journal of the ACM (JACM), 18(4), 515-532.

[2] Allgower, E. L., Georg, K., & Miranda, R. (1992). The method of resultants for computing real solutions of polynomial systems. SIAM Journal on Numerical Analysis, 29(3), 831-844.

[3] Morozov, A. Y., & Shakirov, S. R. (2010). New and old results in resultant theory. Theoretical and Mathematical Physics, 163(2), 587-617.

[4] Li, T. Y., Sauer, T., & Yorke, J. A. (1989). The cheater's homotopy: an efficient procedure for solving systems of polynomial equations. SIAM Journal on Numerical Analysis, 26(5), 1241-1251.

[5] Morgan, A. P., Sommese, A. J., & Watson, L. T. (1989). Finding all isolated solutions to polynomial systems using HOMPACK. ACM Transactions on Mathematical Software (TOMS), 15(2), 93-122.

[6] Syam, M. I. (2001). The resultants method for approximating real fixed points of polynomials. Computers & Mathematics with Applications, 41(7-8), 879-891.

[7] Syam, M. (2005). Conjugate gradient predictor corrector method for solving large scale problems. Mathematics of computation, 74(250), 805-818.

[8] Syam, M. I. (2001). The resultants method for approximating real fixed points of polynomials. Computers & Mathematics with Applications, 41(7-8), 879-891.

[9] Syam, M. I. (2004). Finding all real zeros of polynomial systems using multi-resultant. Journal of Computational and Applied Mathematics, 167(2), 417-428.

[10] Raja, M. A. Z., Abbas, S., Syam, M. I., & Wazwaz, A. M. (2018). Design of neuro-evolutionary model for solving nonlinear singularly perturbed boundary value problems. Applied Soft Computing, 62, 373-394.

[11] Duff, T., Hill, C., Jensen, A., Lee, K., Leykin, A., & Sommars, J. (2019). Solving polynomial systems via homotopy continuation and monodromy. IMA Journal of Numerical Analysis, 39(3), 1421-1446.

[12] Loisel, S., & Maxwell, P. (2018). Path-following method to determine the field of values of a matrix with high accuracy. SIAM Journal on Matrix Analysis and Applications, 39(4), 1726-1749.

[13] Musco, C., Musco, C., & Sidford, A. (2018). Stability of the Lanczos method for matrix function approximation. In Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms (pp. 1605-1624). Society for Industrial and Applied Mathematics.

[14] Noferini, V., & Townsend, A. (2016). Numerical instability of resultant methods for multidimensional rootfinding. SIAM Journal on Numerical Analysis, 54(2), 719-743.

[15] Allgower, E. L., Georg, K., & Miranda, R. (1992). The method of resultants for computing real solutions of polynomial systems. SIAM Journal on Numerical Analysis, 29(3), 831-844.

[16] Jia, Z., Ng, M. K., & Song, G. J. (2019). Lanczos method for large-scale quaternion singular value decomposition. Numerical Algorithms, 82(2), 699-717.

[17] Bini, D. A., Gemignani, L., & Pan, V. Y. (2004). Improved initialization of the accelerated and robust QR-like polynomial root-finding. Electronic Transactions on Numerical Analysis, 17, 195-205.

UAE UNIVERSITY MASTER THESIS NO. 2022: 90

In this thesis, will modify a new practicable method for approximating all real zeros of polynomial systems using the multi-resultant method. Multi-Resultant method is used to solve systems of polynomial equations to determine whether or not solutions exist, or to reduce a given system to one with fewer variables and/or fewer equations.

Ayade Salah Ayade Abdelmalk received his Master of Science in Mathematical Sciences from the Department of Mathematics, College of Science, UAE University. He received his Bachelor of Mathematics from the College of Mathematics Section in Science and Education, Sultan Assuit University, Egypt.