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A SUPPORT THEOREM FOR A WAVE EQUATION

Aysha Khaled Alshamsi

United Arab Emirates University

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College of Science

Department of Mathematical Sciences

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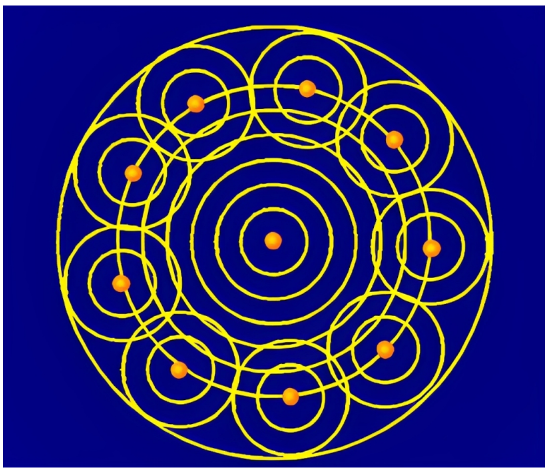

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Declaration of Original Work

I, Aysha Khaled Alshamsi, the undersigned, a graduate student at The United Arab Emirates University (UAEU), and the author of this thesis, entitled "*A Support Theorem for a Wave Equation*", hereby, solemnly declare that this is the original research work done by me under the supervision of Dr. Salem Ben Said, in the College of Science at the UAEU. This work has not previously formed the basis for the award of any academic degree, diploma or a similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my thesis have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this thesis.

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Approval of The Master Thesis

This Master Thesis is approved by the following Examining Committee Members:

1) Advisor (Committee Chair): Salem Ben Said
   Title: Associate Professor
   Department of Mathematical Sciences
   College of Science

   Signature: Date: May 4,2023

2) Member: Youssef El Khatib
   Title: Professor
   Department of Mathematical Sciences
   College of Science

   Signature: Date: May 4,2023

3) Member (External): Hatem Mejjaoli
   Title: Professor
   Taibah University Al Madinah AL Munawarah, KSA

   Signature: Date: May 4,2023

This Master Thesis is accepted by:

Dean of the College of Science: Professor Maamar Benkraouda

Signature: ______________________

Date: 4/7/2023

Dean of the College of Graduate Studies: Professor Ali Al-Marzouqi

Signature: ______________________

Date: July 4, 2023

Abstract

It is well known that the fundamental solution to the classical wave equation $\Delta u(x,t) - \partial_{tt} u(x,t) = 0$ is supported on the light cone $\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : ||x|| = |t|\}$ if and only if the dimension n is odd and ≥ 3. Because we are living in a 3-dimensional world we can hear each other clearly; One has a pure propagator without residual waves. In this thesis we consider the wave equation

$$2\|x\|\Delta_k u_k(x,t) - \partial_{tt} u_k(x,t) = 0, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

where Δ_k is a second order differential and difference operator. First, we prove the existence and the uniqueness of the solution $u_k(x,t)$. Second, we search for the condition on the parameter k and the dimension n for the fundamental solution to be supported on the light cone $\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \sqrt{2||x||} = |t|\}$. Our approach is based heavily on the representation theory of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, where we construct a new representation ω_k of $\mathfrak{sl}(2,\mathbb{R})$ acting on the Schwartz space $S(\mathbb{R}^n)$. Finally, we prove that ω_k lifts to give raise to a unitary representation of a simply connected Lie group with Lie algebra $\mathfrak{sl}(2,\mathbb{R})$.

Keywords: Dunkl operators, wave equation, conservation of total energy, generalized Fourier transform, convolution structure, Huygens' principle, Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, representation theory Lie algebras, integrability of infinitisimal representations.

Title and Abstract (in Arabic)

النظرية الداعمة للمعادلة الموجية

الملخص

من المعروف أن الحل الأساسي لمعادلة الموجة الكلاسيكية $\Delta u(x,t) - \partial_{tt} u(x,t) = 0$ يتم دعمه على المخروط الضوئي $\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : ||x|| = |t|\}$ إذا وفقط إذا كانت الأبعاد n فردية واكبر او تساوي 3. لأننا نعيش في عالم ثلاثي الأبعاد، يمكننا سماع بعضنا البعض بوضوح. وبذلك يكون لدينا ناقل نقي بدون موجات باقية.

في هذه الأطروحة ندرس معادلة الموجة

$$2||x||\Delta_k u_k(x,t) - \partial_{tt} u_k(x,t) = 0, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

حيث Δ_k هو عامل تفاضلي وفرقي من الدرجة الثانية. أولاً، نثبت وجود وفردية الحل $u_k(x,t)$. ثانيًا، نبحث عن الشرط على العامل k والأبعاد n لدعم الحل الأساسي على المخروط الضوئي $\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \sqrt{2||x||} = |t|\}$. يعتمد نهجنا بشكل كبير على نظرية التمثيل لجبر ليّ $\mathfrak{sl}(2,\mathbb{R})$ حيث نقوم ببناء تمثيلًا جديدًا ω_k ل $\mathfrak{sl}(2,\mathbb{R})$ يعمل على الفضاء الشوارزي $S(\mathbb{R}^n)$. أخيرًا، نثبت أن ω_k تتكامل لتعطي تمثيلًا موحدًا لمجموعة لي بسيطة ذات جبر ليّ $\mathfrak{sl}(2,\mathbb{R})$.

كلمات البحث الرئيسية: عوامل دانكل، معادلة الموجة، حفظ الطاقة الكلية، تحويل فورييه العام، هيكل التكرار، مبدأ هويغنز، جبر ليّ $\mathfrak{sl}(2,\mathbb{R})$، نظرية التمثيل لجبر ليّ، تكامل التمثيلات متناهية الصغر.

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Most importantly, I am grateful for my family's unconditional support.

Dedication

To my beloved family.

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Chapter 1: Introduction

1.1 The Classical Wave Equation

One of the most fascinating features of partial differential equations is the second order wave equation, which is the canonical example of a hyperbolic PDE. In n dimensions, the equation takes the form

$$\Delta u(x,t) - \partial_{tt} u(x,t) = 0$$

Where Δ is the Laplacian operator $\dfrac{\partial^2}{\partial x_1^2} + \cdots + \dfrac{\partial^2}{\partial x_n^2}$. Since the wave equation is of second order in t, a well-posed initial value problem for this equation would normally involve two initial conditions such as $u(x,0) = f(x)$ and $\partial_t u(x,0) = g(x)$.

The wave equation has numerous applications. The classical 1-dimension example is the vibration of an ideal string, and in two-dimension (2D) this becomes the vibration of an ideal membrane or drum. In three-dimensional (3D), the most famous example is the propagation of sound waves in a gas or liquid. A curious property known as Huygens principle is as follows: In his wave theory of light, Huygens implicitly proposed that the fundamental solution of the wave equation supported in the future cone $C_+ = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : ||x|| \le |t|\}$ is actually supported on the light cone

$$\partial C_+ = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : ||x|| = |t|\}$$

for $n = 3,5,7,\dots$ (See some historical remarks in [1]).

In [2] Hadamard searched for other second-order hyperbolic equations that satisfied this remarkable fact. He proved the necessary condition that n is odd, and gave a sufficient condition which is highly ineffective to the point that no non-trivial example could be found from it. Here trivial examples refer to those that could be obtained from the wave equation above by a non-singular change of variables or multiplication by a non-zero function. The belief that no non-trivial equation with Huygens principle exist is known as Hadamard's conjecture, and for $n = 3$ it was proved independently by Mathisson [3] and Asgeirsson [4]:

- The solution to the wave equation in 1-dimension is given by d'Alembert's:

$$u(x,t) = \frac{1}{2}(f(x+t)+f(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds.$$

- The solution to the 2-dimensional wave equation is:

$$u\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, t\right) = \iint_D \frac{f\binom{y_1}{y_2}}{\sqrt{t^2-(y_1-x_1)^2-(y_2-x_2)^2}}\frac{dy_1dy_2}{2\pi} + \frac{\partial}{\partial t}\left(\iint_D \frac{g\binom{y_1}{y_2}}{\sqrt{t^2-(y_1-x_1)^2-(y_2-x_2)^2}}\frac{dy_1dy_2}{2\pi}\right),$$

where $D = \left\{\binom{y_1}{y_2} : (x_1-y_1)^2+(x_2-y_2)^2 \le t^2\right\}$

- Kirchhoff's formula gives the solution to the wave equation in the 3-spatial dimension.

$$u(x,t) = \frac{1}{4\pi t^2}\iint_{S(x,t)} f(s)ds + \frac{\partial}{\partial t}\left(\frac{1}{4\pi t^2}\iint_{S(x,t)} g(s)ds\right)$$

where $S \subset \mathbb{R}^3$ is the sphere of center x and radius $|t|$.

- The solution to the wave equation in odd dimensions is given by

$$u(x,t) = \frac{1}{c_n}\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\int_{\partial B(x,t)} f(s)ds\right) + \frac{1}{c_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\int_{\partial B(x,t)} g(s)ds\right),$$

where $c_n = 1\cdot 3\cdot 5\cdot 7\cdots(n-2)$. Here $\partial B(x,t)$ is the boundary of the ball of center x and radius $|t|$.

- Using the method of descent, the solution to the wave equation in even dimensions is given by

$$u(x,t) = \frac{1}{c_n}\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n\int_{B(x,t)} \frac{f(y)}{\sqrt{t^2-\|y-x\|^2}}dy\right) + \frac{1}{c_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n\int_{B(x,t)} \frac{g(y)}{\sqrt{t^2-\|y-x\|^2}}dy\right).$$

As a surprise, Stellmacher [5] found the first counterexample to Hadamard's conjecture when $n = 5$ with the example

$$\Delta - \frac{2}{x_1^2} - \partial_{tt}.$$

One interesting family of examples comes from Calogero-Moser systems of which Stellmacher's example is a particular example. It takes the form

$$\Delta - \frac{1}{2} \sum_{\alpha \in \mathcal{R}} \frac{k_\alpha(k_\alpha + 1)}{\langle \alpha, \alpha \rangle} - \partial_{tt},$$

where Berest and Veselov [2] proved that Huygens' principle holds when n is odd and $n \geq 3 + \sum_{\alpha \in \mathcal{R}} k_\alpha$. Here $\mathcal{R} \subset \mathbb{R}^n$ is a root system and $(k_\alpha)_\alpha$ is a set of parameters.

1.2 Motivation

Dunkl theory is an evolution of Euclidean Fourier analysis and the theory of special functions in several variables. In the late 1970s, it gradually became apparent that radial analysis on flat symmetric spaces and the theory of special functions in one variable are closely related. Generally speaking, it turns out that spherical functions on flat symmetric spaces are written in terms of Bessel functions.

A number of attempts were made in the 80's to generalize the above connection to higher ranks. This subject is partly motivated by harmonic analysis on flat symmetric spaces and the growing interest in special functions of several variables. The crucial breakthrough came with the discovery of the rational Dunkl operators introduced by Dunkl in [6]. These operators are differential-reflection operators associated with finite reflection groups on some finite-dimensional Euclidean space. Dunkl's theory has been enriched by this discovery. This theory is based on the work of Koornwinder [7], Heckman [8], Opdam [9], and Dunkl [6]. Following a series of papers, Dunkl developed the so-called Dunkl transform, which is a theory of integral transforms in several variables related to reflection groups. There has been considerable interest in this theory since it encompasses harmonic analysis on flat symmetric spaces and spherical functions in several variables in a unified manner. See, for example, [10], [11], [12], [13] and [14].

Apart from Fourier analysis and multivariable special functions, Dunk theory is also deeply interconnected with algebra (degenerate Hecke algebras) and probability (Feller processes with jumps). Additionally, Dunkl operators can serve as useful tools for studying quantum many body systems of the Calogero-Moser type. In recent years, such models have gained considerable attention in mathematical physics. A comprehensive bibliography is contained in [15].

In [16] Ben Saïd, Kobayashi and Ørsted gave a far reaching generalization of the Dunkl theory through introducing a positive real parameter a as a deformation parameter of Dunkl theory. In particular, a (k,a)-generalized Fourier transform $\mathcal{F}_{k,a}$ has been constructed and acts on a concrete Hilbert function space deforming $L^2(\mathbb{R}^n)$. The parameter k is a multiplicity function coming from Dunkl's theory. If $a = 2$, then we recover the Dunkl Fourier analysis [6], while the case $a = 1$ provides a new framework and is of particular interest, since it is related to the Laguerre semigroup and the minimal unitary representations of $O(n+1;2)$ in Kobayashi and Mano's works [17] and [18]. In [16] and [19], Ben Saïd, Kobayashi and Ørsted developed this original approach which, as evidenced by the literature, has received increasing interest from international researchers. Examples include [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30] and [31]. Several questions were addressed in detail, but many additional problems were left unsolved.

In this thesis we are dealing with a challenging problem that falls within the line of research described above, that is associated with the generalized Fourier transform $\mathcal{F}_{k,a}$ with $a = 1$. Our main objective is to study the support properties of the unique solution to the the wave equation

$$2\|x\|\Delta_k u_k(x,t) - \partial_{tt} u_k(x,t) = 0.$$

Our approach is based heavily on the representation theory of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, where we construct a new representation ω_k of $\mathfrak{sl}(2,\mathbb{R})$ acting on the Schwartz space $S(\mathbb{R}^n)$. Further, we prove that ω_k integrates to give raise to a unitary representation of a simply connected Lie group with Lie algebra $\mathfrak{sl}(2,\mathbb{R})$.

1.3 Framework and Results

Let G be the Coxeter group associated with a root system $\mathcal{R}$ in $\mathbb{R}^n$. For a G-invariant real function k on $\mathcal{R}$, we write Δ_k for the Dunkl Laplacian on $\mathbb{R}^n$. The operator $\|x\|\Delta_k$ is symmetric on the Hilbert space $L^2(\mathbb{R}^n, \vartheta_k(x)dx)$ consisting of square integrable functions on $\mathbb{R}^n$ against the measure

$$\vartheta_k(x)dx = \|x\|^{\alpha-2} \prod_{\alpha\in\mathcal{R}} |\langle\alpha,n\rangle|^{k_\alpha}.$$

Then $\vartheta_k(x)$ has a degree of homogeneity $2\langle k\rangle - 1$ where $\langle k\rangle = \frac{1}{2}\sum_{\alpha\in\mathcal{R}} k_\alpha$.

The generalized Fourier transform $\mathcal{F}_{k,1}$ is defined by

$$\mathcal{F}_{k,1}(f)(\xi) = c_k \int_{\mathbb{R}^n} f(y)B_k(x,y)\vartheta_k(y)dy$$

where

$$B_k(x,y) = V_k\left(\widetilde{J}_{\langle k\rangle+\frac{n-3}{2}}(\sqrt{2rt(1+\langle\theta',\cdot\rangle)})\right)(\theta'')$$

with V_k is the Dunkl intertwining operator and $x = r\theta'$, $y = t\theta''$. Here $\widetilde{J}$ denotes the normalized Bessel function

$$\widetilde{J}_\nu(w) = \sum_{l=0}^{\infty} \frac{(-1)^l w^{2l}}{2^{2l}l!\Gamma(\nu+l+1)}.$$

In particular,

$$B_k(0,y) = 1, \quad |B_k(x,y)| \le 1, \quad \|x\|\,\Delta_k B_k(x,y) = -\|y\|B_k(x,y).$$

We pin down that $\mathcal{F}_{k,1}$ in a unitary operator from $L^2(\mathbb{R}^n, \vartheta_k(x)dx)$ onto itself, and the inverse transform is given by $\mathcal{F}_{k,1}^{-1} = \mathcal{F}_{k,1}$.

For suitable functions on $\mathbb{R}^n$, in [23] the authors defined the translation operator τ_y by

$$\tau_y f(x) = c_k \int_{\mathbb{R}^n} B_k(x,\xi)B_k(y,\xi)\mathcal{F}_{k,1}f(\xi)\vartheta_k(\xi)d\xi.$$

For $n+2\langle k\rangle - 2 > 0$, the authors proved that τ_y extends to a bounded operator on the space $L^p_{\text{rad}}(\mathbb{R}^n, \vartheta_k(x)dx)$, of radical function in $L^p(\mathbb{R}^n, \vartheta_k(x)dx)$, with $1 \le p \le 2$

$$\|\tau_y f\|_{L^p_k} \le \|f\|_{L^p_k}, \qquad 1 \le p \le 2.$$

By means of the translation operator, a convolution operator $\circledast$ was defined on the space $L^p(\mathbb{R}^n, \vartheta_k(x)dx)$ by

$$f \circledast g(x) = c_k \int_{\mathbb{R}^n} f(y)\tau_x g(y)\vartheta_k(y)dy.$$

In particular $f \circledast g = g \circledast f$ and $\mathcal{F}_{k,1}(f \circledast g) = \mathcal{F}_{k,1}(f)\mathcal{F}_{k,1}(g)$. They proved that the map $f \mapsto f \circledast g$, with $g \in L^1_{\mathrm{rad}}(\mathbb{R}^n, \vartheta_k(x)dx)$, extends to the space $L^p(\mathbb{R}^n, \vartheta_k(x)dx)$ with

$$\|f \circledast g\|_{L^p_k} \leq c_k \|f\|_{L^p_k} \|g\|_{L^1_k}$$

for every $1 \leq p \leq \infty$.

This thesis was motivated by the paper [32] duo to Ben Said and Ørsted which studies the Huygens principle for the so-called Dunkl wave equation $\Delta_k u(x,t) - \partial_{tt}u(x,t) = 0$. They proved that the fundamental solution is supported on the light cone $\Big\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \|x\| = |t|\Big\}$ if and only if $n + 1 + \langle k\rangle = 4, 6, 8, \ldots$ Notice that n needs not be odd, in contrast with Hadamard's and Petrovsky's conditions for the differential case.

Consider the following Cauchy problem

$$\begin{cases} 2\,\|x\|\,\Delta_k u(x,t) - \partial_{tt}u(x,t) = 0 \\ u(x,0) = f(x), \;\; \partial_t u(x,0) = g(x), \end{cases}$$

where the initial data (f,g) belongs to the Schwartz space $S(\mathbb{R}^n)$.

First we prove that there exits a unique solution to the above Cauchy problem given by

$$u(x,t) = P^{11}_{k,t} \circledast f(x) + P^{12}_{k,t} \circledast g(x),$$

where

$$P^{11}_{k,t}(x) = \mathcal{F}_k^{-1}(\cos(t\sqrt{2\|\cdot\|}))(x), \qquad P^{12}_{k,t}(x) = \mathcal{F}_k^{-1}(\sin(t\sqrt{2\|\cdot\|})/\sqrt{2\|\cdot\|})(x).$$

Our construction of the solution uses Fourier analysis related to the transform $\mathcal{F}_k := \mathcal{F}_{k,1}$.

To prove the uniqueness of $u(x,t)$, we show that the total energy

$$E_k[u](t) = \int_{\mathbb{R}^n} \Big(|\partial_t u(x,t)|^2 + |\sqrt{2\|x\|\Delta_k}u(x,t)|^2\Big)\vartheta_k(x)dx$$

is indeed independent of the time t. Therefore if u_1 and u_2 are two solutions to the above Cauchy problem, then $E_k(u_1-u_2)(t)=0$, implying that $\partial_t(u_1-u_2)(x,t)=0$, which gives $(u_1-u_2)(x,t)=(u_1-u_2)(x,0)=0$ that is $u_1(x,t)=u_2(x,t)$.

Using $P^{11}_{k,t}$ and $P^{12}_{k,t}$ we define the distributions P^{11}_k and P^{12}_k on the Schwartz space $S(\mathbb{R}^{n+1})\simeq S(\mathbb{R}^n)\otimes S(\mathbb{R})$ by

$$P^{ij}_k(\psi_1\circledast\psi_2)=\int_{\mathbb{R}}P^{ij}_{k,t}(\psi_1)\psi_2(t)dt,$$

for $\psi_1\in S(\mathbb{R}^n)$ and $\psi_2\in S(\mathbb{R})$. From the construction of the solution $u(x,t)$, we deduce easily that

$$(2\|x\|\Delta_k-\partial_{tt})P^{ij}_k=0.$$

Further, for $\lambda>0$ and $\psi\in S(\mathbb{R}^{n+1})$, we introduce $S^x_\lambda\psi(x,t)=\psi(\lambda^2x,t)$, $S^t_\lambda\psi(x,t)=\psi(x,\lambda t)$, and we set $S_\lambda=S^x_\lambda\circ S^t_\lambda$. By duality, the operator S_λ acts on distributions in the standard way. We prove that P^{11}_k and P^{12}_k satisfy

$$S_\lambda P^{ij}_k=\lambda^{1+j-i}\,P^{ij}_k.$$

Now saying that P^{11}_k and P^{12}_k are supported on the light cone

$$\partial C_+=\left\{(x,t)\in\mathbb{R}^n\times\mathbb{R}\mid\sqrt{2\|x\|}=|t|\right\},$$

is equivalent to say that

$$\left(2\|x\|-|t|^2\right)^m P^{ij}_k=0,$$

for some integer m (see [33]). Next we shall connect the above proved properties of P^{11}_k and P^{12}_k to a representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$.

We introduce the following differential-difference operators on $\mathbb{R}^n\setminus\{0\}$ by

$$\mathbb{E}^+_k:=i\|x\|,\qquad \mathbb{E}^-_k:=i\|x\|\Delta_k,\qquad \mathbb{H}_k:=n+2\langle k\rangle-1+2\sum_{i=1}^n x_i\partial_i.$$

We prove that $\mathbb{E}^+_k$, $\mathbb{E}^-_k$ and $\mathbb{H}_k$ form an $\mathfrak{sl}_2$-triple, that is

$$[\mathbb{E}^+_k,\mathbb{E}^-_k]=\mathbb{H}_k,\quad[\mathbb{H}_k,\mathbb{E}^+_k]=2\mathbb{E}^+_k,\quad[\mathbb{H}_{n,1},\mathbb{E}^-_k]=-2\mathbb{E}^-_k.$$

By means of the above operators we construct a representation ω_k of the 3-dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ by mapping

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \mathbb{H}_k, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \mathbb{E}_k^+, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto \mathbb{E}_k^-.$$

In the light of the above constructed representation ω_k together with the properties of the distributions P_k^{11} and P_k^{12} mentioned earlier, we conclude that proving Huygens' principle for the wave equation $2\,\|x\|\,\Delta_k u(x,t) - \partial_{tt} u(x,t) = 0$ is equivalent to show that P_k^{11} and P_k^{12} generates a finite dimensional representation $\omega_k(\mathfrak{sl}(2,\mathbb{R}))$. Using a result in [16] about the branching decomposition of $S(\mathbb{R}^n)$ under the action of $G \times \mathfrak{sl}(2,\mathbb{R})$, we deduce that Huygens' principle fails when $2\langle k\rangle - 1/2 \notin \mathbb{Z}$, which leaves the likelihood that the wave equation may satisfies Huygens' principle when $2\langle k\rangle - 1/2 \in \mathbb{Z}$. Then we were able to prove that ω_k generates a finite dimensional representation of $\mathfrak{sl}(2,\mathbb{R})$ if and only if (recall that $\langle k\rangle \in \mathbb{R}$)

$$2n + 4\langle k\rangle - 5 \in 2\mathbb{N}.$$

It is remarkable to see that when the multiplicity function k is zero, then $2\,\|x\|\,\Delta - \partial_{tt} = 0$ does not satisfy the Huygens' principle, where Δ is the classical Laplacian, while if the multiplicity function k is non-zero, then the Huygens principle holds as long as $2n+4\langle k\rangle - 5$ is an even integer.

We close the thesis by answering to the question of integrability of the Lie algebra representation ω_k constructed above. Note that the integrability fact is not obvious, since in infinite dimensions, the existence of a group representation is not guaranteed from the existence of a Lie algebra representation. By means of a beautiful result due to E. Nelson [34], we prove that ω_k exponentiates to a unique unitary representation of the universal covering group of $SL(2,\mathbb{R})$.

Chapter 2: Dunkl Operators

2.1 Introduction

Dunkl's discovery in 1989 of the differential difference operators that now bear his name is one of the most important recent developments in the theory of special functions associated with root systems. In this chapter, we will provide an introduction to the theory of Dunkl operators, and to give some of their properties. We do not intend to give a complete survey, but rather focus on those aspects which will be important in the context of this thesis. Our main references for this chapter are [10, 35–37].

2.2 Root Systems and Coxeter Groups

Let $\langle\cdot,\cdot\rangle$ be the standard Euclidean scalar product in $\mathbb{R}^n$. For $x\in\mathbb{R}^n$, denote $\|x\| = \langle x,x\rangle^{1/2}$. Given any vector α in $\mathbb{R}^n\backslash\{0\}$, we denote by s_α the orthogonal reflection in the hyperplane $H_\alpha = \{y\in\mathbb{R}^n : \langle\alpha,y\rangle = 0\}$. More precisely,

$$s_\alpha(x) = x - 2\frac{\langle\alpha,x\rangle}{\langle\alpha,\alpha\rangle}\alpha, \quad \forall x\in\mathbb{R}^n.$$

Lemma 2.1. *For any $v\in\mathbb{R}^n\backslash\{0\}$, the reflection s_v preserves the inner product $\langle\cdot,\cdot\rangle$.*

Proof. *By definition we have*

$$\begin{aligned}
& \langle s_v(x), s_v(y)\rangle \\
= \ & \langle x - 2\frac{\langle x,v\rangle}{\langle v,v\rangle}v, y - 2\frac{\langle y,v\rangle}{\langle v,v\rangle}v\rangle \\
= \ & \langle x,y\rangle + \left\langle x, -2\frac{\langle y,v\rangle}{\langle v,v\rangle}v\right\rangle + \left\langle -2\frac{\langle x,v\rangle}{\langle v,v\rangle}v, y\right\rangle + \left\langle -2\frac{\langle x,v\rangle}{\langle v,v\rangle}v, -2\frac{\langle y,v\rangle}{\langle v,v\rangle}v\right\rangle \\
= \ & \langle x,y\rangle - 2\frac{\langle y,v\rangle}{\langle v,v\rangle}\langle x,v\rangle - 2\frac{\langle x,v\rangle}{\langle v,v\rangle}\langle v,y\rangle + 2\frac{\langle y,v\rangle}{\langle v,v\rangle}\cdot 2\frac{\langle x,v\rangle}{\langle v,v\rangle}\langle v,v\rangle \\
= \ & \langle x,y\rangle - 4\frac{\langle x,v\rangle\langle y,v\rangle}{\langle v,v\rangle} + 4\frac{\langle x,v\rangle\langle y,v\rangle}{\langle v,v\rangle} \\
= \ & \langle x,y\rangle.
\end{aligned}$$

□

Definition 2.1. *A subset $\mathcal{R}$ of $\mathbb{R}^n$ is called a root system if it satisfies the following axioms:*

(a) $\mathcal{R}$ is finite, does not contain 0, and spans $\mathbb{R}^n$;

(b) for all $\alpha, \beta \in \mathcal{R}$, we have $s_\alpha(\beta) \in \mathcal{R}$.

If in addition $\mathcal{R} \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in \mathcal{R}$, then $\mathcal{R}$ is called reduced.

Example 2.1. *Let $\{e_1, \ldots, e_n\}$ be the standard basis vectors of $\mathbb{R}^n$.*

(1) For $n \geq 2$ the reduced root system of type A_{n-1} is given by

$$\mathcal{R} = \left\{\pm(e_i - e_j) : 1 \leq i < j \leq n\right\}.$$

(2) For $n \geq 2$ the reduced root system of type B_n is defined by

$$\mathcal{R} = \left\{\pm(e_i \pm e_j) : 1 \leq i < j \leq n\right\} \cup \{\pm e_i : 1 \leq i \leq n\}.$$

2.3 Finite Coxeter Groups

A matrix $s = (s_{ij})_{i,j=1}^n$ is called orthogonal if $ss^{\mathrm{T}} = I_n$, where s^{T} denotes the transpose of s and I_n is the $n \times n$ identity matrix. For $\alpha \in \mathcal{R}$, writing

$$s_\alpha = I_n - 2\left(\alpha\alpha^{\mathrm{T}}\right)^{-1}\alpha^{\mathrm{T}}\alpha$$

shows that $s_\alpha = s_\alpha^{\mathrm{T}}$ and $s_\alpha s_\alpha^{\mathrm{T}} = I_n$ (note that $\alpha\alpha^{\mathrm{T}} = \langle\alpha, \alpha\rangle$). This implies that s_α belongs to the group of orthogonal matrices $\mathrm{O}(n)$.

Definition 2.2. *Given a root system $\mathcal{R}$, we will denote by G the subgroup of $\mathrm{O}(n)$ generated by the reflections $\{s_\alpha : \alpha \in \mathcal{R}\}$. We say that G is the Coxeter group associated with $\mathcal{R}$.*

Example 2.2. *Denote by S_n the symmetric group of all the permutations of length n.*

(1) *For the root system* $A_{n-1} = \left\{\pm(e_i - e_j), 1 \le i < j \le n\right\}$*, the corresponding Coxeter group is given by* $G \simeq S_n$

(2) *For the root system* $B_n = \left\{\pm(e_i \pm e_j) : 1 \le i < j \le n\right\} \cup \{\pm e_i : 1 \le i \le n\}$*, we have* $G \simeq S_n \ltimes \{\pm 1\}^n$.

Lemma 2.2. *The following hold true:*

(1) *If* $\alpha \in \mathcal{R}$ *then* $-\alpha \in \mathcal{R}$.

(2) *For any root system* $\mathcal{R}$ *in* $\mathbb{R}^n$*, the Coxeter group G is finite.*

Proof. *(1) The first statement follows from the fact that* $s_\alpha(\alpha) = -\alpha$.
(2) As the elements of G permute $\mathcal{R}$*, there exists a natural homomorphism* $\psi : G \to S(\mathcal{R})$ *of G into the symmetric group* $S(\mathcal{R})$ *of* $\mathcal{R}$*, given by* $\psi(g)(\alpha) := g\alpha$*. This homomorphism is injective: indeed, each reflection* s_α*, and therefore also each element* $g \in G$*, fixes the orthogonal complement of the subspace spanned by* $\mathcal{R}$*. If also* $g\alpha = \alpha$ *for all* $\alpha \in \mathcal{R}$*, then g must be the identity. Thus, G is a subgroup of the symmetric group. As a subgroup of a finite group, G is finite.*

Definition 2.3. (multiplicity function). *A multiplicity function* $k : \mathcal{R} \to \mathbb{C}, \alpha \mapsto k(\alpha)$*, is a series of parameters assigned to each disjoint part of a root system* $\mathcal{R}$*. That is, if* $\alpha, \beta, \gamma \in \mathcal{R}$ *such that* $s_\alpha(\beta) = \gamma$*, then* $k(\beta) = k(\gamma)$*. In other words, the multiplicities assigned to two different roots are different if they cannot be related by a series of reflections.*

Example 2.3. *In the case where the root system*

$$\mathcal{R} = B_2 = \{\pm(e_1 - e_2), \pm(e_1 + e_2), \pm e_1, \pm e_2\}$$

there is no root that reflects the roots $\pm e_1$ *and* $\pm e_2$ *into the roots* $\pm(e_1 - e_2)$ *and* $\pm(e_1 + e_2)$*. Therefore, the multiplicity function for* B_2 *can take only two different values:* $k(\pm e_1) = k(\pm e_2) = k_1$ *and* $k(\pm(e_1 - e_2)) = k(\pm(e_1 + e_2)) = k_2$.

2.4 Dunkl Operators

We will assume that $\mathcal{R}$ is a normalized reduced root system with $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in \mathcal{R}$. We will denote by K^+ the set of multiplicity functions k such that $k_\alpha \geq 0$ for every $\alpha \in \mathcal{R}$.

Definition 2.4. (see [38]). *For $\xi \in \mathbb{R}^n$ and $k \in K^+$, the Dunkl operator $T_\xi := T_\xi(k)$ is defined on $C^1(\mathbb{R}^n)$ by*

$$T_\xi f(x) = \partial_\xi f(x) + \frac{1}{2} \sum_{\alpha \in \mathcal{R}} k_\alpha \langle \alpha, \xi \rangle \left\{ \frac{f(x) - f(s_\alpha x)}{\langle \alpha, x \rangle} \right\}$$

where $\partial_\xi = \langle \xi, \nabla \rangle$ is the directional derivative associated to ξ, with $\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right)$. Notice that when $k \equiv 0$, the Dunkl operator T_ξ reduces to ∂_ξ.

Example 2.4. *There is only one root system of rank 1, consisting of two nonzero vectors $\mathcal{R} = \{\pm\sqrt{2}\}$, which is a root system of type A_1. The corresponding Coxeter group is $G = \{\pm id\}$. The Dunkl operator with multiplicity parameter $k \in \mathbb{C}$ is given by*

$$T f(x) = f'(x) + k \frac{f(x) - f(-x)}{x}.$$

Example 2.5. *Dunkl operators of type A_{n-1}. Suppose $G = S_n$ with root system of type A_{n-1}. As all transpositions σ_{ij} are conjugate in S_n, the vector space of multiplicity functions is one-dimensional. The Dunkl operators associated with the parameter $k \geq 0$ are given by*

$$T_i = \partial_i + k \cdot \sum_{j \neq i} \frac{1 - \sigma_{ij}}{x_i - x_j} \quad (i = 1, \ldots, n).$$

Example 2.6. *Suppose $\mathcal{R}$ is a root system of type B_n. There are two conjugacy classes of reflections in G, leading to multiplicity functions of the form $k = (k_0, k_1)$. The associated Dunkl operators are given by*

$$T_i = \partial_i + k_1 \sum_i \frac{1 - \sigma_i}{x_i} + k_0 \cdot \sum_{j \neq i} \left[\frac{1 - \sigma_{ij}}{x_i - x_j} + \frac{1 - \tau_{ij}}{x_i + x_j} \right] \quad (i = 1, \ldots, n),$$

where $\tau_{ij} := \sigma_{ij} \sigma_i \sigma_j$.

Let $\mathcal{P} := \mathbb{C}[\mathbb{R}^n]$ be the $\mathbb{C}$-algebra of polynomial functions on $\mathbb{R}^n$, and $\mathcal{P}_m \subset \mathcal{P}$ be the space of polynomials of degree m. It is well know that :

(D1) The Dunkl operator T_ξ is homogeneous of degree -1 on $\mathcal{P}$, that is, $T_\xi p \in \mathcal{P}_{m-1}$ for all $p \in \mathcal{P}_m$.

(D2) The Dunkl operator T_ξ leaves the spaces $C_c^\infty(\mathbb{R}^n)$ and $S(\mathbb{R}^n)$ invariant.

The above two claims follow directly from the fundamental theorem of calculus

$$\frac{f(x)-f(s_\alpha x)}{\langle \alpha, x\rangle} = \int_0^1 \partial_\alpha f(x - t\langle \alpha, x\rangle \alpha)dt.$$

Theorem 2.1. *If $f, g \in C^1(\mathbb{R}^n)$, we have*

$$\begin{aligned} T_\xi(fg)(x) =& g(x)T_\xi f(x) + f(x)T_\xi g(x) \\ &- \frac{1}{2}\sum_{\alpha\in\mathcal{R}} k_\alpha \frac{\langle \alpha, \xi\rangle}{\langle \alpha, x\rangle}\{f(x)-f(s_\alpha x)\}\{g(x)-g(s_\alpha x)\}. \end{aligned}$$

Proof. *Using definition 2.4. we have*

$$\begin{aligned} T_\xi(fg)x &= \partial_\xi(fg)(x) + \frac{1}{2}\sum k_\alpha \frac{\langle \alpha, \xi\rangle}{\langle \alpha, x\rangle}\Big\{f(x)g(x) - f(s_\alpha x)g(s_\alpha x)\Big\} \\ &= \partial_\xi f(x)g(x) + \partial_\xi g(x)f(x) + \frac{1}{2}\sum k_\alpha \frac{\langle \alpha, \xi\rangle}{\langle \alpha, x\rangle}\Big\{f(x)g(x) - f(s_\alpha x)g(s_\alpha x)\Big\} \\ &= \partial_\xi f(x)g(x) + \frac{1}{2}\sum k_\alpha \frac{\langle \alpha, \xi\rangle}{\langle \alpha, x\rangle}\Big\{f(x)g(x) - g(x)f(s_\alpha x)\Big\} \\ &\quad -\frac{1}{2}\sum k_\alpha \frac{\langle \alpha, \xi\rangle}{\langle \alpha, x\rangle}\Big\{f(x)g(x) - g(x)f(s_\alpha x)\Big\} \\ &\quad +\partial_\xi g(x)f(x) + \frac{1}{2}\sum k_\alpha \frac{\langle \alpha, \xi\rangle}{\langle \alpha, x\rangle}\Big\{f(x)g(x) - f(x)g(s_\alpha x)\Big\} \\ &\quad -\frac{1}{2}\sum k_\alpha \frac{\langle \alpha, \xi\rangle}{\langle \alpha, x\rangle}\Big\{f(x)g(x) - f(x)g(s_\alpha x)\Big\} \\ &\quad +\frac{1}{2}\sum k_\alpha \frac{\langle \alpha, \xi\rangle}{\langle \alpha, x\rangle}\Big\{f(x)g(x) - f(s_\alpha x)g(s_\alpha x)\Big\} \end{aligned}$$

This finishes the proof.

As a corollary we obtain the following important property.

Corollary 2.1. *Assume that f and g are in $C^1(\mathbb{R}^n)$ and at least one of them is G invariant. Then,*

$$T_\xi[fg](x) = g(x)T_\xi f(x) + f(x)T_\xi g(x).$$

It is well known that $\partial_\xi\partial_\eta = \partial_\eta\partial_\xi$. The most remarkable fact about the Dunkl operators is that a similar fact holds true.

Theorem 2.2. *(see [35, 39]) For fixed $k \in K^+$, the Dunkl operators commute,*

$$T_\xi T_\eta = T_\eta T_\xi \quad \textit{for all } \xi, \eta \in \mathbb{R}^n.$$

Proof. *Using the bracket $[\cdot,\cdot]$ for the commutator of two operators, i.e. $[A,B] = A \circ B - B \circ A$. Let us write the Dunkl operator as $T_\xi = \partial_\xi + \frac{1}{2}\sum_{\alpha\in\mathcal{R}} k_\alpha\langle\alpha,\xi\rangle\Delta_\alpha$ where $\Delta_\alpha f(x) = \frac{f(x)-f(r_\alpha x)}{\langle\alpha,x\rangle}$. For arbitrary $\xi,\eta \in \mathbb{R}^n$ we can have*

$$\left[T_\xi, T_\eta\right] = I + II + III$$

where

$$\begin{aligned}
I &= \left[\partial_\xi, \partial_\eta\right] = 0,\\
II &= \frac{1}{2}\sum_{\alpha\in\mathcal{R}} k_\alpha \left\{\langle\alpha,\xi\rangle\left[\Delta_\alpha,\partial_\eta\right] + \langle\alpha,\eta\rangle\left[\partial_\xi,\Delta_\alpha\right]\right\},\\
III &= \frac{1}{4}\sum_{\alpha,\beta\in\mathcal{R}} k_\alpha k_\beta\langle\alpha,\xi\rangle\langle\beta,\eta\rangle\left[\Delta_\alpha,\Delta_\beta\right].
\end{aligned}$$

For $\alpha \in \mathcal{R}$, we have

$$\left[\partial_\xi,\Delta_\alpha\right] = \frac{\langle\alpha,\xi\rangle}{\langle\alpha,\cdot\rangle}\left\{r_\alpha\partial_\alpha - \Delta_\alpha\right\},$$

and

$$\left[\Delta_\alpha,\partial_\eta\right] = -\frac{\langle\alpha,\eta\rangle}{\langle\alpha,\cdot\rangle}\left\{r_\alpha\partial_\alpha - \Delta_\alpha\right\}.$$

Therefore,

$$II = \frac{1}{2}\sum_{\alpha\in\mathcal{R}} k_\alpha\frac{\langle\alpha,\xi\rangle\langle\alpha,\eta\rangle}{\langle\alpha,\cdot\rangle}\left\{r_\alpha\partial_\alpha - \Delta_\alpha\right\}(-1+1) = 0.$$

The term III can be written as

$$III = \frac{1}{4}\sum_{\alpha,\beta\in\mathcal{R}} k_\alpha k_\beta \{\langle\alpha,\xi\rangle\langle\beta,\eta\rangle - \langle\alpha,\eta\rangle\langle\beta,\xi\rangle\}\Delta_\alpha\Delta_\beta.$$

The fact that part $III = 0$ *is due to the following crucial fact proved by Dunkl.*

Theorem 2.3. *Suppose* $B(\cdot,\cdot)$ *is a bilinear form on* $\mathbb{R}^n$ *such that*

$$B(s_\alpha\lambda, s_\alpha\mu) = B(\mu,\lambda) \qquad \forall\lambda,\mu\in\mathbb{R}^n, \forall\alpha\in R\cap \operatorname{span}\langle\lambda,\mu\rangle.$$

If $w\in G$ *is a pure rotation (i.e.* $\dim \operatorname{Im}(w-\mathrm{id}) = 2$*) then*

$$\sum_{\alpha,\beta\in\mathcal{R}, s_\alpha s_\beta = w} k_\alpha k_\beta B(\alpha,\beta)\Delta_\alpha\Delta_\beta = 0.$$

This finishes the proof of Theorem 2.2..

2.4.1 The Dunkl Laplacian

The constant coefficient differential operator $p(\partial)$, with $p\in\mathbb{C}[\mathbb{R}^n]$, has a well defined Dunk-analogue $p(T)$ defined for a monomial $m(x) = x_1^{d_1}\cdots x_n^{d_n}$ by $m(T) = T_{e_1}^{d_1}\cdots T_{e_n}^{d_n}$. Here $e_1,\ldots,e_n$ is an orthonormal basis for $\mathbb{R}^n$. For $p_0(x) = \|x\|^2 = \sum_{j=1}^n x_j^2$, we write $\Delta_k := p_0(T)$. The operator Δ_k is the so-called Dunkl Laplacian.

Theorem 2.4. *The Dunkl Laplacian operator can be written as*

$$\Delta_k f(x) = \Delta f(x) + \sum_{\alpha\in\mathcal{R}} k_\alpha \left\{\frac{\langle\nabla f(x),\alpha\rangle}{\langle\alpha,x\rangle} - \frac{f(x)-f(r_\alpha x)}{\langle\alpha,x\rangle^2}\right\}$$

where $\nabla = \left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right)$ *denotes the usual gradient operator.*

Proof. *Suppose* $\{e_1,,....e_n\}$ *is an orthonormal basis for* $\mathbb{R}^n$. *Recall that* $T_{e_j} = \partial_{e_j} + \frac{1}{2}\sum_{\alpha\in\mathcal{R}} k_\alpha\langle\alpha,e_j\rangle\Delta_\alpha$,

where $\Delta_\alpha f(x) = \dfrac{f(x)-f(r_\alpha x)}{\langle\alpha,x\rangle}$.

Thus,

$$
\begin{aligned}
\sum_{j=1}^{n} T_{e_j}^2 &= \sum_{j=1}^{n} \left\{ \partial_{e_j} + \frac{1}{2} \sum_{\alpha\in\mathcal{R}} k_\alpha \left\langle \alpha, e_j \right\rangle \Delta_\alpha \right\}^2 \\
&= \sum_{j=1}^{n} \left\{ \partial_{e_j}^2 + \frac{1}{2} \sum_{\alpha\in\mathcal{R}} k_\alpha \left\langle \alpha, e_j \right\rangle \left(\partial_{e_j}\Delta_\alpha + \Delta_\alpha \partial_{e_j} \right) \right. \\
&\quad \left. + \frac{1}{4} \sum_{\alpha,\beta\in\mathcal{R}} k_\alpha k_\beta \left\langle \alpha, e_j \right\rangle \left\langle \beta, e_j \right\rangle \Delta_\alpha \Delta_\beta \right\} \\
&= \Delta + \frac{1}{2} \sum_{\alpha\in\mathcal{R}} k_\alpha \left(\partial_\alpha \Delta_\alpha + \Delta_\alpha \partial_\alpha \right) + \frac{1}{4} \sum_{\alpha,\beta\in\mathcal{R}} k_\alpha k_\beta \langle \alpha, \beta \rangle \Delta_\alpha \Delta_\beta \\
&= \Delta + \frac{1}{2} \sum_{\alpha\in\mathcal{R}} k_\alpha \left(\partial_\alpha \Delta_\alpha + \Delta_\alpha \partial_\alpha \right) + \frac{1}{4} \sum_{\alpha,\beta\in\mathcal{R},\alpha\neq\beta} k_\alpha k_\beta \langle \alpha, \beta \rangle \Delta_\alpha \Delta_\beta .
\end{aligned}
$$

The above third term vanishes due to Theorem 2.3. and the fact that $\Delta_\alpha^2 = 0$. *Further,*

$$
\begin{aligned}
\partial_\alpha \Delta_\alpha + \Delta_\alpha \partial_\alpha &= [\partial_\alpha, \Delta_\alpha] + 2\Delta_\alpha \partial_\alpha \\
&= \frac{2}{\langle \alpha, \cdot \rangle} \{ s_\alpha \partial_\alpha - \Delta_\alpha \} + \frac{2}{\langle \alpha, \cdot \rangle} (1 - s_\alpha) \partial_\alpha \\
&= \frac{2}{\langle \alpha, \cdot \rangle} \{ \partial_\alpha - \Delta_\alpha \} .
\end{aligned}
$$

This finishes the proof of the statement.

Example 2.7. *For* $n = 1$, *we have* $\mathcal{R} = \{\pm\sqrt{2}\}$ *and* $G = \{\pm \mathrm{id}\}$. *In this case,*

$$
T f(x) = f'(x) + k \, \frac{f(x) - f(-x)}{x}
$$

and

$$
\Delta_k f(x) = T^2 f(x) = f''(x) + \frac{2k}{x} \, f'(x) - 2k \, \frac{f(x) - f(-x)}{x^2} .
$$

Proof.

$$
\begin{aligned}
T^2 f(x) = & \left(f'(x) + \frac{k}{x}(f(x) - f(-x)) \right)' + \frac{k}{x} \left(f'(x) + \frac{k}{x}(f(x) - f(-x)) \right) \\
& - \frac{k}{x} \left(f'(-x) + \frac{k}{-x}(f(-x) - f(x)) \right)
\end{aligned}
$$

$$
\begin{aligned}
&= f''(x) + \frac{k}{x}\left(f'(x) - f'(-x)\right) - \frac{k}{x^2}\left(f(x) - f(-x)\right) + \frac{k}{x}f'(x) \\
&\quad + \frac{k^2}{x^2}\left(f(x) - f(-x)\right) - \frac{k}{x}f'(-x) + \frac{k^2}{x^2}\left(f(-x) - f(x)\right) \\
&= f''(x) + \frac{k}{x}f'(x) - \frac{k}{x}f'(-x) - \frac{k}{x^2}f(x) + \frac{k}{x^2}f(-x) + \frac{k}{x}f'(x) \\
&\quad + \frac{k^2}{x^2}f(x) - \frac{k^2}{x^2}f(-x) - \frac{k}{x}f'(-x) + \frac{k^2}{x^2}f(-x) - \frac{k^2}{x^2}f(x) \\
&= f''(x) + \frac{2k}{x}f'(x) - \frac{k}{x^2}f(x) + \frac{k}{x^2}f(-x) \\
&= f''(x) + \frac{2k}{x}f'(x) + \frac{k}{x^2}\left(f(-x) - f(x)\right).
\end{aligned}
$$

2.4.2 The Dunkl Intertwining Operator

Dunkl's intertwining operator is an isomorphism on $\mathcal{P} = \mathbb{C}[\mathbb{R}^n]$ which intertwines the commutative algebra of Dunkl operators with the algebra of partial differential operators with constant coefficients.

Theorem 2.5. *Let $k \in K^+$. There exists a unique linear isomorphism (intertwining operator) V_k on $\mathcal{P}$ such that*

$$T_\xi V_k = V_k \partial_\xi, \quad V_k \mathbf{1} = \mathbf{1}, \quad V_k \mathcal{P}_m = \mathcal{P}_m.$$

An explicit representation for V_k is known so far only in some special cases. However, for a further development of the theory it is crucial to extend the domain of V_k a way from polynomials. For instance, Trimèche proved that V_k induces a homeomorphism of $C(\mathbb{R}^n)$ and also that of $C^\infty(\mathbb{R}^n)$. The most important property of V_k is the following result:

Theorem 2.6. *(see [40]) For each $x \in \mathbb{R}^n$ there exists a unique probability measure μ_x^k on $\mathbb{R}^n$ such that*

$$V_k f(x) = \int_{\mathbb{R}^n} f(y)\, d\mu_x^k(y).$$

Example 2.8. *(see [38]) For $n = 1$ and $k > 0$, the integral representation of V_k is given by*

$$V_k f(x) = \frac{\Gamma(k+1/2)}{\Gamma(1/2)\Gamma(k)} \int_{-1}^{1} f(xt)(1-t)^{k-1}(1+t)^k dt.$$

Below we can verify that $T\left(V_k f\right)=V_k\left(f'\right)$. *Indeed, let* $b_k=\frac{\Gamma(k+1/2)}{\Gamma(1/2)\Gamma(k)}$. *Then,*

$$\begin{aligned}
TV_kf(x) &= \partial_x(V_kf)+k\frac{V_kf(x)-V_kf(-x)}{x}\\
&= b_k\int_{-1}^{1} tf'(xt)(1-t)^{k-1}(1+t)^k dt \quad + \\
&\quad b_k\frac{k}{x}\int_{-1}^{1} f(xt)(1-t)^{k-1}(1+t)^k dt - b_k\frac{k}{x}\int_{-1}^{1} f(-xt)(1-t)^{k-1}(1+t)^k dt\\
&= b_k\int_{-1}^{1} tf'(xt)(1-t)^{k-1}(1+t)^k dt\\
&\quad +b_k\frac{k}{x}\int_{-1}^{1} f(xt)(1+t)^k(1-t)^k\left\{\frac{1}{1-t}-\frac{1}{1+t}\right\}dt\\
&= b_k\int_{-1}^{1} tf'(xt)(1-t)^{k-1}(1+t)^k dt \quad +b_k\int_{-1}^{1}\left(\frac{f(xt)}{x}\right)2kt(1-t^2)^{k-1}dt.
\end{aligned}$$

Further, an integration by parts gives

$$\begin{aligned}
TV_kf(x) &= b_k\int_{-1}^{1} f'(xt)\left(t+(1-t)\right)(1-t)^{k-1}(1+t)^k dt\\
&= V_kf'(x).
\end{aligned}$$

That is $T\left(V_kf\right)=V_k\left(f'\right)$.

2.5 The Dunkl Kernel and The Dunkl Transform

2.5.1 The Dunkl Kernel

For a fixed $y\in\mathbb{R}^n$ we search for a function f solving

$$\begin{cases} T_\xi f(x)=\langle\xi,y\rangle\, f(x), & \forall\xi\in\mathbb{R}^n,\\ f(0)=1. \end{cases}$$

If $k\equiv 0$ then the solution to this problem is $f(x)=e^{\langle x,y\rangle}$.

For fixed $y\in\mathbb{R}^n$ define

$$E_k(x,y):=V_k(e^{\langle\,\cdot\,,\,y\rangle})(x), \qquad x\in\mathbb{R}^n.$$

Indeed we may rewrite $E_k(x,y)$ as

$$E_k(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} V_k \langle \cdot, y \rangle^n (x).$$

Then:

i) The homogeneity of V_k implies $E_k(0,y) = 1$.

ii) By the intertwining property, we have

$$T_\xi V_k \langle \cdot, y \rangle^n (x) = V_k \partial_\xi \langle \cdot, y \rangle^n (x) = n \langle \xi, y \rangle V_k \langle \cdot, y \rangle^{n-1} (x).$$

Thus

$$T_\xi E_k(x,y) = \langle \xi, y \rangle \sum_{n=1}^{\infty} \frac{1}{(n-1)!} V_k \langle \cdot, y \rangle^{n-1} (x) = \langle \xi, y \rangle E_k(x,y).$$

The uniqueness follows by standard arguments. Hence we have proved:

Theorem 2.7. *Let $y \in \mathbb{R}^n$ and $k \in K^+$. Then $E_k(\cdot, y)$ is the unique solution of the system*

$$\begin{cases} T_\xi f(x) = \langle \xi, y \rangle \, f(x) \\ f(0) = 1. \end{cases}$$

Example 2.9. *For $n = 1$, we have*

$$\begin{aligned} E_k(x,y) = V_k(e^y)(x) &= \frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k)} \int_{-1}^{1} e^{txy} (1+t)(1-t^2)^{k-1} dt \\ &= e^{xy} \, {}_1\mathrm{F}_1(k; 2k+1; -2xy), \end{aligned}$$

where

$${}_1\mathrm{F}_1(a;b;t) = \sum_{k=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{t^n}{n!}.$$

We collect some basic properties of the Dunkl kernel E_k.

Proposition 2.1. *([41, 42]) The Dunkl kernel E_k satisfies: For $x, y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, and $g \in G$,*

a) $E_k(x,y) = E_k(y,x)$.

b) $E_k(\lambda x,y) = E_k(x,\lambda y)$ and $E_k(gx,gy) = E_k(x,y)$.

c) $E_k(x,y) > 0$.

d) $|E_k(x,-iy)| \leq 1$.

2.5.2 The Dunkl Transform

Let $L^1(\mathbb{R}^n, \vartheta_k(x)dx)$ be the space of integrable functions on $\mathbb{R}^n$ with respect to the measure

$$\vartheta_k(x)dx = \prod_{\alpha\in\mathcal{R}} |\langle\alpha,x\rangle|^{k_\alpha} dx.$$

The Dunkl kernel gives rise to an integral transform.

Definition 2.5. *The Dunkl transform associated to a root system $\mathcal{R}$ and a multiplicity function $k \in K^+$ is defined on $L^1(\mathbb{R}^n, \vartheta_k(x)dx)$ by*

$$\mathcal{F}_{k,2}(f)(\xi) = c_k \int_{\mathbb{R}^n} f(x)E_k(x,-i\xi) \prod_{\alpha\in\mathcal{R}} |\langle\alpha,x\rangle|^{k_\alpha} dx$$

for some positive constant c_k.

The following are the main properties for the Dunkl transform; they are in complete analogy to the corresponding results for the Euclidean Fourier transform.

Theorem 2.8. *(see [6, 43])*

1) *(Plancherel Theorem) The Dunkl transform $\mathcal{F}_{k,2}$ has a unique extension to an isometric isomorphism of $L^2(\mathbb{R}^n, \vartheta_k(x)dx)$.*

2) *(L^1-inversion) If $f \in L^1(\mathbb{R}^n, \vartheta_k(x)dx)$ and $\mathcal{F}_{k,2}f \in L^1(\mathbb{R}^n, \vartheta_k(x)dx)$, then*

$$f = \mathcal{F}_{k,2}^{-1}\mathcal{F}_{k,2}f \quad a.e,$$

   where $\mathcal{F}_{k,2}^{-1}f(\xi) := \mathcal{F}_{k,2}f(-\xi)$.

3) *The Dunkl transform $\mathcal{F}_{k,2}$ is injective on $L^1(\mathbb{R}^n, \vartheta_k(x)dx)$.*

4) The Dunkl transform $\mathcal{F}_{k,2}$ is an homeomorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

Chapter 3: The $(\mathbf{k},\mathbf{a})$-Generalized Fourier Transform

3.1 Introduction

In the previous chapter we introduced the Dunkl transform $\mathcal{F}_{k,2}$. In the present chapter we will introduce the generalized Fourier transform $\mathcal{F}_{k,a}$ where $a>0$. This transform was introduced by Ben Said, Ørsted and Kobayashi in [16]. The main properties of $\mathcal{F}_{k,a}$ will be given, such as the Plancherel formula and the inversion formula. The framework of this thesis concerns with the case $a=1$. We close this chapter by a convolution structure associated with $\mathcal{F}_{k,a}$ when $a=1$.

3.2 The Kernel $\mathbf{B_{k,a}(x,y)}$

In this section we will introduce the kernel of the generalized Fourier transform. When $a=2$, this kernel reduces to the Dunkl kernel introduced in the previous chapter.

Let us introduce the normalized I-Bessel function $\widetilde{I}_\nu(w)$ defined by

$$\widetilde{I}_\lambda(w) := \left(\frac{w}{2}\right)^{-\lambda} I_\lambda(w) = \sum_{\ell=0}^{\infty} \frac{w^{2\ell}}{2^{2\ell}\ell!\Gamma(\lambda+\ell+1)}$$
$$= \frac{1}{\sqrt{\pi}\Gamma\left(\lambda+\frac{1}{2}\right)} \int_{-1}^{1} e^{wt}\left(1-t^2\right)^{\lambda-\frac{1}{2}} dt.$$

The normalized Bessel function of the first kind $\widetilde{J}_\nu(\omega)$ is given by

$$\widetilde{J}_\nu(\omega) := \left(\frac{\omega}{2}\right)^{-\nu} J_\nu(\omega) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \omega^{2\ell}}{2^{2\ell}\ell!\gamma(\nu+\ell+1)}.$$

From the definition of the normalized I-Bessel function we have

$$\widetilde{J}_\nu(\omega) = \widetilde{I}_\nu(-i\omega) = \widetilde{I}_\nu(i\omega).$$

Definition 3.1. *The Gegenbauer polynomials $C_m^\alpha(t)$ are defined for $\alpha>0$ and $m\in\mathbb{N}$ by*

$$C_m^\alpha(t) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{[m/2]} (-1)^k \frac{\Gamma(m-k+\alpha)}{k!(m-2k)!}(2t)^{m-2k},$$

In [19], the authors proved that there exists a constant $B(\nu) > 0$ such that

$$\sup_{-1\le t\le 1}\left|\frac{1}{\nu}C_m^\nu(t)\right| \le B(\nu)m^{2\nu-1} \quad \text{for any } m\in\mathbb{N}.$$

For $b>0$, let us consider the following infinite sum:

$$\mathcal{I}(b,\nu;w;t) = \frac{\Gamma(b\nu+1)}{\nu}\sum_{m=0}^{\infty}(m+\nu)\left(\frac{w}{2}\right)^{bm}\widetilde{I}_{b(m+\nu)}(w)C_m^\nu(t).$$

Lemma 3.1. *1) The above summation converges absolutely and uniformly on any compact subset of*

$$U := \{(b,\nu,w,t)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{C}\times[-1,1] : 1+b\nu>0\}.$$

In particular, $\mathcal{I}(b,\nu;w;t)$ is a continuous function on U.

2) (Special value at $w=0$)

$$\mathcal{I}(b,\nu;0;t)\equiv 1. \tag{3.1}$$

Proof. *1) It is sufficient to show that for a sufficiently large m_0 the summation over m ($\ge m_0$) converges absolutely and uniformly on any compact set of U. It is known that*

$$|\widetilde{I}_\lambda(w)| \le \frac{e^{|\mathrm{Re}\, w|}}{\Gamma(\lambda+1)},$$
$$\left|\frac{1}{\nu}C_m^\nu(t)\right| \le B(\nu)m^{2\nu-1} \quad \textit{for any } m\ge 1.$$

Then,

$$\begin{aligned}
&\frac{1}{\nu}\sum_{m=m_0}^{\infty}\left|(m+\nu)\left(\frac{w}{2}\right)^{bm}\widetilde{I}_{b(m+\nu)}(w)C_m^\nu(t)\right| \\
&\le \sum_{m=m_0}^{\infty}\left|\frac{(m+\nu)B(\nu)w^{bm}e^{|\mathrm{Re}\, w|}m^{2\nu-1}}{2^{bm}\Gamma(bm+b\nu+1)}\right| \\
&= B(\nu)e^{|\mathrm{Re}\, w|}\sum_{m=m_0}^{\infty}\frac{(1+\frac{\nu}{m})m^{2\nu}}{\Gamma(bm+b\nu+1)}\left(\left|\frac{w}{2}\right|^b\right)^m.
\end{aligned}$$

Since $b>0$, $\Gamma(bm+b\nu+1)$ grows faster than any other term in each summand as m goes to infinity, and consequently, the last sum converges. Furthermore, the convergence *is uniform on any compact set of parameters* (b,ν,w). *Hence, we have proved the first assertion.*

2) Since $b>0$, *the summand defining* $\mathcal{I}$ *vanishes at* $w=0$ *for any* $m>0$, *and therefore*

$$\begin{aligned}\mathcal{I}(b,\nu;0;t) &= \frac{\Gamma(b\nu+1)}{\nu}\cdot\nu\cdot\widetilde{I}_{b\nu}(0)\cdot C_0^{\nu}(t)\\ &= 1.\end{aligned}$$

Thus, the second assertion is proved.

Example 3.1. *The special values at* $b=1,2$ *are given by*

$$\begin{aligned}\mathcal{I}(1,\nu;w;t) &= e^{wt},\\ \mathcal{I}(2,\nu;w;t) &= \Gamma\Big(\nu+\frac{1}{2}\Big)\widetilde{I}_{\nu-\frac{1}{2}}\left(\frac{w(1+t)^{\frac{1}{2}}}{\sqrt{2}}\right).\end{aligned}$$

Definition 3.2. *Introduce the following continuous function of* $t\in[-1,1]$ *with parameters* $r,s>0$, *and* $z\in\mathbb{C}^{+}\backslash i\pi\mathbb{Z}$ *:*

$$h_{k,a}(r,s;z;t) := \frac{\exp\left(-\frac{1}{a}\left(r^a+s^a\right)\coth(z)\right)}{\sinh(z)^{\frac{2\langle k\rangle+n+a-2}{a}}}\mathcal{I}\left(\frac{2}{a},\frac{2\langle k\rangle+n-2}{2};\frac{2(rs)^{\frac{a}{2}}}{a\sinh(z)};t\right),$$

where

$$\langle k\rangle := \frac{1}{2}\sum_{\alpha\in\mathcal{R}}k_\alpha.$$

Example 3.2. *For* $a=1,2$ *we respectively have:*

$$h_{k,a}(r,s;z;t) = \frac{\exp(-\frac{1}{a}(r^a+s^a)\coth(z))}{\sinh(z)^{\frac{2\langle k\rangle+n+a-2}{a}}} \times\begin{cases}\Gamma\Big(\langle k\rangle+\dfrac{n-1}{2}\Big)\widetilde{I}_{\langle k\rangle+\frac{n-3}{2}}\Big(\dfrac{\sqrt{2}(rs)^{\frac{1}{2}}}{\sinh z}(1+t)^{\frac{1}{2}}\Big)\\ \exp\Big(\dfrac{rst}{\sinh z}\Big).\end{cases}$$

For $x, y \in \mathbb{R}$, we define the kernel function $\Lambda_{k,a}(x,y;z)$ by

$$\Lambda_{k,a}(r\omega, s\eta; z) = \widetilde{V}_k(h_{k,a}(r,s;z;\cdot))(\omega,\eta)$$

where $\widetilde{V}_k$ is given in terms of the Dunkl intertwining operator by

$$(\widetilde{V}_k h)(x,y) := (V_k h_y)(x) = \int_{\mathbb{R}^n} h(\langle \xi, y\rangle) d\mu_x^k(\xi).$$

Definition 3.3. *For a multiplicity function $k \geqslant 0$, $a > 0$ such that $2\langle k\rangle + n > max(1, 2-a)$ we introduce the kernel $B_{k,a}(\xi,x)$ by*

$$B_{k,a}(x,y) = e^{i\pi((2\langle k\rangle+1+a-2)/2a)}\Lambda_{k,a}\left(x,y;i\frac{\pi}{2}\right).$$

Example 3.3. *For $n = 1, a > 0, k \geqslant 0$ such that $2k > 1-a$, we can write the kernel $B_{k,a}$ as follows*

$$B_{k,a}(x,y) = \Gamma\left(\frac{2k+a-1}{a}\right)\left(\widetilde{J}_{\frac{(2k-1)}{a}}\left(\frac{2}{a}|xy|^{\frac{a}{2}}\right) + \frac{xy}{(ai)^{\frac{2}{a}}}\widetilde{J}_{\frac{(2k+1)}{a}}\left(\frac{2}{a}|xy|^{\frac{a}{2}}\right)\right).$$

Example 3.4. *(see [19]) For arbitrary dimension n, and $a = 1,2$ we get respectively following the formula:*

$$h_{k,a}(r,s;\frac{\pi i}{2};t) = \begin{cases} \Gamma\left(\langle k\rangle + \frac{n-1}{2}\right)e^{-\frac{\pi i}{2}(2\langle k\rangle+n-1)}\widetilde{J}_{\langle k\rangle+\frac{n-3}{2}}(\sqrt{2}(rs)^{\frac{1}{2}}(1+t)^{\frac{1}{2}}) \\ e^{-\frac{\pi i}{2}(\langle k\rangle+\frac{n}{2})}e^{-irst}. \end{cases}$$

In the polar coordinates $x = r\omega$ and $y = s\eta$, the kernel $B_{k,a}(x,y)$, with $a = 1,2$ is given respectively by

$$B_{k,a}(r\omega, s\eta) = \begin{cases} \Gamma\left(\langle k\rangle + \frac{n-1}{2}\right)\left(\widetilde{V}_k\left(\widetilde{J}_{\langle k\rangle+\frac{n-3}{2}}\left(\sqrt{2rs(1+\cdot)}\right)\right)\right)(\omega,\eta) \\ \left(\widetilde{V}_k\left(e^{-irs\cdot}\right)\right)(\omega,\eta). \end{cases}$$

As one can see form the example above that the kernel $B_{k,2}(x,y)$ coincides with the Dunkl kernel at $E(x,-iy)$ (see Chapter 2). The following important result was proved in [19]

Theorem 3.1. *Let $k \in K^+$ be a non-negative root multiplicity function, $a = 1$ or 2, and $x, y \in \mathbb{R}^n$. Then $\left|B_{k,a}(x,y)\right| \leqslant 1$.*

In analogy with the Dunkl kernel properties stated in the previous chapter, we shall list the main properties of the kernel $B_{k,a}$ for every $a > 0$. Recall that the Dunkl kernel corresponds to the case $a = 2$.

Recall from the previous chapter the Dunkl Laplacian operator Δ_k, and let E be the Euler operator defined by

$$E = \sum_{j=0}^{\infty} x_j \partial_{x_j}.$$

Theorem 3.2. *The kernel $B_{k,a}(\xi, x)$ solves the following differential-difference equations on $\mathbb{R}^n \times \mathbb{R}^n$*

$$E^x B_{k,a}(\xi, x) = E^\xi B_{k,a}(\xi, x)$$
$$\|\xi\|^{2-a} \Delta_k B_{k,a}(\xi, x) = -\|x\|^a B_{k,a}(\xi, x),$$
$$\|x\|^{2-a} \Delta_k B_{k,a}(\xi, x) = -\|\xi\|^a B_{k,a}(\xi, x).$$

The superscripts in E^x and E^ξ denote the relevant variable.

Properties 3.1. *On $\mathbb{R}^n \times \mathbb{R}^n$, the kernel $B_{k,a}(\cdot,\cdot)$ satisfies the following properties*

(1) $B_{k,a}(\lambda x, \xi) = B_{k,a}(x, \lambda \xi)$ *for every* $\lambda \in \mathbb{R}$.

(2) $B_{k,a}(gx, g\xi) = B_{k,a}(x, \xi)$ *for every* $g \in G$.

(3) $B_{k,a}(\xi, x) = B_{k,a}(x, \xi)$.

(4) $B_{k,a}(0, x) = 1$.

3.3 The $(\mathbf{k},\mathbf{a})$-Generalized Fourier Transform

Dunkl theory is a generalization of the Euclidean Fourier analysis, where the role of orthogonal groups, which provides the underline structure for the ordinary Fourier analysis, is played by finite reflection groups, and the Lebesgue measure is replaced by a weighted measure invariant under the reflection group and parameterized by a multiplicity function k. This theory started at the beginning of the 90s. Later on, S. Ben Said, T. Kobayashi and B. Ørsted [16] gave a far reaching generalization of the Dunkl theory by introducing a parameter $a > 0$ which arises from the interpolation of two $\mathfrak{sl}(2,\mathbb{R})$ actions. This provides the so-called (k,a)-generalized Fourier transform which includes the Fourier transform ($k = 0$ and $a = 2$), the Dunkl transform ($k > 0$ and $a = 2$), the Kobayashi-Mano transform ($k = 0$ and $a = 1$), and a new unitary operator ($k > 0$ and $a = 1$) having a rich structure, as much as the Dunkl transform.

For $a > 0$ and a multiplicity function k on the root system $\mathcal{R}$, we introduce the following normalization constant

$$c_{k,a} := \left(\int_{\mathbb{R}^n} \exp\left(-\frac{1}{a}\|x\|^a \right) \vartheta_{k,a}(x)dx \right)^{-1},$$

where the density function $\vartheta_{k,a}(x)$ on $\mathbb{R}^n$ is given by

$$\vartheta_{k,a}(x) := \|x\|^{a-2} \prod_{\alpha\in\mathcal{R}} |\langle \alpha, x\rangle|^{k_\alpha}.$$

Denote by $L^2(\mathbb{R}^n, \vartheta_{k,a}(x)dx)$ the space of square integrable functions on $\mathbb{R}^n$ with respect to the weighted measure $\vartheta_{k,a}(x)dx$. The (k,a)-generalized Fourier transform $\mathcal{F}_{k,a}$ is defined on $L^2\left(\mathbb{R}^n, \vartheta_{k,a}(x)dx\right)$ by

$$\mathcal{F}_{k,a}f(\xi) = c_{k,a} \int_{\mathbb{R}^n} B_{k,a}(\xi, x) f(x) \vartheta_{k,a}(x)dx.$$

For $a = 1$ or 2, the explicit expression of the kernel $B_{k,a}$ can be deduced from the previous part. In particular, for $a = 1$ or 2, the unitary operator $\mathcal{F}_{k,a}$ includes some known transforms as special cases:

- the Euclidean Fourier transform ($a = 2$, $k \equiv 0$),
- the Hankel transform (Kobayashi-Mano) ($a = 1$, $k \equiv 0$),
- the Dunkl transform (Dunkl) ($a = 2$, $k > 0$),
- a new Dunk type transform ($a = 1$, $k > 0$).

Next we will discuss basic properties of $\mathcal{F}_{k,a}$ for general k and a.

Theorem 3.3. *Let $a>0$ and k be a non-negative multiplicity function on the root system $\mathcal{R}$ satisfying the inequality $a+2\langle k\rangle+n>2$.*

1) (Plancherel formula) *The (k,a)-generalized Fourier transform $\mathcal{F}_{k,a}$ is a unitary operator from $L^2(\mathbb{R}^n,\vartheta_{k,a}(x)dx)$ into itself. That is, $\mathcal{F}_{k,a}$ is a bijective linear operator satisfying*

$$\|\mathcal{F}_{k,a}(f)\|_k=\|f\|_k$$

for any $f\in L^2(\mathbb{R}^n,\vartheta_{k,a}(x)dx)$.

2) *The (k,a)-generalized Fourier transform $\mathcal{F}_{k,a}$ is of finite order if and only if $a\in\mathbb{Q}$. If a is of the form $a=\frac{q}{q'}$, where q and q' are positive integers, then*

$$\mathcal{F}_{k,a}^{2q}=\mathrm{id}.$$

Theorem 3.4. (Inversion formula [19]). *Let r be a non negative integer and suppose the inequality $2\langle k\rangle+n>2-a$ with $a=1/r$ is satisfied. We pin down that $\mathcal{F}_{k,1/r}$ is a unitary operator on $L^2(\mathbb{R}^n,\vartheta_{k,1/r}(x)dx)$. The inversion formula is given as*

$$\mathcal{F}_{k,\frac{1}{r}}^{-1}=\mathcal{F}_{k,\frac{1}{r}}.$$

Theorem 3.5. *The unitary operator $\mathcal{F}_{k,a}$ satisfies the following intertwining relations on a dense subspace of $L^2\left(\mathbb{R}^n,\vartheta_{k,a}(x)dx\right)$.*

1) $\mathcal{F}_{k,a}\circ E=-(E+n+2\langle k\rangle+a-2)\circ\mathcal{F}_{k,a}$.

2) $\mathcal{F}_{k,a}\circ\|x\|^a=-\|x\|^{2-a}\Delta_k\circ\mathcal{F}_{k,a}$.

3) $\mathcal{F}_{k,a}\circ\|x\|^{2-a}\Delta_k=-\|x\|^a\circ\mathcal{F}_{k,a}$.

Here E denotes the Euler operator

$$E=\sum_{j=0}^{n}x_j\partial_{x_j}$$

and Δ_k denotes the Dunkl Laplacian operator

Example 3.5. *Suppose $n=1$, $a>0$, and $k\geq 0$ such that $2k>1-a$. Then*

$$\begin{aligned} B_{k,a}(x,y) &= e^{i\frac{\pi}{2}(\frac{2k+a-1}{a})}\Lambda_{k,a}(x,y;i\frac{\pi}{2}) \\ &= \Gamma\Big(\frac{2k+a-1}{a}\Big)\Big(\widetilde{J}_{\frac{2k-1}{a}}\Big(\frac{2}{a}|xy|^{\frac{a}{2}}\Big)+\frac{xy}{(ai)^{\frac{2}{a}}}\widetilde{J}_{\frac{2k+1}{a}}\Big(\frac{2}{a}|xy|^{\frac{a}{2}}\Big)\Big), \end{aligned}$$

where

$$\widetilde{J}_{\nu}(w)=\frac{1}{\Gamma(\alpha+1)}\,{}_0\mathrm{F}_1(\alpha+1;-\frac{w^2}{4})$$

is the normalised J-Bessel function. Thus, for $f\in L^2(\mathbb{R},|x|^{2k+a-2}dx)$, the integral transform $\mathcal{F}_{k,a}$ takes the form

$$\begin{aligned} \mathcal{F}_{k,a}f(y) &= 2^{-1}a^{-(\frac{2k-1}{a})} \\ &\quad \int_{\mathbb{R}} f(x)\Big(\widetilde{J}_{\frac{2k-1}{a}}\Big(\frac{2}{a}|xy|^{\frac{a}{2}}\Big)+\frac{xy}{(ai)^{\frac{2}{a}}}\widetilde{J}_{\frac{2k+1}{a}}\Big(\frac{2}{a}|xy|^{\frac{a}{2}}\Big)\Big)|x|^{2k+a-2}dx. \end{aligned}$$

3.4 The Translation Operator

In this section we will study a translation operator associated with the generalized Fourier transform $\mathcal{F}_{k,1}$, i.e. when $a=1$. It is worth mentioning that the n-dimensional translation operator, and therefore a convolution structure, are available only when $a=1$. When $n=1$, these structure are available only when $a=2/m$ where m is a nonzero integer. In this section we will state the main properties of the translation operator and the convolution product, such as the positivity-preserving operator acting on a suitable space of radial functions on $\mathbb{R}^n$.

3.4.1 Translation Operator

For the classical Fourier analysis, the translation operator $\tau_y, y\in\mathbb{R}^n$, is defined for a suitable function f by $(\tau_y f)(x)=f(x-y)$. Further, τ_y is a positive operator (in the sense that $f\geqslant 0$ on $\mathbb{R}^n$ implies $\tau_y f\geqslant 0$ on $\mathbb{R}^n$) and $\|\tau_y f\|_{L^p(\mathbb{R}^n,dx)}=\|f\|_{L^p(\mathbb{R}^n,dx)}$ for all $f\in L^p(\mathbb{R}^n,dx)$ with $1\leqslant p\leqslant\infty$. An elementary property establishes that the translation operator and the multiplication by characters correspond one to each other under the Euclidean Fourier transform $\mathcal{F}$,

$$\mathcal{F}(\tau_y f)(\xi)=\mathrm{e}^{i\langle y,\xi\rangle}\mathcal{F}f(\xi).$$

In analogy with the Euclidean case, a suitable replacement for the translation operator must be introduced. A such generalized translation operator acting on functions defined on $\mathbb{R}$ has been introduced and studied in [16] using a product formula for the function $B_{k,1}$. A such product formula is not available on $\mathbb{R}^n$, and therefore, alternatively, in [23] defined the translation operator $f \mapsto \tau_y f$, $y \in \mathbb{R}^n$, for $f \in L^2(\mathbb{R}^n, \vartheta_k(x)dx)$ by

$$\mathcal{F}_k(\tau_y f)(\xi) := B_k(y,\xi)\mathcal{F}_k f(\xi), \quad \xi \in \mathbb{R}^n.$$

The above definition makes sense as $\mathcal{F}_k = \mathcal{F}_{k,1}$ is an isometry on $L^2(\mathbb{R}^n, \vartheta_k(x)dx)$ onto itself and the fact that $|B_k(y,\xi)| \leqslant 1$.

Let

$$\mathcal{L}_k^1(\mathbb{R}^n) := \Big\{ f \in L^1(\mathbb{R}^n, \vartheta_k(x)dx) \text{ such that } \mathcal{F}_k f \in L^1(\mathbb{R}^n, \vartheta_k(x)dx \Big\}.$$

We pin down that $\mathcal{L}_k^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n, \vartheta_k(x)dx) \cap L^\infty(\mathbb{R}^n, \vartheta_k(x)dx)$ and, therefore, $\mathcal{L}_k^1(\mathbb{R}^n)$ is a subspace of $L^2(\mathbb{R}^n, \vartheta_k(x)dx)$.

Definition 3.4. *For $f \in \mathcal{L}_k^1(\mathbb{R}^n)$, the translation τ_y can also be described by*

$$\tau_y f(x) = c_k \int_{\mathbb{R}^n} B_k(x,\xi) B_k(y,\xi) \mathcal{F}_k f(\xi) \vartheta_k(\xi) d\xi,$$

Example 3.6. *Than rank one case. Let $x \in \mathbb{R}$ and $f \in \mathcal{C}_b(\mathbb{R})$. For $m = 2 \in \mathbb{N}\backslash\{0\}$ and $k > \frac{1}{4}$, the translation operator τ_y^k is given explicitly by*

$$\tau_x^k f(y) = \int_{\mathbb{R}} f(z) d\nu_k^{x,y}(z), \quad y \in \mathbb{R},$$

where

$$d\nu_k^{x,y}(z) = \begin{cases} \mathcal{K}_k(x,y,z) d\mu_k(z) & \text{if} \quad xy \neq 0 \\ d\delta_x(z) & \text{if} \quad y = 0 \\ d\delta_y(z) & \text{if} \quad x = 0 \end{cases}$$

Here the kernel $\mathcal{K}_k(x,y,z)$ is compactly supported where

$$\operatorname{supp}(\nu_k^{x,y}) \subset \Big\{ z \in \mathbb{R} : |\sqrt{|x|} - \sqrt{|y|} < \sqrt{|z|} < \sqrt{|x|} + \sqrt{|y|} \Big\}.$$

Moreover, the the kernel $\mathcal{K}_k(x,y,z)$ is explicitly given by

$$\mathcal{K}_k(x,y,z) = \frac{M_k}{4} K_B^{2k-1}\left(|x|^{\frac{1}{2}}, |y|^{\frac{1}{2}}, |z|^{\frac{1}{2}}\right)$$
$$\times \{1 + \xi_k(x,y,z) + \xi_k(z,x,y) + \xi_k(y,z,x)\}.$$

Here K_B^{α}, and $\xi_{k,n}$ are defined by

$$K_B^{\alpha}(u,v,w) = 2^{-2\alpha+1} \frac{\left\{\left[(u+v)^2 - w^2\right]\left[w^2 - (u-v)^2\right]\right\}^{\alpha-\frac{1}{2}}}{(uvw)^{2\alpha}},$$

$$\xi_k(x,y,z) = \frac{2\,\mathrm{sgn}(xy)}{(4k-2)} C^{2k-1}(\sigma_{x,y,z}),$$

where

$$C^{\alpha}(t) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{1} (-1)^k \frac{\Gamma(2-k+\alpha)}{k!(2-2k)!} (2t)^{2-2k},$$

and

$$\sigma_{x,y,z} = \frac{|x|+|y|-|z|}{2|xy|^{\frac{1}{2}}}.$$

Here are some basic properties of the translation operator:

Theorem 3.6. *Let $f \in \mathcal{L}_k^1(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n, \vartheta_k(x)dx) \cap L^{\infty}(\mathbb{R}^n, \vartheta_k(x)dx)$. Then*

1) For every $x,y \in \mathbb{R}^n$, we have $\tau_y f(x) = \tau_x f(y)$.

2) For every $y \in \mathbb{R}^n$, the operator τ_y satisfies

$$\int_{\mathbb{R}^n} \tau_y f(x) g(x) \vartheta_k(x) dx = \int_{\mathbb{R}^n} f(x) \tau_y g(x) \vartheta_k(x) dx. \tag{3.2}$$

3) $\tau_0 f = f$

Proof. *(i) This follows from the symmetry of the kernel $B_k(x,y) = B_k(y,x)$*

(ii) To prove it assume first that both f and g are from $\mathcal{S}(\mathbb{R}^n)$.

Hence,

$$\int_{\mathbb{R}^n} \mathcal{F}_k f(x) g(x) \vartheta_k(x) dx = \int_{\mathbb{R}^n} f(x) \mathcal{F}_k g(x) \vartheta_k(x) dx.$$

Then both integrals in (ii) are well defined. From the definition

$$\begin{aligned}
&\int_{\mathbb{R}^n} \tau_y f(x) g(x) \vartheta_k(x) dx \\
= &\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} B_k(x,\xi) B_k(y,\xi) \mathcal{F}_k(f) \vartheta_k(\xi) d\xi \right) g(x) \vartheta_k(x) dx \\
= &\int_{\mathbb{R}^n} \mathcal{F}_k(f)(\xi) \mathcal{F}_k g(\xi) B_k(y,\xi) \vartheta_k(x) d\xi
\end{aligned}$$

We also have

$$\begin{aligned}
&\int_{\mathbb{R}^n} f(x) \tau_y g(x) \vartheta_k(x) dx \\
= &\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} B_k(x,\xi) B_k(y,\xi) \mathcal{F}_k(g) \vartheta_k(x)(\xi) d\xi \right) f(x) \vartheta_k(x) dx \\
= &\int_{\mathbb{R}^n} \mathcal{F}_k(f)(\xi) \mathcal{F}_k(g)(\xi) B_k(y,\xi) \vartheta_k(\xi) d\xi.
\end{aligned}$$

This proves (ii) when both f and g are from $\mathcal{S}(\mathbb{R}^n)$.

(iii) Follows directly from Definition 3.4.

Below is one of the main result in [23] where the translation operator is extended to the space $L^p_{\text{rad}}(\mathbb{R}^n, \vartheta_k(x)dx)$ of radial functions in $L^p(\mathbb{R}^n, \vartheta_k(x)dx)$.

Theorem 3.7. *1) For every $y \in \mathbb{R}^n$, the translation τ_y is a positive operator on the space of bounded and positive functions in $L^p_{\text{rad}}(\mathbb{R}^n, \vartheta_k(x)dx)$. Further, $\|\tau_y f\|_{L^1_k} = \|f\|_{L^1_k}$.*

2) For every $y \in \mathbb{R}^n$, the translation τ_y, initially defined on the intersection of the spaces $L^1(\mathbb{R}^n, \vartheta_k(x)dx) \cap L^\infty(\mathbb{R}^n, \vartheta_k(x)dx)$, extends as a bounded operator to the space $L^p_{\text{rad}}(\mathbb{R}^n, \vartheta_k(x)dx)$ for every $1 \leqslant p \leqslant 2$, with

$$\|\tau_y f\|_{L^p_k} \leqslant \|f\|_{L^p_k}.$$

3.5 The Convolution Structure

We define a convolution operator $\circledast$ on the space $L^2(\mathbb{R}^n, \vartheta_k(x)dx)$ by

$$f \circledast g(x) := c_k \int_{\mathbb{R}^n} f(y) \tau_x g(y) \vartheta_k(y) dy.$$

The above integral is finite as $\tau_x g \in L^2(\mathbb{R}^n, \vartheta_k(x)dx)$ for $g \in L^2(\mathbb{R}^n, \vartheta_k(x)dx)$. By one of the previous definitions of the generalized translation operator, we may rewrite the convolution $\circledast$ as

$$f \circledast g(x) = c_k \int_{\mathbb{R}^n} \mathcal{F}_k f(\xi) \mathcal{F}_k g(\xi) B_k(x,\xi) \vartheta_k(\xi) d\xi.$$

In particular, we obtain

$$f \circledast g = g \circledast f \quad \text{and} \quad \mathcal{F}_k(f \circledast g) = \mathcal{F}_k f \cdot \mathcal{F}_k g.$$

To end this section, let us note that the map $f \mapsto f \circledast g$, where $g \in L^1_{\text{rad}}(\mathbb{R}^n, \vartheta_k(x)dx)$, extends to $L^p(\mathbb{R}^n, \vartheta_k(x)dx)$ for every $1 \leqslant p \leqslant \infty$, with

$$\|f \circledast g\|_{L^p_k} \leqslant c_k \|f\|_{L^p_k} \|g\|_{L^1_k}$$

for every $f \in L^p(\mathbb{R}^n, \vartheta_k(x)dx)$. Indeed, by Theorem 3.7.,

$$\begin{aligned} |f \circledast g(x)| &\leqslant c_k \int_{\mathbb{R}^n} |f(y)||\tau_x g(y)|\vartheta_k(y)dy \\ &\leqslant c_k \|g\|_{L^1_k}^{\frac{1}{p'}} \left(\int_{\mathbb{R}^n} |f(y)|^p |\tau_x g(y)| \vartheta_k(y) dy \right)^{\frac{1}{p}}, \end{aligned}$$

with $\frac{1}{p} + \frac{1}{p'} = 1$. Thus,

$$\begin{aligned} \|f \circledast g\|_{L^p_k} &\leqslant c_k \|g\|_{L^1_k}^{\frac{1}{p'}} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(y)|^p |\tau_x g(y)| \vartheta_k(x) \vartheta_k(y) dx dy \right)^{\frac{1}{p}} \\ &= c_k \|g\|_{L^1_k}^{\frac{1}{p'}} \left(\int_{\mathbb{R}^n} |f(y)|^p \left(\int_{\mathbb{R}^n} |\tau_x g(y)| \vartheta_k(x) dx \right) \vartheta_k(y) dx dy \right)^{\frac{1}{p}} \\ &\leqslant c_k \|g\|_{L^1_k}^{\frac{1}{p'} + \frac{1}{p}} \|f\|_{L^p_k}. \end{aligned}$$

Chapter 4: Linear Lie Groups and Lie Algebras

4.1 Introduction

In this chapter we start with the theory of linear Lie group and the corresponding Lie algebras. Then, we introduce the representation theory of Lie algebras and their fundamental properties. Finally, we recall Nelson's theorem about the integrability of representations of Lie algebras, which will play a crucial role in Chapter 6.

4.2 Linear Lie Groups

On $M(n,\mathbb{R})$ we consider the norm

$$\|A\| := \sqrt{\langle A,A\rangle}, \quad \langle A,B\rangle := \operatorname{tr}\left(A^T B\right).$$

Clearly we have $\|AB\| \leq \|A\|\|B\|$. The group

$$GL(n,\mathbb{R}) := \{A \in M(n,\mathbb{R}) \mid \det(A) \neq 0\}.$$

The group $GL(n,\mathbb{R})$ is an open set in $M(n,\mathbb{R})$ as $\det : M(n,\mathbb{R}) \longrightarrow \mathbb{R}$ is a continuous map and $GL(n,\mathbb{R})$ is the complement of the inverse image of the closed subset $\{0\} \subset \mathbb{R}$.

Definition 4.1. *A group G is called a linear Lie group if there is an $n \in \mathbb{N}$ such that G is isomorphism to a closed subgroup of $GL(n,\mathbb{R})$.*

Example 4.1. *(1) The general linear groups (over $\mathbb{R}$ or $\mathbb{C}$) are themselves linear Lie groups. Indeed, $GL(n,\mathbb{C})$ is a subgroup of itself. Furthermore, if A_m is a sequence of matrices in $GL(n,\mathbb{C})$ and A_m converges to A, then by the definition of $GL(n,\mathbb{C})$, either A is in $GL(n,\mathbb{C})$, or A is not invertible.*

(2) The special linear group is by definition

$$SL(n,\mathbb{R}) = \{g \in GL(n,\mathbb{R}) \mid \det(g) = 1\}$$

if A_n is a sequence of matrices with determinant one and A_n converges to A, then A also has determinant one, because the determinant is a continuous function. Thus, $SL(n,\mathbb{R})$ is a closed subgroup of $GL(n,\mathbb{C})$.

4.3 Lie Algebras

Definition 4.2. *Let $\mathbf{K}$ be $\mathbb{R}$ or $\mathbb{C}$. A $\mathbf{K}$-vector space $\mathfrak{g}$ provided with a $\mathbf{K}$-bilinear map*

$$\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g}, \quad (X,Y)\mapsto[X,Y]$$

is called a Lie algebra if

$$[X,Y]=-[Y,X], \quad \forall X,Y\in\mathfrak{g},$$

and the Jacobi identity

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$$

holds. The dimension of $\mathfrak{g}$ as a $\mathbf{K}$-vector space is called the dimension of the Lie algebra $\mathfrak{g}$.

Example 4.2.

1. *Every associative algebra $\mathfrak{g}$ is a Lie algebra with the braket $[X,Y]:=XY-YX$ for all $X,Y\in\mathfrak{g}$.*
2. *Let $\mathfrak{g}=\mathbb{R}^3$ and let $[\cdot,\cdot]:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3$ be given by*

   $$[X,Y]=X\times Y,$$

   where $X\times Y$ is the cross product .Then $\mathfrak{g}$ is a Lie algebra.

The exponential of a matrix $X\in M(n,\mathbf{K})$ is defined by

$$\exp(X)=\sum_{n=0}^{\infty}\frac{X^n}{n!}.$$

where X^0 is defined to be the identity matrix I.

Lemma 4.1. *For $X, Y \in \mathfrak{gl}(n,\mathbb{R}) = M(n,\mathbb{R})$ and $k \longrightarrow \infty$, we have*

(i) $\exp\left(\frac{X}{k}\right)\exp\left(\frac{Y}{k}\right) = \exp\left(\frac{X+Y}{k} + \frac{[X,Y]}{2k^2} + O\left(\frac{1}{k^3}\right)\right).$

(ii) $\exp\left(\frac{X}{k}\right)\exp\left(\frac{Y}{k}\right)\exp\left(-\frac{X}{k}\right)\exp\left(-\frac{Y}{k}\right) = \exp\left(\frac{[X,Y]}{k} + O\left(\frac{1}{k^3}\right)\right).$

Proof.

(i) Since $k \longrightarrow \infty$, we have

$$\begin{aligned}\exp\left(\frac{X}{k}\right)\exp\left(\frac{Y}{k}\right) &= \left(I + \frac{X}{k} + \frac{X^2}{2k^2} + O\left(\frac{1}{k^3}\right)\right)\left(I + \frac{Y}{k} + \frac{Y^2}{2k^2} + O\left(\frac{1}{k^3}\right)\right) \\ &= I + \frac{X+Y}{k} + \frac{X^2+2XY+Y^2}{2k^2} + O\left(\frac{1}{k^3}\right).\end{aligned}$$

Thus

$$\begin{aligned}&\log\left[\exp\left(\frac{X}{k}\right)\exp\left(\frac{Y}{k}\right)\right] \\ =&\left\{\frac{X+Y}{k} + \frac{X^2+2XY+Y^2}{2k^2}\right\} - \frac{1}{2}\left\{\frac{X+Y}{k}\right\}^2 + O\left(\frac{1}{k^3}\right) \\ =&\frac{X+Y}{k} + \frac{[X,Y]}{2k^2} + O\left(\frac{1}{k^3}\right).\end{aligned}$$

(ii) By part (i) we have

$$\begin{aligned}&\exp\left(\frac{X}{k}\right)\exp\left(\frac{Y}{k}\right)\exp\left(-\frac{X}{k}\right)\exp\left(-\frac{Y}{k}\right) \\ =&\left\{I + \frac{X+Y}{k} + \frac{X^2+2XY+Y^2}{2k^2} + O\left(\frac{1}{k^3}\right)\right\} \\ &\left\{I - \frac{X+Y}{k} + \frac{X^2+2XY+Y^2}{2k^2} + O\left(\frac{1}{k^3}\right)\right\} \\ =&I + \frac{[X,Y]}{k^2} + O\left(\frac{1}{k^3}\right).\end{aligned}$$

Thus

$$\log\left[\exp\left(\frac{X}{k}\right)\exp\left(\frac{Y}{k}\right)\exp\left(-\frac{X}{k}\right)\exp\left(-\frac{Y}{k}\right)\right] = \frac{[X,Y]}{k^2} + O\left(\frac{1}{k^3}\right).$$

Lemma 4.2. *For $X,Y \in \mathfrak{gl}(n,\mathbb{R}) = M(n,\mathbb{R})$ and $k \longrightarrow \infty$, we have*

(i) $\lim_{k\longrightarrow\infty} \left[\exp\left(\frac{X}{k}\right)\exp\left(\frac{Y}{k}\right)\right]^k = \exp(X+Y)$.

(ii) $\lim_{k\to\infty} \left[\exp\left(\frac{X}{k}\right)\exp\left(\frac{Y}{k}\right)\exp\left(-\frac{X}{k}\right)\exp\left(-\frac{Y}{k}\right)\right]^{k^2} = \exp([X,Y])$.

Proof. *Using Lemma 4.1. we obtain*

(i)

$$\left[\exp\left(\frac{X}{k}\right)\exp\left(\frac{Y}{k}\right)\right]^k = \exp\left([X,Y]+O\left(\frac{1}{k}\right)\right)$$
$$\longrightarrow \exp([X,Y]) \quad \textit{as } k \longrightarrow \infty$$

(ii) Using Lemma 4.1. we obtain

$$\left[\exp\left(\frac{X}{k}\right)\exp\left(\frac{Y}{k}\right)\exp\left(-\frac{X}{k}\right)\exp\left(-\frac{Y}{k}\right)\right]^{k^2}$$
$$= \exp\left([X,Y]+O\left(\frac{1}{k}\right)\right)$$
$$\longrightarrow \exp([X,Y]) \quad \textit{as } k \longrightarrow \infty.$$

Definition 4.3. *Let G be a linear Lie group contained in $GL(n,\mathbb{R})$. Henceforth we will write*

$$\mathfrak{g} := \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid \exp(tX) \in G \textit{ for all } t \in \mathbb{R}\},$$

where exp is the exponential map for matrices.

Properties 4.1. *For $\lambda \in \mathbb{R}$ and $X,Y \in \mathfrak{g}$, we have*

(i) $\lambda X \in \mathfrak{g}$

(ii) $X+Y \in \mathfrak{g}$

(iii) $[X,Y] \in \mathfrak{g}$.

Proof.

(i) we observe that $\exp(t(\lambda X)) = \exp((t\lambda)X)$.

(ii) For $X, Y \in \mathfrak{g}$ we have $\left[\exp\left(\frac{tX}{k}\right)\exp\left(\frac{tY}{k}\right)\right]^k \in G$. Since G is by definition closed, it follows from Lemma 4.2. that $\exp(t(X+Y)) \in G$. That is $X+Y \in \mathfrak{g}$.

(iii) similar to (ii), we have

$\lim_{k\to\infty}\left[\exp\left(\frac{tX}{k}\right)\exp\left(\frac{tY}{k}\right)\exp\left(-\frac{tX}{k}\right)\exp\left(-\frac{tY}{k}\right)\right]^{k^2} = \exp(t[X,Y]) \in G$. That is $[X,Y] \in \mathfrak{g}$.

In view of the above properties, we can state the following:

Theorem 4.1. *The set $\mathfrak{g}$ provided with the bilinear map*

$$[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}, \quad (X,Y) \mapsto [X,Y] = XY - YX$$

is a real Lie algebra, called the Lie algebra of G.

Example 4.3. *the Lie Algebra of $O(n,\mathbb{R}) = \{g \in GL(n,\mathbb{R}) \mid {}^t gg = g^t g = I_n\}$ is $o(n) = \left\{X \in M_n(\mathbb{R}) : X = -X^T\right\}$. We have*

$$\begin{aligned} X \in o(n) &\Longleftrightarrow e^{tX} \in O(n)\forall t \\ &\Longleftrightarrow \left(e^{tX}\right)^{-1} = \left(e^{tX}\right)^T \forall t \\ &\Longleftrightarrow e^{-tX} = e^{tX^T}\forall t \\ &\Longleftrightarrow -X = X^T. \end{aligned}$$

Example 4.4. *Recall that*

$$SL(n,\mathbb{R}) = \{X \in GL(n,\mathbb{R}) \mid \det(g) = 1\}.$$

Its Lie algebra $\mathfrak{sl}(n,\mathbb{R})$ is the space of matrices X such that $\det(\exp(tX)) = 1$, for all $t \in \mathbb{R}$. Using the fact that $\det(\exp(tX)) = \exp(t\,\mathrm{tr}(X))$, thus

$$\mathfrak{sl}(n,\mathbb{R}) = \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid \mathrm{tr}(X) = 0\}.$$

4.3.1 One-parameter Subgroups

Let G be a linear Lie group and $\mathcal{H}$ a complex Hilbert space.

Definition 4.4. *A one-parameter subgroup γ of a topological group G is a continuous homomorphism $\gamma : \mathbb{R} \longrightarrow G, t \mapsto \gamma(t)$. If the one-parameter subgroup $\gamma = \gamma(t)$ is differentiable in t, we assign to it an infinitesimal generator X_γ by*

$$X_\gamma := \frac{d}{dt}\gamma(t)_{|t=0}.$$

Let $\mathfrak{g}$ be the Lie algebra of a linear Lie group G. For every $X \in \mathfrak{g}$, the following map $\gamma : \mathbb{R} \longrightarrow G, t \mapsto \exp(tX)$ is a one parameter subgroup. This is due to the fact that the map $t \mapsto \exp(tX)$ is continuous (even differentiable) and $\exp(A+B) = \exp(A)\exp(B)$ for commuting $A, B \in M(n, \mathbb{R})$. In this example we have X as infinitesimal generator.

4.3.2 Representation of Lie Groups

Definition 4.5. *Let G be a linear Lie group and $\mathcal{H}$ a complex Hilbert space. A representation $(\pi, \mathcal{H})$ of G (with identity e) is a group homomorphism from G to the set $GL(\mathcal{H})$ of linear bounded operators in $\mathcal{H}$, that is*

$$\pi(g_1 g_2) = \pi(g_1)\pi(g_2), \quad \pi(e) = \text{ identity transformation,}$$

such that for all $v \in \mathcal{H}$ the map

$$g \mapsto \pi(g)v$$

is continuous. This means that for any $g_0 \in G$ we have

$$\|\pi(g)v - \pi(g_0)v\| \longrightarrow 0 \quad \text{as } g \to g_0,$$

for all $v \in \mathcal{H}$. The above condition is called the strong continuity property.

Definition 4.6. *(1) A representation $(\pi, \mathcal{H})$ is called trivial if $\pi(g) = \mathrm{id}_{\mathcal{H}}$ for all $g \in G$.*

(2) A representation is said to be unitary if $\pi(G)$ consists of unitary operators. Recall that a unitary operator T on a Hilbert space is an operator such that $TT^* = T^*T = I$, where T^* is the Hilbert space adjoint operator of T.

Definition 4.7. *A finite dimensional representation of a topological group G on a vector space V over $\mathbb{R}$ or $\mathbb{C}$ is a Lie group homomorphism*

$$\pi : G \longrightarrow GL(n, \mathbf{K}), \quad \mathbf{K} = \mathbb{R} \text{ or } \mathbb{C},$$

where $\dim(V) = n$. *An n-dimensional representation of a group G is a prescription π associating to each $g \in G$ a matrix $\pi(g) = A(g) \in GL(n, \mathbf{K})$ such that*

$$A\left(gg'\right) = A(g)A\left(g'\right)$$

holds for all $g, g' \in G$.

Example 4.5. *Let $(\pi, \mathbb{C}^n)$ be the following representation $\pi(g) = g$, where $\pi(g)$ will act by matrix multiplication on the vector space $\mathbb{C}^n$.*

4.3.3 Representation of Lie Algebras

Definition 4.8. *Let $\mathfrak{g}$ be a Lie algebra on a field $\mathbf{K}$, and V a $\mathbf{K}$-vector space. A representation (ω, V) of $\mathfrak{g}$ is a homomorphism of $\mathfrak{g}$ into* $\mathrm{End}(V)$, *the space of endomorphisms of V, i.e.*

$$\omega(\alpha X + \beta Y) = \alpha\omega(X) + \beta\omega(Y), \quad \alpha, \beta \in \mathbf{K},$$
$$\omega([X,Y]) = [\omega(X), \omega(Y)] = \omega(X)\omega(Y) - \omega(Y)\omega(X).$$

In analogy with group representations, a representation (ω, V) of a Lie algebra $\mathfrak{g}$ is called irreducible if there is no nontrivial $\mathfrak{g}$-invariant subspace in V.

Each Lie algebra has the trivial representation $\omega(X) = 0$ for all $X \in \mathfrak{g}$. This is trivially an irreducible representation.

Example 4.6. *Let $\mathfrak{g}$ be an arbitrary Lie algebra. For $x \in \mathfrak{g}$, define the operator* $\mathrm{ad}(x)$ *on $\mathfrak{g}$ via*

$$\mathrm{ad}(x)y := [x, y], \quad y \in \mathfrak{g}.$$

Using Jacobi identity we have

$$\begin{aligned}\mathrm{ad}([x,y])(z) = [[x,y],z] &= [x,[y,z]] - [y,[x,z]] \\ &= \mathrm{ad}(x)(\mathrm{ad}(y)(z)) - \mathrm{ad}(y)(\mathrm{ad}(x)(z)) \\ &= (\mathrm{ad}(x)\,\mathrm{ad}(y) - \mathrm{ad}(y)\,\mathrm{ad}(x))(z) \\ &= [\mathrm{ad}(x), \mathrm{ad}(y)](z).\end{aligned}$$

Thus $(\mathrm{ad}, \mathfrak{g})$ *is a representation of* $\mathfrak{g}$*; it is called the adjoint representation of the Lie algebra* $\mathfrak{g}$.

A representation of a Lie group G gives rise to a representation of its Lie algebra $\mathfrak{g}$ in a natural way.

Definition 4.9. *Let* $(\pi, \mathcal{H})$ *be a representation of* G *on a Hilbert space* $\mathcal{H}$*. We call a vector* $v \in \mathcal{H}$ *smooth for the representation* π *if the map* $g \mapsto \pi(g)v$ *is a smooth function from* G *to* $\mathcal{H}$*. The set of smooth vectors for* π *form a subspace* $\mathcal{H}^\infty$ *of* $\mathcal{H}$*. Let* $v \in \mathcal{H}^\infty$ *and* $X \in \mathfrak{g}$*, we define*

$$f(X) = \pi(\exp(X))v$$

Then f *is of class* C^∞*. Put*

$$d\pi(X)v := df(0)X.$$

It follows that

$$\lim_{t\to 0} \frac{\pi(\exp(tX))v - v}{t} = d\pi(X)v.$$

Clearly $d\pi(X)$ *as a linear mapping of* $\mathcal{H}^\infty$ *into* $\mathcal{H}$ *and it depends linearly on* X.

Theorem 4.2. *Let* π *be a representation of a linear Lie group* G *on a Hilbert space* $\mathcal{H}$*. For* $X \in \mathfrak{g}$*, define a linear mapping from* $\mathcal{H}^\infty$ *into* $\mathcal{H}$ *by*

$$d\pi(X)v = \lim_{t\to 0} \frac{\pi(\exp(tX))v - v}{t}.$$

For every $X \in \mathfrak{g}, d\pi(X)$ *leaves* $\mathcal{H}^\infty$ *stable, and* $d\pi$ *is a representation of* $\mathfrak{g}$ *on* $\mathcal{H}^\infty$*. We call it the derived (or the infinitesimal) representation of* π.

Proof. *For $v \in \mathcal{H}^\infty$, we have*

$$\pi(g)d\pi(X)v = \lim_{t\to 0}\frac{\pi(g\exp(tX))v - \pi(g)v}{t}$$
$$= \frac{d}{dt}\bigg|_{t=0}\pi(g\exp(tX))v.$$

Since the map $g \mapsto \pi(g)v$ is a C^∞-map it follows that $g \mapsto \pi(g)d\pi(X)v$ is smooth and, therefore $d\pi(X)v \in \mathcal{H}^\infty$. Hence $\mathcal{H}^\infty$ is stable by $d\pi(X)$, for all $X \in \mathfrak{g}$. Next we will show that $d\pi([X,Y]) = [d\pi(X), d\pi(Y)]$ for all $X, Y \in \mathfrak{g}$. Let

$$c(t) := \exp\left((-\operatorname{sg} t)|t|^{1/2}X\right)\exp\left(-|t|^{1/2}Y\right)\exp\left((\operatorname{sg} t)|t|^{1/2}X\right)\exp\left(|t|^{1/2}Y\right)$$
$$= \exp\left((\operatorname{sg} t)|t|[X,Y] + o\left(|t|^{3/2}\right)\right).$$

The function $t \mapsto c(t)$ is a C^1-curve from $\mathbb{R}$ to G, with $c'(0) = [X,Y]$. For $v \in \mathcal{H}^\infty$ the map $t \mapsto \pi(c(t))v$ has differential $d\tilde{f}(e)([X,Y])$ at $t = 0$, where $\tilde{f}(g) = \pi(g)v$. That is

$$\lim_{t\to 0}\frac{\pi(c(t))v - v}{t} = d\pi([X,Y])v.$$

Hence

$$\lim_{t\to 0}\frac{\pi\left(c\left(t^2\right)\right)v - v}{t^2} = d\pi([X,Y])v.$$

Using the strong continuity of the representation $(\pi, \mathcal{H})$ of G, we obtain

$$\lim_{t\to 0}\frac{\pi(\exp(tX)\exp(tY))v - \pi(\exp(tY)\exp(tX))v}{t^2}$$
$$= \lim_{t\to 0}\pi(\exp(tY))\pi(\exp(tX))\frac{\pi\left(c\left(t^2\right)\right)v - v}{t^2}$$
$$= d\pi([X,Y])v.$$

On the other hand, the map

$$(s,t) \mapsto \pi(\exp(sX)\exp(tY))v$$

is of class C^∞. This is due to the fact that the map is the composition of the following C^∞ maps

$$(s,t) \mapsto (\exp sX, \exp tY), \quad (g_1, g_2) \mapsto g_1g_2, \quad g \mapsto \pi(g)v.$$

In particular, for each $v \in \mathcal{H}$, the map $(s,t) \mapsto \langle \pi(\exp(sX)\exp(tY))v, v\rangle$ is of class C^∞. Hence

$$\begin{aligned}\langle d\pi(X)d\pi(Y)v, w\rangle &= \frac{\partial}{\partial s}\frac{\partial}{\partial t}\langle \pi(\exp sX \exp tY)v, w\rangle_{|s=t=0} \\ &= \lim_{t\to 0}\left\langle t^{-2}\{\pi(\exp(tX)\exp(tY)) - \pi(\exp(tX)) - \pi(\exp(tY)) + I\}v, w\right\rangle.\end{aligned}$$

Replace X by Y and vice versa, we obtain

$$\langle d\pi(Y)d\pi(X)v, w\rangle = \lim_{t\to 0}\left\langle t^{-2}\{\pi(\exp(tY)\exp(tX)) - \pi(\exp(tX)) - \pi(\exp(tY)) + I\}v, w\right\rangle.$$

Thus

$$\langle\{d\pi(X)d\pi(Y) - d\pi(Y)d\pi(X)\}v, w\rangle = \lim_{t\to 0}\left\langle t^{-2}\{\pi(\exp(tX)\exp(tY)) - \pi(\exp(tY)\exp(tX))\}v, w\right\rangle.$$

In conclusion $d\pi([X,Y]) = d\pi(X)d\pi(Y) - d\pi(Y)d\pi(X)$.

4.3.4 Integrability of Infinitesimal Representations: The Infinite Dimensional Case

In the case where $\mathcal{H}$ is a Hilbert space, the situation is less obvious since the infinitesimal operators are not in general bounded and therefore are not defined on the entire $\mathcal{H}$. Next we will give Nelson's theorem which establishes when a representation of a Lie algebra $\mathfrak{g}$, given in terms of skewsymmetric operators, can be seen as the infinitesimal representation of a unitary representation of a simply connected Lie group G having $\mathfrak{g}$ as a Lie algebra.

Theorem 4.3. ((see [34])). *Let $\mathfrak{g}$ be a real Lie algebra of dimension m and $\mathcal{H}$ be a Hilbert space. Let $\{X_1, \ldots, X_m\}$ be a basis of the Lie algebra $\mathfrak{g}$. Assume that $\omega(X_1), \ldots, \omega(X_m)$ are skew-symmetric operators which have a common invariant dense domain D. If the operator $\omega\left(X_1^2 + \cdots + X_m^2\right)$ is essentially self-adjoint, then there is on $\mathcal{H}$ a unique unitary representation π of the universal covering Lie group G with Lie algebra $\mathfrak{g}$,*

$$\omega(X) = \frac{\mathrm{d}}{\mathrm{dt}}_{|t=0}\pi(\exp(tX)).$$

Chapter 5: Wave Equation

5.1 Introduction

In this chapter we consider the wave equation

$$2\|x\|\Delta_k u_k(x,t) - \partial_{tt} u_k(x,t) = 0, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

where Δ_k is a second order differential and difference operator. First, we prove the existence and the uniqueness of the solution $u_k(x,t)$. Second, we search for the condition on the parameter k and the dimension n for the fundamental solution to be supported on the light cone $\left\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \sqrt{2\|x\|} = |t|\right\}$. Our approach is based heavily on the representation theory of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, where we construct a new representation ω_k of $\mathfrak{sl}(2,\mathbb{R})$ acting on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

5.2 The Existence and Uniqueness of the Solution

For $k \in K^+$ and $f \in C^\infty(\mathbb{R}^n)$, recall from Chapter 2 the Dunkl Laplacian operator

$$\Delta_k f(x) = \Delta f(x) + \sum_{\alpha \in \mathcal{R}} k_\alpha \left\{ \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(s_\alpha x)}{\langle \alpha, x \rangle^2} \right\}$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$ denotes the usual gradient operator.

We will consider the following wave equation

$$2\|x\|\Delta_k^x u_k(x,t) - \partial_{tt} u_k(x,t) = 0$$

where t is any real number and $x \in \mathbb{R}^n$. Here the superscript in Δ_k^x denotes the relevant variable. The fact that we consider the operator $\|x\|\Delta_k$ in the equation above is due to the fact that this differential-difference operator goes very well with the generalized Fourier transform $\mathcal{F}_{k,1}$ that we are dealing with in this thesis. More precisely,

$$\mathcal{F}_{k,1}(\|x\|\Delta_k f)(\xi) = -\|\xi\|\mathcal{F}_{k,1}(f)(\xi).$$

Let us recall that when $n = 1,2,3$, the classical wave equation $\partial_{tt}u(x,t) = \Delta u(x,t)$ describe roughly vibrations of a string, a drum, and sound waves in air, respectively.

The initial conditions we give are for both u_k and $\partial_t u_k$,

$$u_k(x,0) = f(x), \qquad \partial_t u_k(x,0) = g(x).$$

We will take f and g in the Schwartz space $S(\mathbb{R}^n)$, although the solution we will obtain allows much more general choice. We will prove that these initial conditions (called Cauchy data) determine a unique solution without any growth conditions.

Through out this chapter we will assume that $k \in K^+$ and $n \geq 1$. For $t \in \mathbb{R}$, denote by $P_{k,t}$ the 2×2 matrix of tempered distributions on $\mathbb{R}^n$

$$\begin{aligned} P_{k,t} &= \begin{pmatrix} P_{k,t}^{11} & P_{k,t}^{12} \\ P_{k,t}^{21} & P_{k,t}^{22} \end{pmatrix} \\ &:= \begin{pmatrix} \mathcal{F}_k(\cos(t\sqrt{2\|\cdot\|}) & \mathcal{F}_k(\sin(t\sqrt{2\|\cdot\|})/\sqrt{2\|\cdot\|}) \\ \mathcal{F}_k(-\sqrt{2\|\cdot\|}\sin(t\sqrt{2\|\cdot\|}) & \mathcal{F}_k(\cos(t\sqrt{2\|\cdot\|})) \end{pmatrix}. \end{aligned}$$

Let $U_k(x,0) := \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}$, where the initial data $(f,g) \in S(\mathbb{R}^n) \times S(\mathbb{R}^n)$. Define the vector column $U_k(x,t)$ by

$$\begin{aligned} U_k(x,t) &:= \left\{P_{k,t} \circledast U_k(\cdot,0)\right\}(x) \\ &= \left\{\begin{pmatrix} P_{k,t}^{11} & P_{k,t}^{12} \\ P_{k,t}^{21} & P_{k,t}^{22} \end{pmatrix} \circledast \begin{pmatrix} f \\ g \end{pmatrix}\right\}(x). \end{aligned} \tag{5.1}$$

Applying Fourier transform $\mathcal{F}_k$ with respect to x to the above identity, we obtain

$$\mathcal{F}_k\left(U_k(\cdot,t)\right)(\xi) = e^{t\mathbb{A}}\mathcal{F}_k\left(U_k(\cdot,0)\right)(\xi),$$

where

$$\mathbb{A} := \begin{pmatrix} 0 & 1 \\ -2\|\xi\| & 0 \end{pmatrix}.$$

That is $\mathcal{F}_k(U_k(\cdot,t))(\xi)$ is a solution to the following ordinary differential equation

$$\begin{aligned}\partial_t\mathcal{F}_k(U_k(\cdot,t))(\xi) &= \mathbb{A}\mathcal{F}_k(U_k(\cdot,t))(\xi)\\ &= \begin{pmatrix} 0 & 1 \\ -2\|\xi\| & 0 \end{pmatrix}\mathcal{F}_k(U_k(\cdot,t))(\xi).\end{aligned}$$

Since $-2\|\xi\|\mathcal{F}_k(f)(\xi) = \mathcal{F}_k(2\|x\|\Delta_k f)(\xi)$, and the fact that the generalized Fourier transform $\mathcal{F}_k$ is injective, we conclude that

$$\partial_t U_k(x,t) = \begin{pmatrix} 0 & 1 \\ 2\|x\|\Delta_k & 0 \end{pmatrix} U_k(x,t).$$

Hence, if we set $U_k(x,t) = \begin{pmatrix} u_k(x,t) \\ v_k(x,t) \end{pmatrix}$, then

$$\partial_t u_k(x,t) = v(x,t)$$

and $v_k(x,t)$ satisfies

$$\partial_t v_k(x,t) = 2\|x\|\Delta_k u_k(x,t).$$

In conclusion, the function u_k satisfies the wave equation

$$\partial_{tt} u_k(x,t) = 2\|x\|\Delta_k u_k(x,t).$$

Lemma 5.1. *The above constructed solution $u_k(x,t)$ satisfies the initial data. That is $u_k(x,0) = f(x)$ and $\partial_t u_k(x,0) = g(x)$.*

Proof. *Denote by δ the Dirac distribution.Then,*

$$\mathcal{F}_k(\delta) = c_{k,1}\int_{\mathbb{R}^n} B_{k,1}(\xi,x)\delta(x)\vartheta_k(x)dx = \langle \delta, B_{k,1}(\xi,\cdot)\rangle = B_{k,1}(\xi,0) = 1.$$

This implies $\mathcal{F}_k^{-1}(\cos(t\sqrt{2\|\cdot\|})) \to \delta$ as $t \to 0$. On the other hand, one may check easily that $\mathcal{F}_k^{-1}(\sin(t\sqrt{2\|\cdot\|})/\sqrt{2\|\cdot\|}) \to 0$ as $t \to 0$. Using the continuity of the of *the convolution* $\circledast$*, we conclude that*

$$u_k(x,t) \to (\delta \circledast f)(x) \text{ as } t \to 0,$$

where

$$\begin{aligned} \delta \circledast f(x) &= c_k \int_{\mathbb{R}^n} \delta(y)\tau_x f(y)\vartheta_k(y)dy \\ &= \langle \delta, \tau_x f(y) \rangle = \tau_x f(0) = \tau_0 f(x) = f(x). \end{aligned}$$

For the derivative

$$\partial_t u(x,t) = P_{k,t}^{21} \circledast f(x) + P_{k,t}^{22} \circledast g(x)$$

where $P_{k,t}^{21} = \mathcal{F}^{-1}(-\sqrt{2\|\cdot\|}\sin t\sqrt{2\|\cdot\|})$ *and* $P_{k,t}^{22} = \mathcal{F}^{-1}(\cos(t\sqrt{2\|\cdot\|}))$*. Using the same approach, we deduce that* $P_{k,t}^{21} \longrightarrow 0$ *and* $P_{k,t}^{22} \longrightarrow \delta$ *as* $t \longrightarrow 0$*. Hence*

$$\lim_{t\to 0} \partial_t u(x,t) = (\delta \circledast g)(x) = g(x).$$

One of the main problems in the theory of partial differential equations is the uniqueness of the solution. We claim that the above constructed solution $u_k(x,t)$ is unique. To prove our claim we need the following lemma.

Define the total energy of the solution $u_k(x,t)$ at time t by

$$E_k[u_k](t) = \int_{\mathbb{R}^n} \left(|\partial_t u_k(x,t)|^2 + |\sqrt{2\|x\|\Delta_k} u_k(x,t)|^2 \right) \vartheta_k(x)dx.$$

Lemma 5.2. *Assume that the Cauchy data* (f,g) *are Schwartz functions. The total energy* $E_k[u_k]$ *is finite and independent of the time* t*.*

Proof. *Using the Plancherel Theorem 3.3. for the Fourier transform* $\mathcal{F}_k$ *and the fact that*

$$\mathcal{F}_k\left(\sqrt{2\|x\|\Delta_k} u_k(\cdot,t)\right)(\xi) = \sqrt{2\|\xi\|}\mathcal{F}_k(u_k(\cdot,t))(\xi)$$

it follows that

$$E_k\left[u_k\right](t) = \int_{\mathbb{R}^N} \left\{ \left|\partial_t \mathcal{F}_k\left(u_k(\cdot,t)\right)(\xi)\right|^2 + 2\|\xi\| \left|\mathcal{F}_k\left(u_k(\cdot,t)\right)(\xi)\right|^2 \right\} v_k(\xi) d\xi$$

From our construction of the solution

$$u_k(x,t) = P_{k,t}^{11} \circledast f(x) + P_{k,t}^{12} \circledast g(x),$$

we obtain

$$\mathcal{F}_k\left(u_k(\cdot,t)\right)(\xi) = \cos(t\sqrt{2\|\xi\|})\mathcal{F}_k f(\xi) + \frac{\sin(t\sqrt{2\|\xi\|})}{\sqrt{2\|\xi\|}}\mathcal{F}_k g(\xi), \quad \textit{for all } t \in \mathbb{R},$$

we conclude that

$$\begin{aligned} \left|\mathcal{F}_k\left(u_k(\cdot,t)\right)(\xi)\right|^2 = {} & \cos^2(t\sqrt{2\|\xi\|}) \left|\mathcal{F}_k f(\xi)\right|^2 + \frac{\sin^2(t\sqrt{2\|\xi\|})}{2\|\xi\|} \left|\mathcal{F}_k g(\xi)\right|^2 \\ & + \sqrt{2}\frac{\cos(t\sqrt{\|\xi\|})\sin(t\sqrt{\|\xi\|})}{\sqrt{\|\xi\|}} \operatorname{Rel}\left(\mathcal{F}_k f(\xi)\overline{\mathcal{F}_k g(\xi)}\right), \end{aligned}$$

and

$$\begin{aligned} \left|\partial_t \mathcal{F}_k\left(u_k(\cdot,t)\right)(\xi)\right|^2 = {} & \cos^2(t\sqrt{2\|\xi\|}) \left|\mathcal{F}_k g(\xi)\right|^2 + 2\|\xi\| \sin^2(t\sqrt{2\|\xi\|}) \left|\mathcal{F}_k f(\xi)\right|^2 \\ & - 2\sqrt{2\|\xi\|} \cos(t\sqrt{2\|\xi\|}) \sin(t\sqrt{2\|\xi\|}) \operatorname{Rel}\left(\mathcal{F}_k f(\xi)\overline{\mathcal{F}_k g(\xi)}\right). \end{aligned}$$

Then we have

$$\begin{aligned} E_k\left[u_k\right](t) &= \int_{\mathbb{R}^n} \left\{ 2\|\xi\| \left|\mathcal{F}_k f(\xi)\right|^2 + \left|\mathcal{F}_k g(\xi)\right|^2 \right\} \vartheta_k(\xi) d\xi \\ &= \int_{\mathbb{R}^n} \left\{ \left|\sqrt{2\|x\|\Delta_k} f(x)\right|^2 + |g(x)|^2 \right\} \vartheta_k(x) dx. \end{aligned}$$

This finishes the proof of the lemma. □

Let us return to the uniqueness of the solution u_k to the wave equation

$$\partial_{tt} u_k(x,t) - 2\|x\|\Delta_k u_k(x,t) = 0.$$

Assume that the wave equation has two distinct solutions, say $u_k^{(1)}$ and $u_k^{(2)}$, with the same initial data (f,g). Therefore, their difference $Z_k = u_k^{(1)} - u_k^{(2)}$ solves the same wave equation with the initial data

$$Z(x,0) = u_k^{(1)}(x,0) - u_2^{(2)}(x,0) = 0, \qquad \partial_t Z(x,0) = \partial_t u_k^{(1)}(x,0) - \partial_t u_k^{(2)}(x,0) = 0.$$

Therefore, using the above lemma where we proved that the total energy of a solution of the wave equation depends only on the initial data, we deduce that

$$E_k\left[Z_k\right](t) = E_k\left[u_k^{(1)} - u_k^{(2)}\right](t) = 0.$$

Using the definition of the total energy, we deduce that $\partial_t\left(u_k^{(1)} - u_k^{(2)}\right)(x,t) = 0$ for every $t \in \mathbb{R}$. Therefore, $\left(u_k^{(1)} - u_k^{(2)}\right)(x,t)$ is a constant function with respect to t. Hence,

$$\left(u_k^{(1)} - u_k^{(2)}\right)(x,t) = \left(u_k^{(1)} - u_k^{(2)}\right)(x,0) = 0.$$

This proves that the solutions of the wave equation are uniquely determined by the initial Cauchy data.

The following main theorem will summarize the above computations.

Theorem 5.1. *The solution $u_k(x,t)$ to the deformed wave equation is uniquely given by*

$$u_k(x,t) = \left(P_{k,t}^{11} \circledast f\right)(x) + \left(P_{k,t}^{12} \circledast g\right)(x),$$

where $P_{k,t}^{11}$ and $P_{k,t}^{12}$ are the tempered distributions on $\mathbb{R}^n$ given by

$$P_{k,t}^{11} = \mathcal{F}_k^{-1}(\cos(t\sqrt{2\|\cdot\|})), \quad P_{k,t}^{12} = \mathcal{F}_k^{-1}(\sin(t\sqrt{2\|\cdot\|})/\sqrt{2\|\cdot\|}).$$

Before studying the support of the solution u_k and of the propagators, let us make some observations regarding the estimate and the limit of $u_k(\cdot,t)$ in the space $L^2(\mathbb{R}^n, \upsilon_k(x)dx)$ of square integrable functions with respect to $\upsilon_k(x)dx$. We restrict our attention to the L^2-behaviors because these are the most physically interesting quantities.

Proposition 5.1. *The unique solution $u_k(x,t)$ to the wave equation satisfies:*

1. *For all $t \in \mathbb{R}$, we have the following Strichartz-type inequality*

$$\|u_k(\cdot,t)\|_k \leqslant \|f\|_k + \left\|(-2\|x\|\Delta_k)^{-1/2} g\right\|_k.$$

2. *As $|t| \to \infty$, the function $t \mapsto \|u_k(\cdot,t)\|_k$ has a finite limit depending in the initial data*

$$\lim_{|t|\to\infty} \|u_k(\cdot,t)\|_k^2 = \frac{1}{2}\|f\|_k^2 + \frac{1}{2}\left\|(-2\|x\|\Delta_k)^{-1/2} g\right\|_k^2.$$

   In particular, if $\|u_k(\cdot,t)\|_k \to 0$ as $|t| \to \infty$, then $u_k = 0$

Proof. *1) Using the Plancherel formula for the generalized Fourier transform we get*

$$\begin{aligned}
& \int_{\mathbb{R}^n} |u(x,t)|^2 \vartheta_k(x)dx \\
= & \int_{\mathbb{R}^n} |\mathcal{F}_k u(\cdot,t)(\xi)|^2 \vartheta_k(\xi)d\xi \\
= & \int_{\mathbb{R}^n} \left(\cos(t\sqrt{2\|\xi\|})\mathcal{F}_k f(\xi) + \frac{\sin(t\sqrt{2\|\xi\|})}{\sqrt{2\|\xi\|}}\mathcal{F}_k g(\xi)\right) \vartheta_k(\xi)d\xi \\
& \left(\cos(t\sqrt{2\|\xi\|})\overline{\mathcal{F}_k f(\xi)} + \frac{\sin(t\sqrt{2\|\xi\|})}{\sqrt{2\|\xi\|}}\overline{\mathcal{F}_k g(\xi)}\right) \vartheta_k(\xi)d\xi \\
= & \int_{\mathbb{R}^n} \cos^2(t\sqrt{2\|\xi\|})|\mathcal{F}_k f(\xi)|^2 \vartheta_k(\xi)d\xi \\
& + \int_{\mathbb{R}^n} \frac{\sin^2(t\sqrt{2\|\xi\|})}{2\|\xi\|} |\mathcal{F}_k g(\xi)|^2 \vartheta_k(\xi)d\xi \\
& + \int_{\mathbb{R}^n} \cos(t\sqrt{2\|\xi\|})\frac{\sin(t\sqrt{2\|\xi\|})}{\sqrt{2\|\xi\|}}\mathcal{F}_k f(\xi)\overline{\mathcal{F}_k g(\xi)}\vartheta_k(\xi)d\xi \\
& \int_{\mathbb{R}^n} \cos(t\sqrt{2\|\xi\|})\frac{\sin(t\sqrt{2\|\xi\|})}{\sqrt{2\|\xi\|}}\mathcal{F}_k g(\xi)\overline{\mathcal{F}_k f(\xi)}\vartheta_k(\xi)d\xi \\
\leq & \int_{\mathbb{R}^n} |\mathcal{F}_k f(\xi)|^2 \vartheta_k(\xi)d\xi + \int_{\mathbb{R}^n} \left|\frac{\mathcal{F}_k g(\xi)}{\sqrt{2\|\xi\|}}\right|^2 \vartheta_k(\xi)d\xi \\
& + \int_{\mathbb{R}^n} |\mathcal{F}_k f(\xi)| \left|\frac{\mathcal{F}_k g(\xi)}{\sqrt{2\|\xi\|}}\right| \vartheta_k(\xi)d\xi \\
\leq & \|\mathcal{F}_k f\|_k^2 + \left\|\frac{\mathcal{F}_k g}{\sqrt{2\|\xi\|}}\right\|_k^2 + 2\|\mathcal{F}_k f\|_k \left\|\frac{\mathcal{F}_k g}{\sqrt{2\|\xi\|}}\right\|_k
\end{aligned}$$

$$= \left(\|\mathcal{F}_k f\|_k + \left\|\frac{\mathcal{F}_k g}{\sqrt{2\|\xi\|}}\right\|_k\right)^2.$$

Therefore, we proved

$$\begin{aligned}
\|u(x,t)\|_k &\leq \left(\|\mathcal{F}_k f\|_k + \left\|\frac{\mathcal{F}_k g}{\sqrt{2\|\xi\|}}\right\|_k\right) \\
&= \left(\|\mathcal{F}_k f\|_k + \left\|\mathcal{F}_k((-2\|x\|\Delta_k)^{-\frac{1}{2}} g)\right\|_k\right) \\
&= \|f\|_k + \left\|(-2\|x\|\Delta_k)^{-\frac{1}{2}} g\right\|_k.
\end{aligned}$$

□

2) Below we will use the familiar trigonometric identities for double angles,

$$\begin{aligned}
&\int_{\mathbb{R}^n} |u(x,t)|^2 \vartheta_k(x) dx \\
&= \int_{\mathbb{R}^n} |\mathcal{F}_k u(\cdot,t)(\xi)|^2 \vartheta_k(\xi) d\xi \\
&= \int_{\mathbb{R}^n} \left(\cos(t\sqrt{2\|\xi\|})\mathcal{F}_k f(\xi) + \frac{\sin(t\sqrt{2\|\xi\|})}{\sqrt{2\|\xi\|}} \mathcal{F}_k g(\xi)\right) \\
&\quad \left(\cos(t\sqrt{2\|\xi\|})\overline{\mathcal{F}_k f(\xi)} + \frac{\sin(t\sqrt{2\|\xi\|})}{\sqrt{2\|\xi\|}} \overline{\mathcal{F}_k g(\xi)}\right) \vartheta_k(\xi) d\xi \\
&= \int_{\mathbb{R}^n} \cos^2(t\sqrt{2\|\xi\|}) |\mathcal{F}_k f(\xi)|^2 \vartheta_k(\xi) d\xi + \int_{\mathbb{R}^n} \frac{\sin^2(t\sqrt{2\|\xi\|})}{2\|\xi\|} |\mathcal{F}_k g(\xi)|^2 \vartheta_k(\xi) d\xi \\
&\quad + \int_{\mathbb{R}^n} \cos(t\sqrt{2\|\xi\|}) \frac{\sin(t\sqrt{2\|\xi\|})}{\sqrt{2\|\xi\|}} \mathcal{F}_k f(\xi) \overline{\mathcal{F}_k g(\xi)} \vartheta_k(\xi) d\xi \\
&\quad + \int_{\mathbb{R}^n} \cos(t\sqrt{2\|\xi\|}) \frac{\sin(t\sqrt{2\|\xi\|})}{\sqrt{2\|\xi\|}} \mathcal{F}_k g(\xi) \overline{\mathcal{F}_k f(\xi)} \vartheta_k(\xi) d\xi \\
&= \int_{\mathbb{R}^n} |\mathcal{F}_k f(\xi)|^2 \cos^2(t\sqrt{2\|\xi\|}) \vartheta_k(\xi) d\xi + \int_{\mathbb{R}^n} \frac{|\mathcal{F}_k g(\xi)|^2}{2\|\xi\|} \sin^2(t\sqrt{2\|\xi\|}) \vartheta_k(\xi) d\xi \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^n} \sin(t\sqrt{2\|\xi\|}) \left(\mathcal{F}_k f(\xi) \frac{\overline{\mathcal{F}_k g(\xi)}}{\sqrt{2\|\xi\|}} + \overline{\mathcal{F}_k f(\xi)} \frac{\mathcal{F}_k g(\xi)}{\sqrt{2\|\xi\|}}\right) \vartheta_k(\xi) d\xi.
\end{aligned}$$

Using classical trigonometric identities, we get

$$\int_{\mathbb{R}^n} |u(x,t)|^2 \vartheta_k(x) dx$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{R}^n} |\mathcal{F}_k f(\xi)|^2 \left(1+\cos(2t\sqrt{2\|\xi\|})\right) \vartheta_k(\xi) d\xi \\
&\quad +\frac{1}{2}\int_{\mathbb{R}^N} \frac{|\mathcal{F}_k g(\xi)|^2}{2\|\xi\|} \left(1-\cos(2t\sqrt{2\|\xi\|}\right) \vartheta_k(\xi) d\xi \\
&\quad +\frac{1}{2}\int_{\mathbb{R}^n} \sin(2t\sqrt{2\|\xi\|}) \left(\mathcal{F}_k f(\xi)\frac{\overline{\mathcal{F}_k g(\xi)}}{\sqrt{2\|\xi\|}} + \overline{\mathcal{F}_k f(\xi)}\frac{\mathcal{F}_k g(\xi)}{\sqrt{2\|\xi\|}}\right) \vartheta_k(\xi) d\xi \\
&= \frac{1}{2}\int_{\mathbb{R}^n} |\mathcal{F}_k f(\xi)|^2 \vartheta_k(\xi) d\xi + \frac{1}{2}\int_{\mathbb{R}^n} \frac{|\mathcal{F}_k g(\xi)|^2}{2\|\xi\|} \vartheta_k(\xi) d\xi \\
&\quad +\frac{1}{2}\int_{\mathbb{R}^n} |\mathcal{F}_k f(\xi)|^2 \cos(2t\sqrt{2\|\xi\|}) \vartheta_k(\xi) d\xi \\
&\quad -\frac{1}{2}\int_{\mathbb{R}^n} \frac{|\mathcal{F}_k g(\xi)|^2}{2\|\xi\|} \cos(2t\sqrt{2\|\xi\|}) \vartheta_k(\xi) d\xi \\
&\quad +\frac{1}{2}\int_{\mathbb{R}^n} \sin(2t\sqrt{2\|\xi\|}) \left(\mathcal{F}_k f(\xi)\frac{\overline{\mathcal{F}_k g(\xi)}}{\sqrt{2\|\xi\|}} + \overline{\mathcal{F}_k f(\xi)}\frac{\mathcal{F}_k g(\xi)}{\sqrt{2\|\xi\|}}\right) \vartheta_k(\xi) d\xi.
\end{aligned}
$$

Now, since f and g are integrable functions, by Riemann-Lebesgue Lemma for cosine and sine Fourier transforms, we have

$$
\lim_{|t|\to\infty} \|u_k(\cdot,t)\|_k^2 = \frac{1}{2}\|f\|_k^2 + \frac{1}{2}\left\|(-2\|x\|\Delta_k)^{-1/2} g\right\|_k^2.
$$

Recall that our solution

$$
u_k(x,t) = \left(P_{k,t}^{11} \circledast f\right)(x) + \left(P_{k,t}^{12} \circledast g\right)(x),
$$

Hence, if $\|u_k(\cdot,t)\| \to 0$ as $|t| \to \infty$, then $f = 0$ and $g = 0$. Therefore, $u_k = 0$ □

We shall now discuss the strict Huygens principle which will hold only under a condition involving the dimension n and the multiplicity function k. Our approach uses the representation theory of the group $SL(2,\mathbb{R})$, following [32, 33].

Our next task is to investigate certain symmetries and invariance of the wave equation under consideration, which are reflected in symmetries and invariance of the propagators $P_{k,t}^{ij}$. To do so, we define the 2×2 matrix

$$
P_k = \begin{pmatrix} P_k^{11} & P_k^{12} \\ P_k^{21} & P_k^{22} \end{pmatrix}
$$

of entryways distributions on $\mathbb{R}^{n+1}$, where

$$P_k^{ij}(\psi_1 \otimes \psi_2) := \int_{\mathbb{R}} P_{k,t}^{ij}(\psi_1)\, \psi_2(t)dt, \quad i,j = 1,2,$$

for $\psi_1 \in \mathcal{S}(\mathbb{R}^n)$ and $\psi_2 \in \mathcal{S}(\mathbb{R})$. Here we used the fact that $\mathcal{S}\left(\mathbb{R}^{n+1}\right) \simeq \mathcal{S}(\mathbb{R}^n)\, \hat{\otimes} \mathcal{S}(\mathbb{R})$.

The following statement follows directly from the construction of the solution $u_k(x,t)$.

Proposition 5.2. *We have*

$$\left(\|x\|\Delta_k - \frac{1}{2}\partial_{tt}\right) P_k^{ij} = 0, \quad i,j = 1,2.$$

Recall that G stands for the Coxeter group associated with the root system $\mathcal{R}$ (see Chapter 2). For $h \in G$, $\psi \in \mathcal{S}(\mathbb{R}^{n+1})$, and for each $t \in \mathbb{R}$, denote by π_x the unitary action of G on $\psi(\cdot,t)$ given by

$$\pi_x(h)\psi(x,t) := \psi(h^{-1}\cdot x,t).$$

By duality, we have the action π_x^* of G on tempered distributions by the rule

$$\pi_x^*(h)(T)(\psi) = T(\pi_x(h)^{-1}\psi),$$

for $\psi \in \mathcal{S}(\mathbb{R}^{n+1})$ and $T \in \mathcal{S}'(\mathbb{R}^{n+1})$. Further, let τ be the operation of time-reflection $\tau(x,t) = (x,-t)$, and denote by

$$\pi_t(\tau)\psi(x,t) := \psi(x,-t).$$

Similarly as for π_x^*, we obtain the action π_t^* on distributions.

Begin with a solution $u_k(x,t)$ to the Cauchy problem under consideration with Cauchy data (f,g). Then $\pi_x(h)u_k(x,t)$ solves the wave equation with initial data $(\pi_x(h)f, \pi_x(h)g)$. The analogue of the above construction of the solution $u_k(x,t)$ reads

$$\pi_x(h)U_k(x,t) = \left\{P_{k,t} \circledast \pi_x(h)U_k(\cdot,0)\right\}(x).$$

This amounts to

$$U_k(x,t) = \pi_x^*(h)\left\{P_{k,t} \circledast \pi_x(h)U_k(\cdot,0)\right\}(x) = \left\{\pi_x^*(h)P_{k,t} \circledast U_k(\cdot,0)\right\}(x),$$

which implies

$$\pi_x^*(h)P_{k,t}^{ij} = P_{k,t}^{ij}, \qquad i,j = 1,2.$$

Plugging this into the definition of P_k^{ij}, we conclude that

$$\pi_x^*(h)P_k^{ij} = P_k^{ij}, \qquad i,j = 1,2.$$

For the operation of time-reflection, clearly $\pi_t(\tau)u_k(x,t) = u_k(x,-t)$ solves the wave equation with Cauchy data $(f,-g)$. Thus, the analogue of (5.1) reads

$$\begin{bmatrix} u_k(x,-t) \\ -(\partial_t u_k)(x,-t) \end{bmatrix} = P_{k,t} \circledast \begin{bmatrix} f \\ -g \end{bmatrix},$$

which we may rewrite as

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} U_k(x,-t) = P_{k,t} \circledast \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} U_k(x,0). \tag{5.2}$$

On the other hand, from (5.1), it follows that $U_k(x,-t) = P_{k,-t} \circledast U_k(x,0)$. Comparing this with the above identity, we obtain

$$P^{ij}_{k,-t} = (-1)^{i-j} P^{ij}_{k,t} \qquad \text{for } i,j = 1,2,$$

which implies

$$\pi_t^*(\tau) P^{ij}_k = (-1)^{i-j} P^{ij}_k \qquad \text{for } i,j = 1,2.$$

From the time-reflection action on the propagators, it is clear that time is reversible, except for a minus sign that may appear when the second Cauchy datum g or its Fourier transform are involved. So the past is determined by the present as well as the future.

Now,we will investigate the symmetries of the propagators under a dilation operator. As a result, we can determine the homogeneity of the distribution P^{ij}_k, with $i,j = 1,2$. For a function $f = f(x,t)$ on $\mathbb{R}^{n+1}$ and $\lambda > 0$, let

$$S^x_\lambda f(x,t) = f(\lambda^2 x,t),$$
$$S^t_\lambda f(x,t) = f(x,\lambda t),$$

where the superscript denotes the relevant variable. Set

$$S_\lambda = S^x_\lambda \circ S^t_\lambda.$$

where the superscript denotes the relevant variable. By duality, the operators S_λ, S^x_λ and S^t_λ act on distributions in the standard way.

Observe that if $u_k(x,t)$ satisfies the wave equation with initial condition (f,g) then $S_\lambda u_k(x,t)$ solves the wave equation with initial data $\left(S_\lambda^x f(x), \lambda S_\lambda^x g(x)\right)$. Thus,

$$S_\lambda U_k(x,t) = P_{k,t} \circledast \begin{pmatrix} S_\lambda^x f \\ \lambda S_\lambda^x g \end{pmatrix} \tag{5.3}$$

Further,

$$\begin{aligned}
S_\lambda U_k(x,t) &= S_\lambda \begin{pmatrix} u_k(x,t) \\ v_k(x,t) \end{pmatrix} = S_\lambda \begin{pmatrix} u_k(x,t) \\ \partial_t u_k(x,t) \end{pmatrix} \\
&= \begin{pmatrix} S_\lambda u_k(x,t) \\ \partial_t \left\{ S_\lambda u_k(x,t) \right\} \end{pmatrix} = \begin{pmatrix} u_k(\lambda^2 x, \lambda t) \\ \lambda \left\{ \partial_t u_k \right\} (\lambda^2 x, \lambda t) \end{pmatrix} \\
&= \begin{pmatrix} u_k \\ \lambda \partial_t u_k \end{pmatrix} (\lambda^2 x, \lambda t) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} u_k \\ \partial_t u_k \end{pmatrix} (\lambda^2 x, \lambda t) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \left\{ P_{k,\lambda t} \circledast \begin{pmatrix} f \\ g \end{pmatrix} \right\} (\lambda^2 x) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} S_\lambda^x \left\{ P_{k,\lambda t} \circledast \begin{pmatrix} f \\ g \end{pmatrix} \right\} (x) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \left\{ S_\lambda^x P_{k,\lambda t} \circledast \begin{pmatrix} S_\lambda^x f \\ S_\lambda^x g \end{pmatrix} \right\} (x) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \left\{ S_\lambda^x P_{k,\lambda t} \circledast \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} S_\lambda^x f \\ \lambda S_\lambda^x g \end{pmatrix} \right\} (x).
\end{aligned}$$

Hence, by comparing Equation (5.3) with above we deduce that

$$S_\lambda^x P_{k,\lambda t}^{ij} = \lambda^{j-i} P_{k,t}^{ij}, \qquad i,j = 1,2.$$

This result will allow us to find the degree of the homogeneity of the distributions P_k^{ij}, with $i,j = 1,2$. For $\psi_1 \in \mathcal{S}(\mathbb{R}^n)$ and $\psi_2 \in \mathcal{S}(\mathbb{R})$, by duality, we have

$$S_\lambda \left(P_k^{ij} \right) (\psi_1 \otimes \psi_2) \quad = \quad P_k^{ij} \left(S_{\lambda^{-1}}^x (\psi_1) \otimes S_{\lambda^{-1}}^t (\psi_2) \right)$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} P^{ij}_{k,t}\left(S^x_{\lambda^{-1}}(\psi_1)\right) S^t_{\lambda^{-1}}(\psi_2)(t)dt \\
&= \lambda \int_{\mathbb{R}} P^{ij}_{k,\lambda t}\left(S^x_{\lambda^{-1}}(\psi_1)\right) \psi_2(t)dt \\
&= \lambda \int_{\mathbb{R}} S^x_{\lambda}\left(P^{ij}_{k,\lambda t}(\psi_1)\right) \psi_2(t)dt \\
&= \lambda^{1+j-i} \int_{\mathbb{R}} P^{ij}_{k,t}(\psi_1)\,\psi_2(t)dt \\
&= \lambda^{1+j-i} P^{ij}_k(\psi_1 \otimes \psi_2).
\end{aligned}
$$

We summarize the above computations.

Proposition 5.3. *Let $k \in K^+$ and $n \geq 1$.*

(i) *The distribution P^{ij}_k satisfies the deformed wave equation, i.e.*

$$\left(\|x\|\Delta_k - \frac{1}{2}\partial_{tt}\right) P^{ij}_k = 0, \quad i,j = 1,2.$$

(ii) *If $h \in G$ and τ denotes the operation of time-reflection, then*

$$\pi^*_x(h)P^{ij}_k = P^{ij}_k, \qquad \pi^*_t(\tau)P^{ij}_k = (-1)^{i-j}P^{ij}_k, \qquad i,j = 1,2.$$

(iii) *For every $\lambda > 0$, we have*

$$S_\lambda P^{ij}_k = \lambda^{1+j-i} P^{ij}_k, \qquad i,j = 1,2.$$

Next we shall describe the structure of a representation of the universal covering group $\widetilde{SL(2,\mathbb{R})}$ of $SL(2,\mathbb{R})$ on $\mathcal{S}(\mathbb{R}^{n+1})$. This structure, together with the above Propositions, allows to prove that our Cauchy problem satisfies the strict Huygens principle, under a condition involving n and k. We adapt the method of R. Howe for the classical wave equation, i.e. when $k \equiv 0$ (cf. [33]) and Ben Said and Ørsted for the wave equation associated with the Dunkl Laplacian operator [32].

Recall that the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is of dimension 3. We take a basis for the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ as

$$\mathbf{e}^+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{e}^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{h} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The triple $\{\mathbf{e}^+, \mathbf{e}^-, \mathbf{h}\}$ satisfies the commutation relations

$$[\mathbf{e}^+, \mathbf{e}^-] = \mathbf{h}, \qquad [\mathbf{h}, \mathbf{e}^+] = 2\mathbf{e}^+, \qquad [\mathbf{h}, \mathbf{e}^-] = -2\mathbf{e}^-, \tag{5.4}$$

where $[A,B] := AB - BA$. An $\mathfrak{sl}_2$ triple is a triple of non-zero elements in a Lie algebra satisfying the same relation with (5.4).

We recall from Chapter 2 that Δ_k is the Dunkl Laplacian associated with a multiplicity function k on the root system, and that

$$\langle k \rangle = \frac{1}{2} \sum_{\alpha \in \mathcal{R}} k_\alpha.$$

Choose $x_1, x_2, \ldots, x_n$ as the usual system of coordinates on $\mathbb{R}^n$. We introduce the following differential-difference operators on $\mathbb{R}^n$:

$$\mathbb{E}_k^+ := i\|x\|, \qquad \mathbb{E}_k^- := i\|x\|\Delta_k, \qquad \mathbb{H}_k := n + 2\langle k \rangle - 1 + 2\sum_{i=1}^n x_i \partial_i. \tag{5.5}$$

Theorem 5.2. *The operators $\mathbb{E}_k^+$, $\mathbb{E}_k^-$ and $\mathbb{H}_k$ form an $\mathfrak{sl}_2$ triple for any multiplicity function k.*

The proof of the above statement needs the following preparation lemmas.

Lemma 5.3. *We write the Euler operator as*

$$E := \sum_{j=1}^n x_j \partial_j. \tag{5.6}$$

1) The Dunkl Laplacian Δ_k is of degree -2, namely,

$$[E,\Delta_k] = -2\Delta_k \tag{5.7}$$

2) We have

$$\sum_{j=1}^{n} \left(x_j T_j(k) + T_j(k) x_j\right) = n + 2\langle k\rangle + 2E. \tag{5.8}$$

Proof.

1) This statement goes back to G. Heckman [44, Theorem 3.3]

2) Let us write the Dunkl operator as $T_j f = \partial_j f + \sum_{\alpha\in R^+} k_\alpha \langle \alpha, e_j\rangle \triangle_\alpha(f)$ where $\triangle_\alpha(f) = \dfrac{f(x)-f(r_\alpha x)}{\langle \alpha, x\rangle}$. Then,

$$\begin{aligned}
\sum_{j=1}^{n}(x_jT_j+T_jx_j)f &= 2\sum_{j=1}^{n} x_jT_jf + \sum_{j=1}^{n}(T_jx_j - x_jT_j)f \\
&= 2\sum_{j=1}^{n} x_jT_jf + \sum_{j=1}^{n}\partial_j(x_jf) + \sum_{j=1}^{n}\sum_{\alpha\in R^+} k_\alpha\langle\alpha,e_j\rangle\triangle_\alpha(x_jf) \\
&\quad -\sum_{j=1}^{n} x_j\partial_j(f) - \sum_{j=1}^{n}\sum_{\alpha\in R^+} k_\alpha\langle\alpha,e_j\rangle x_j\triangle_\alpha(f) \\
&= 2\sum_{j=1}^{n} x_jT_jf + \sum_{j=1}^{n} f + \sum_{j=1}^{n} x_j\partial_j(f) - \sum_{j=1}^{n} x_j\partial_j(f) \\
&\quad + \sum_{\alpha\in R^+} k_\alpha\langle\alpha,e_j\rangle\,[\triangle_\alpha, x_j]\,f \\
&= 2\sum_{j=1}^{n} x_jT_jf + nf + \sum_{j=1}^{n}\sum_{\alpha\in R^+} k_\alpha\langle\alpha,e_j\rangle\,[\triangle_\alpha, x_j]\,f.
\end{aligned}$$

Now, by G. Heckman, we know that

$[R_\alpha, x_j] = \langle\alpha, e_j\rangle r_\alpha$. It follows,

$$\sum_{j=1}^{n}(x_jT_j+T_jx_j)f = 2\sum_{j=1}^{n} x_jT_jf + nf + \sum_{j=1}^{n}\sum_{\alpha\in R^+} k_\alpha\langle\alpha,e_j\rangle\langle\alpha,e_j\rangle f(r_\alpha x)$$

$$
\begin{aligned}
&= 2\sum_{j=1}^{n} x_j T_j f + nf + 2\sum_{\alpha\in R^+} k_\alpha r_\alpha \\
&= 2\sum_{j=1}^{n} x_j \partial_j + 2\sum_{j=1}^{n} x_j \sum_{\alpha\in R^+} \langle \alpha, e_j\rangle R_\alpha f + nf + 2\sum_{\alpha\in R^+} k_\alpha f(r_\alpha x) \\
&= 2\sum_{j=1}^{n} x_j \partial_j + 2\sum_{\alpha\in R^+} k_\alpha \triangle_\alpha f \sum_j \langle x, e_j\rangle\langle \alpha, e_j\rangle + nf + 2\sum_{\alpha\in R^+} k_\alpha f(r_\alpha x) \\
&= 2\sum_{j=1}^{n} x_j \partial_j + 2\sum_{\alpha\in R^+} k_\alpha \triangle_\alpha f \langle \alpha, x\rangle + nf + 2\sum_{\alpha\in R^+} k_\alpha f(r_\alpha x) \\
&= 2\sum_{j=1}^{n} x_j \partial_j + 2\sum_{\alpha\in R^+} k_\alpha \left(f(x) - f(r_\alpha x)\right) + nf + 2\sum_{\alpha\in R^+} k_\alpha f(r_\alpha x) \\
&= 2\sum_{j=1}^{n} x_j \partial_j + 2\sum_{\alpha\in R^+} k_\alpha f + nf \\
&= 2\sum_{j=1}^{n} x_j \partial_j + 2\langle k\rangle f + nf.
\end{aligned}
$$

Therefore

$$\sum_{j=1}^{n} (x_j T_j + T_j x_j) f = (n + 2\langle k\rangle + 2\sum_{j=1}^{n} x_j \partial_j) f$$

Lemma 5.4. *Suppose $\psi(r)$ is a C^∞ function of one variable. Then we have*

$$[\Delta_k, \psi(\|x\|)] = \psi''(\|x\|) + \|x\|^{-1}\psi'(\|x\|)\left((n + 2\langle k\rangle - 1) + 2E\right).$$

Proof. *Take an arbitrary C^∞ function f on $\mathbb{R}^n$. Assume that g is a G-invariant smooth function. By Chapter 2 we have*

$$
\begin{aligned}
&T_j(k) g = \partial_j g, \qquad (5.9)\\
&T_j(k)(fg) = (T_j(k) f) g + f(\partial_j g),
\end{aligned}
$$

Take an arbitrary C^∞ function ψ on $\mathbb{R}^+$. Clearly we have

$$T_j[\psi(\|x\|)] = x_j \|x\|^{-1}\psi'(\|x\|).$$

Further,

$$\begin{aligned} T_j\left[f(x)\psi(\|x\|)\right] &= T_j(f(x))\psi(\|x\|) + f(x)T_j\left(\psi(\|x\|)\right) \\ &= T_j(f)\psi(\|x\|) + f(x)x_j\|x\|^{-1}\psi'(\|x\|) \end{aligned}$$

Apply the Dunkl operator T_j again to the above we get

$$\begin{aligned} & T_j^2\left[f(x)\psi(\|x\|)\right] \\ = \;& T_j^2(f)\psi(\|x\|) + T_j(f)T_j\psi(\|x\|) + T_j\left\{f(x)x_j\|x\|^{-1}\;\psi'(\|x\|)\right\} \\ = \;& T_j^2(f)\psi(\|x\|) + T_j(f)T_j\psi(\|x\|) + T_j(x_jf(x))\|x\|^{-1}\psi'(\|x\|) \\ & + x_jf(x)T_j\left(\|x\|^{-1}\psi'(x)\right) \\ = \;& T_j^2(f)\psi(\|x\|) + \|x\|^{-1}\psi'(\|x\|)x_jT_j(f) + \|x\|^{-1}\psi'(\|x\|)T_j(x_jf) \\ & + x_jf(x)\partial_j\left\{\|x\|^{-1}\psi'(\|x\|)\right\} \end{aligned}$$

Taking the summation over j, we arrive at

$$\begin{aligned} & \Delta_k\left(f(x)\psi(\|x\|)\right) \\ = \;& \left(\Delta_kf(x)\right)\psi(\|x\|) + \|x\|^{-1}\psi'(\|x\|)\left(2E+n+2\langle k\rangle\right)f(x) \\ & + f(x)E\left(\|x\|^{-1}\psi'(\|x\|)\right). \end{aligned}$$

To finish the proof, one may use the following observation: in the polar coordinate $x = r\omega$, the Euler operator E amounts to $r(\partial/\partial r)$, and

$$r(\partial/\partial r)\left(r^{-1}\psi'(r)\right) = -r^{-1}\psi'(r) + \psi''(r).$$

Denote by $\mathcal{H}_m(\mathbb{R}^n)$ the space of k-harmonic homogeneous polynomials of degree m. We set

$$\lambda_{k,m} := 2m + 2\langle k\rangle + n - 2. \tag{5.10}$$

We begin with the following lemma.

Lemma 5.5. *For all $\psi \in C^\infty(\mathbb{R}_+)$ and $p \in \mathcal{H}_m(\mathbb{R}^n)$, we have*

$$\mathbb{H}_k\Big(p(x)\psi(\|x\|)\Big) = \Big\{(\lambda_{k,m}+1)\psi(\|x\|) + 2\|x\|\psi'(\|x\|)\Big\}p(x), \tag{5.11}$$

$$\Delta_k\Big(p(x)\psi(\|x\|)\Big) = \Big\{(\lambda_{k,m}+1)\|x\|^{-1}\psi'(\|x\|) + \psi''(\|x\|)\Big\}p(x). \tag{5.12}$$

Proof. *The first statement is straightforward because the Euler operator E is of the form $r\frac{\partial}{\partial r}$ in the polar coordinates $x = r\omega$. To see the second statement, we apply Lemma 5.4. to $p(x)$. Now using the fact that $Ep = mp$ and since p is a harmonic polynomial, i.e. $\Delta_k p = 0$, then we get the desired formula.*

Proof. *[Proof of Theorem 5.2] It is clear that the operator $\mathbb{E}_k^+$ is homogeneous of degree 1, while, by Lemma 5.5., the operator $\mathbb{E}_k^-$ is homogeneous of degree -1. Let $E = \sum_{j=1}^N x_j\partial_j$ be the Euler operator. Since $\mathbb{H}_k$ is of the form $2E$ + a constant, the identity $\left[\mathbb{H}_k, \mathbb{E}_k^\pm\right] = \pm 2\mathbb{E}_k^\pm$ is now clear.*

For the brackets $\left[\mathbb{E}_k^+, \mathbb{E}_k^-\right] = \mathbb{H}_k$, we apply Lemma 5.5. to the function $\psi(r) = r$ in order to get

$$\Delta_k \circ \|x\| - \|x\|\Delta_k = (n + 2\langle k\rangle - 1)\|x\|^{-1} + 2\|x\|^{-1}E.$$

Composing with the multiplication operator $\|x\|$, we have

$$\|x\|\Delta_k \circ \|x\| - \|x\|^2\Delta_k = (n + 2\langle k\rangle - 1) + 2E.$$

In view of the definition of the operator $\mathbb{H}_k$, this means that $\left[\mathbb{E}_k^+, \mathbb{E}_k^-\right] = \mathbb{H}_k$. Hence, Theorem 5.2. is proved.

The differential-difference operators $\mathbb{E}_k^+$, $\mathbb{E}_k^-$ and $\mathbb{H}_k$ stabilize $C^\infty(\mathbb{R}^n\backslash\{0\})$, the space $\mathbb{R}^n\backslash\{0\}$ of smooth functions. Thus, we can define an $\mathbb{R}$-linear map

$$\omega_k : \mathfrak{sl}(2,\mathbb{R}) \to \operatorname{End}\left(C^\infty\left(\mathbb{R}^n\backslash\{0\}\right)\right)$$

by setting

$$\omega_k(\mathbf{h}) = \mathbb{H}_k, \quad \omega_k\left(\mathbf{e}^+\right) = \mathbb{E}_k^+, \quad \omega_k\left(\mathbf{e}^-\right) = \mathbb{E}_k^-.$$

Then Theorem 5.2. implies that ω_k is a representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$.

We use the letter L to denote the left regular representation of the Coxeter group G on $C^\infty(\mathbb{R}^n \setminus \{0\})$.

Lemma 5.6. *The two actions L of the Coxeter group G and ω_k of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ commute.*

Proof. *Obviously $L(h)$ commutes with the multiplication operator $\mathbb{E}_k^+ = i\|x\|$. It is well known that $L(h) \circ T_{e_j} \circ L(h) = T_{he_j}$ for all $h \in G$. Therefore, $L(h)$ commutes with the Dunkl Laplacian. Hence, it commutes also with $\mathbb{E}_k^-$. Finally, the commutation relation $[\mathbb{E}_k^+, \mathbb{E}_k^-] = \mathbb{H}_k$ implies $L(h) \circ \mathbb{H}_k = \mathbb{H}_k \circ L(h)$ for all $h \in G$.*

For a complex number λ such that $\operatorname{Rel}\lambda > -1$, we denote by $L_\ell^{(\lambda)}$ the Laguerre polynomial,

$$L_\ell^{(\lambda)}(t) := \frac{(\lambda+1)_\ell}{\ell!} \sum_{j=0}^{\ell} \frac{(-\ell)_j}{(\lambda+1)_j} \frac{t^j}{j!} = \sum_{j=0}^{\ell} \frac{(-1)^j \Gamma(\lambda+\ell+1)}{(\ell-j)!\Gamma(\lambda+j+1)} \frac{t^j}{j!}.$$

Here, $(a)_m := a(a+1)\cdots(a+m-1)$ is the Pochhammer symbol. Below we will state some basic properties of Laguerre polynomials.

Theorem 5.3. *Assume* $\operatorname{Rel}\lambda > -1$.

1) $L_\ell^{(\lambda)}(t)$ is the unique polynomial of degree ℓ satisfying the Laguerre differential equation

$$\Big(t\frac{d^2}{dt^2} + (\lambda+1-t)\frac{d}{dt} + \ell\Big) L_\ell^{(\lambda)}(t) = 0.$$

2) The following recurrence formulas hold:

$$(\ell + t\frac{d}{dt} - t + \lambda + 1)L_\ell^{(\lambda)}(t) = (\ell+1)L_{\ell+1}^{(\lambda)}(t),$$
$$(\ell - t\frac{d}{dt})L_\ell^{(\lambda)}(t) = (\ell+\lambda)L_{\ell-1}^{(\lambda)}(t).$$

3) The following orthogonality property holds:

$$\int_0^\infty L_\ell^{(\lambda)}(t) L_s^{(\lambda)}(t) t^\lambda e^{-t} dt = \delta_{\ell s} \frac{\Gamma(\lambda+\ell+1)}{\Gamma(\ell+1)}.$$

4) $\{L_\ell^{(\lambda)}(t) : \ell \in \mathbb{N}\}$ form an orthogonal basis in $L^2(\mathbb{R}_+, t^\lambda e^{-t} dt)$ if λ is real and $\lambda > -1$.

For $\ell, m \in \mathbb{N}$ and $p \in \mathcal{H}_m(\mathbb{R}^n)$, we introduce the following functions on $\mathbb{R}^n$,

$$\Phi_\ell(p,x) := p(x) L_\ell^{(\lambda_{k,m})}\Big(2\|x\|\Big) \exp\Big(-\|x\|\Big). \tag{5.13}$$

Here, $\lambda_{k,m} = 2m + 2\langle k\rangle + n - 2$, and $L_\ell^{(\lambda)}(t)$ is the Laguerre polynomial. Hence, for $x = r\omega \in \mathbb{R}^n$ ($r > 0$, $\omega \in S^{n-1}$), we have

$$\Phi_\ell(p,x) = p(\omega) r^m L_\ell^{(\lambda_{k,m})}\Big(2r\Big) \exp\Big(-r\Big).$$

We define the following vector space of functions on $\mathbb{R}^n$ by

$$D_k(\mathbb{R}^n) := \mathbb{C}\text{-span}\Big\{\Phi_\ell(p,\cdot) \mid \ell \in \mathbb{N}, m \in \mathbb{N}, p \in \mathcal{H}_m(\mathbb{R}^n)\Big\}. \tag{5.14}$$

Proposition 5.4. *Assume that $2\langle k\rangle + n - 1 > 0$. For $\ell, s, m, n \in \mathbb{N}$, $p \in \mathcal{H}_m(\mathbb{R}^n)$ and $q \in \mathcal{H}_n(\mathbb{R}^n)$.*

1) $\Phi_\ell(p,x) \in C(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, \vartheta_k(x)dx)$.

2) We have the following orthogonality

$$\int_{\mathbb{R}^n} \Phi_\ell(p,x)\overline{\Phi_s(q,x)}\vartheta_k(x)dx = \delta_{m,n}\delta_{\ell,s}\frac{\Gamma(\lambda_{k,m}+\ell+1)}{2^{1+\lambda_{k,m}}\Gamma(\ell+1)}\int_{S^{n-1}} p(\omega)\overline{q(\omega)}\vartheta_k(\omega)d\sigma(\omega).$$

3) $D_k(\mathbb{R}^n)$ is a dense subspace of $L^2(\mathbb{R}^n, \vartheta_k(x)dx)$.

Proof. *The function $x \mapsto \Phi_\ell(p,x)$ is continuous at $x = 0$. Therefore, it is a continuous function on $x \in \mathbb{R}^n$ of exponential decay. On the other hand, in polar coordiantes the measure $\vartheta_k(x)dx$ take the form $r^{2\langle k\rangle+n-2}\vartheta_k(\omega)d\omega$, which is locally integrable under our assumptions on the multiplicity function k. Therefore, $\Phi_\ell(p,x) \in L^2(\mathbb{R}^n, \vartheta_k(x)dx)$. Hence the first statement is proved.*

To see the second and third statements, we use the polar coordinates $x = r\omega$ to rewrite the left-hand side of the integral as

$$\Big(\int_0^\infty L_\ell^{(\lambda_{k,m})}(2r)L_s^{(\lambda_{k,n})}(2r)\exp\Big(-2r\Big)r^{m+n+2\langle k\rangle+n-2}dr\Big)\Big(\int_{S^{n-1}} p(\omega)\overline{q(\omega)}\vartheta_k(\omega)d\sigma(\omega)\Big).$$

Since Δ_k-harmonic polynomials of different degrees are orthogonal to each other, the integration over S^{n-1} vanishes if $m \neq n$.

Suppose that $m = n$. By changing the variable $t := 2r$, we see that the first integration amounts to

$$\frac{1}{2^{1+\lambda_{k,m}}}\int_0^\infty L_\ell^{(\lambda_{k,m})}(t)L_s^{(\lambda_{k,m})}(t)\,t^{\lambda_{k,m}}e^{-t}dt. \tag{5.15}$$

By the orthogonality relation for the Laguerre polynomials, we get

$$\frac{1}{2^{1+\lambda_{k,m}}}\int_0^\infty L_\ell^{(\lambda_{k,m})}(t)L_s^{(\lambda_{k,m})}(t)\,t^{\lambda_{k,m}}e^{-t}dt = \delta_{\ell s}\frac{\Gamma(\lambda_{k,m}+\ell+1)}{2^{1+\lambda_{k,m}}\Gamma(\ell+1)}.$$

Hence, the second statement is proved. The third statement follows from the completeness of the Laguerre polynomials.

We pin down the following statement which is already implicit in the proof of the above Proposition.

Proposition 5.5. *For fixed $m \in \mathbb{N}$ and a multiplicity function k such that $2m+2\langle k\rangle+n-1>0$, we set*

$$f_{\ell,m}(r) := \Big(\frac{2^{\lambda_{k,m}+1}\Gamma(\ell+1)}{\Gamma(\lambda_{k,m}+\ell+1)}\Big)^{1/2} r^m L_\ell^{(\lambda_{k,m})}\Big(2r\Big)\exp(-r). \tag{5.16}$$

Then $\{f_{\ell,m}(r) : \ell \in \mathbb{N}\}$ forms an orthonormal basis in $L^2(\mathbb{R}_+, r^{2\langle k\rangle+n-2}dr)$.

For each $m \in \mathbb{N}$, we take an orthonormal basis $\{h_{j,m}\}_{j\in J_m}$ of the space $\mathcal{H}_m(\mathbb{R}^n)|_{S^{n-1}}$. Here J_m is a finite set of integers. A basis of $\mathcal{H}_m(\mathbb{R}^n)$ is constructed in [10, Corollary 5.1.13]. Proposition 5.4. immediately yields the following statement.

Corollary 5.1. *For $\ell, m \in \mathbb{N}$ and $j \in J_m$, we set*

$$\Phi_{\ell,m,j}(x) := h_{j,m}\Big(\frac{x}{\|x\|}\Big)f_{\ell,m}(\|x\|).$$

Then, the set $\left\{\Phi_{\ell,m,j} \mid \ell \in \mathbb{N}, m \in \mathbb{N}, j \in J_m\right\}$ *forms an orthonormal basis of* $L^2\big(\mathbb{R}^d, \vartheta_k(x)dx\big)$.

Let (ϖ, X) be a representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. A non-zero vector $v \in X$ is a lowest weight vector of weight $\mu \in \mathbb{C}$ if v satisfies

$$\varpi(\mathbf{e}^-)v = 0, \quad \text{and} \quad \varpi(\mathbf{h})v = \mu v.$$

We say (ϖ, X) is a lowest weight module of weight μ if X is generated by such v. For each $\lambda \in \mathbb{C}$, there exists a unique irreducible lowest weight $\mathfrak{sl}(2,\mathbb{R})$-module, to be denoted by $\varpi(\lambda)$, of weight $\lambda + 1$.

Theorem 5.4. *(See [16]) Suppose k is a non-negative root multiplicity function satisfying the inequality* $2\langle k\rangle + n - 1 > 0$. *The representation* $(\omega_k, D_k(\mathbb{R}^n))$ *is decomposed into the direct sum as follows:*

$$D_k(\mathbb{R}^n) \simeq \bigoplus_{m=0}^{\infty} \mathcal{H}_m(\mathbb{R}^n)\Big|_{S^{n-1}} \otimes \varpi(\lambda_{k,m}). \tag{5.17}$$

Here, $\lambda_{k,m} = 2m + 2\langle k\rangle + n - 2$. *The Coxeter group G acts on the first factor, and the Lie algebra* $\mathfrak{sl}(2,\mathbb{R})$ *acts on the second factor of each summand in* (5.17).

Let

$$\begin{aligned}
\mathbb{E}_{n,1} &:= i\Big(\|x\| - \frac{1}{2}t^2\Big), \\
\mathbb{F}_{n,1} &:= i\Big(\|x\|\Delta_k - \frac{1}{2}\partial_{tt}\Big), \\
\mathbb{H}_{n,1} &:= n + 2\langle k\rangle - \frac{1}{2} + 2\sum_{\ell=1}^{n} x_\ell \partial_\ell + t\partial_t.
\end{aligned}$$

Using the fact that $\mathbb{E}_k^+$, $\mathbb{E}_k^-$ and $\mathbb{H}_k$ form an $\mathfrak{sl}_2$ triple, we deduce easily the following commutation relations hold

$$[\mathbb{E}_{n,1}, \mathbb{F}_{n,1}] = \mathbb{H}_{n,1}, \quad [\mathbb{H}_{n,1}, \mathbb{E}_{n,1}] = 2\mathbb{E}_{n,1}, \quad [\mathbb{H}_{n,1}, \mathbb{F}_{n,1}] = -2\mathbb{F}_{n,1}. \tag{5.18}$$

These are the commutation relations of a standard basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Equation (5.18) gives rise to a representation $\omega_{k,n+1}$ of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ on the Schwartz space

$\mathcal{S}(\mathbb{R}^{n+1})$ by setting

$$\omega_{k,n+1}(\mathbf{h}) = \mathbb{H}_{n,1}, \qquad \omega_{k,n+1}(\mathbf{e}^+) = \mathbb{E}_{n,1}, \qquad \omega_{k,n+1}(\mathbf{e}^-) = \mathbb{F}_{n,1}. \tag{5.19}$$

where $\{\mathbf{e}^+, \mathbf{e}^-, \mathbf{h}\}$ is a basis for the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$,

$$\mathbf{e}^+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{e}^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{h} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall that the solution to the wave equation

$$\partial_{tt} u_k(x,t) - 2\|x\|\Delta_k u_k(x,t) = 0$$

is given by

$$u_k(x,t) = \left(P_{k,t}^{11} \circledast f\right)(x) + \left(P_{k,t}^{12} \circledast g\right)(x).$$

Recall also that the Huygens' principle is equivalent to the fact that the propagators P_k^{11} and P_k^{12} are supported on the set z

Proposition 5.6. *A distribution T is supported on the set*

$$\left\{(x,y) \in \mathbb{R}^{p+q} \mid r_{p,q}^2 := \sum_{i=1}^{p} x_i^2 - \sum_{i=1}^{q} y_i^2 = 0\right\}$$

if and only if we can find an integer $m \geq 1$ such that

$$T\left(\left(r_{p,q}^2\right)^m f\right) = 0, \quad \forall f \in C_c^\infty\left(\mathbb{R}^{p+q}\right).$$

Since $\mathcal{C}$ is the locus of zeros of $\|x\| - \frac{1}{2}t^2$, then, P_k^{11} and P_k^{12} are supported on $\mathcal{C}$ if and only if

$$\mathbb{E}_{n,1}^m \cdot P_k^{ij} = 0 \tag{5.20}$$

for some positive integer m (see [33]). Moreover, in the light of Proposition 5.3. we deduce that P_k^{ij} is an eigen-distribution for $\mathbb{H}_{n,1}$. The above facts amount to saying the distribution P_k^{ij} generates a finite-dimensional $\omega_{k,n+1}^*$ for $\mathfrak{sl}(2,\mathbb{R})$ on $\mathcal{S}'(\mathbb{R}^{n+1})$. Thus, the qualitative part of the Huygens principle holds.

Theorem 5.5. *The Huygens principle holds if and only if P_k^{11} and P_k^{12} are supported on the set $\mathcal{C}$ if and only if P_k^{11} and P_k^{12} generate finite-dimensional representations $\omega_{k,n+1}^*$ for $\mathfrak{sl}(2,\mathbb{R})$ on $\mathcal{S}'(\mathbb{R}^{n+1})$.*

Theorem 5.6. *The Huygens principle cannot hold when*

$$2\langle k\rangle - \frac{1}{2} \notin \mathbb{Z}.$$

Proof. *The spectrum of the element $\mathbf{k} := i(\mathbf{e}^- - \mathbf{e}^+)$ acting on $\mathcal{S}'(\mathbb{R}^{n+1})$ via the dual representation $\omega_{k,n+1}^*$ is $2\mathbb{Z}^+ + 2\langle k\rangle + n - \frac{1}{2}$, whereas, it is well known, the spectrum of $\mathbf{k}$ in finite-dimensional representations of $\mathfrak{sl}(2,\mathbb{R})$ is contained in $\mathbb{Z}$.*

The above theorem leaves the likelihood that the wave equation may satisfies Huygens' principle when $2\langle k\rangle - \frac{1}{2} \in \mathbb{Z}$.

Now, using Proposition 5.3., we get

$$\Big\{2\sum_{\ell=1}^{n} x_\ell\partial_\ell + t\partial_t\Big\}P_k^{ij} = (1+j-i)P_k^{ij}.$$

Therefore

$$\mathbb{H}_{n,1}P_k^{ij} = -\left(n+2\langle k\rangle - \frac{1}{2} + i - j - 1\right)P_k^{ij}, \qquad i,j = 1,2.$$

That is P_k^{ij} is an eigen-distributation for $\mathbb{H}_{n,1}$ with eigenvalue $-(n+2\langle k\rangle - \frac{3}{2} + i - j)$. Thus, if we assume $n+2\langle k\rangle - \frac{3}{2} + i - j \in \mathbb{N}$, with $i,j = 1,2$, and keeping in mind that

$$\mathbb{F}_{n,1}\cdot P_k^{ij} = 0,$$

we can conclude that each distribution P_k^{ij}, with $i,j = 1,2$, generates a finite-dimensional $\omega_{k,n+1}^*(\mathfrak{sl}(2,\mathbb{R}))$ on $\mathcal{S}'(\mathbb{R}^{n+1})$ of highest weight $n+2\langle k\rangle - \frac{3}{2} + i - j$. We now summarize all the above computations and discussions.

Proposition 5.7. *Under the assumption*

$$n+2\langle k\rangle - \frac{3}{2} + i - j \in \mathbb{N}, \tag{5.21}$$

the tempered distribution P_k^{ij} *generates an* $\mathfrak{sl}(2,\mathbb{R})$*-module of dimension*

$$d_{i,j}(k) = n + 2\langle k\rangle - \frac{3}{2} + i - j + 1, \qquad i,j = 1,2.$$

By taking into account the condition (5.21) for both P_k^{11} and P_k^{12}, we obtain:

Theorem 5.7. (Strict Huygens' Principle) *Assume that* $k \in K^+$ *and* $n \geq 1$. *The unique solution* $u_k(x,t)$ *to the Cauchy problem*

$$\partial_{tt} u_k(x,t) - 2\|x\|\Delta_k u_k(x,t) = 0, \qquad u(x,0) = f(x), \qquad \partial_t u(x,0) = g(x)$$

depends only on the values of $\tau_x(k)f$ *and* $\tau_x(k)g$ *(and their derivatives) for* $\|x\| = \frac{1}{2}t^2$ *if and only if*

$$n + 2\langle k\rangle - \frac{5}{2} \in \mathbb{N}.$$

The above theorem finishes the question asked in this chapter.

Chapter 6: The Integrability of the Representation ω_k

6.1 Introduction

In the previous chapter we constructed the representation ω_k of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. This infinitesimal representation ω_k was used to investigate the validity of the Huygens principal for the wave equation $\partial_{tt}u_k - 2\|x\|\Delta_k u_k = 0$ with the initial data $u_k(x,0) = f(x)$ and $\partial_t u_k(x,0) = g(x)$. In this chapter we will answer to the question of the integrability of the representation ω_k to give raise to a representation of a simply connected Lie group with Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Our approach uses a famous theorem due to E. Nelson [34]. Note that the integrability fact is not obvious, since in infinite dimensions, the existence of a group representation is not guaranteed from the existence of a Lie algebra representation.

6.2 The Integrability of the Representation ω_k

Our exposition will center around the three dimensional Lie algebra

$$\mathfrak{sl}(2,\mathbb{R}) = \left\{ X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}; a,b,c \in \mathbb{R} \right\}.$$

A basis for $\mathfrak{sl}(2,\mathbb{R})$ can be chosen as $\{\mathbf{h}, \mathbf{e}^+, \mathbf{e}^-\}$, where

$$\mathbf{h} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{e}^+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{e}^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with the commutation relations

$$[\mathbf{h}, \mathbf{e}^+] = 2\mathbf{e}^+, \quad [\mathbf{h}, \mathbf{e}^-] = -2\mathbf{e}^-, \quad [\mathbf{e}^+, \mathbf{e}^-] = \mathbf{h}.$$

Of course, the above three equations characterize the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ completely.

Recall from the previous chapter the following operators on $\mathbb{R}^n$:

$$\mathbb{E}_k^+ := i\|x\|, \qquad \mathbb{E}_k^- := i\|x\|\Delta_k, \qquad \mathbb{H}_k := n + 2\langle k\rangle - 1 + 2\sum_{i=1}^{n} x_i \partial_{x_i},$$

where $\langle k \rangle = \sum_{\alpha \in \mathcal{R}^+} k_\alpha$. We proved in the previous chapter that the operators $\mathbb{E}_k^+$, $\mathbb{E}_k^-$ and $\mathbb{H}_k$ form an $\mathfrak{sl}_2$-triple. For $k \equiv 0$, $\{\mathbb{E}_0^+, \mathbb{E}_0^-, \mathbb{H}_0\}$ is the $\mathfrak{sl}_2$ triple introduced in Kobayashi and Mano [17, 18] where the authors studied the L^2-model of the minimal representation of the double covering group of $SO_0(n+1,2)$. (To be more precise, the formulas in [18] are given for the $\mathfrak{sl}_2$-triple for $\{2\mathbb{E}_0^+, \frac{1}{2}\mathbb{E}_0^-, \mathbb{H}_0\}$ in our notation.)

Finally, recall the representation ω_k of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ defined by

$$\omega_k(\mathbf{h}) = \mathbb{H}_k, \qquad \omega_k(\mathbf{e}^+) = \mathbb{E}_k^+, \qquad \omega_k(\mathbf{e}^-) = \mathbb{E}_k^-.$$

Our goal is to study whether the representation ω_k of $\mathfrak{sl}(2,\mathbb{R})$ is integrable or not. Our approach uses essentially a famous result due to Nelson [34]. This approach was used first by Ben Said in [45]. Let us recall Nelson's theorem.

Theorem 6.1. *Let $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}$ be a basis of a Lie algebra $\mathfrak{g}$ of dimension r, and let ω be a densely defined representation of $\mathfrak{g}$ on a Hilbert space $\mathcal{H}$. Then ω is the derivative of some continuous unitary representation of a Lie group G with Lie algebra $\mathfrak{g}$ if and only if*

(i) For every $X \in \mathfrak{g}$, $\omega(X)$ is a skew-symmetric operator on $\mathcal{H}$,

(ii) The operator $\omega(\mathbf{u_1}^2 + \ldots + \mathbf{u_r}^2)$ is essentially self-adjoint.

To study the structure of the representation ω_k of $\mathfrak{sl}(2,\mathbb{R})$, we will consider a second $\mathfrak{sl}_2$-triple which spans the Lie algebra

$$\mathfrak{su}(1,1) := \left\{ X \in \mathfrak{sl}(2,\mathbb{C}) : X^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X = 0 \right\}.$$

The Lie algebra $\mathfrak{su}(1,1)$ is another real form of $\mathfrak{sl}(2,\mathbb{C})$. It is worth mentioning that $\mathfrak{su}(1,1)$ is isomorphic to the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Indeed, if we consider the Cayley unitary matrix

$$c := \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ 1 & i \end{pmatrix},$$

then, $\mathrm{Ad}(c) : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{su}(1,1)$ defined by $\mathrm{Ad}(c)X = cXc^{-1}$ induces a Lie algebra isomorphism. It is convenient to use either one or the other suitable form of the two isomorphic Lie algebra according to the problem at hand.

We set $\{\mathbf{k},\mathbf{n}^+,\mathbf{n}^-\} := c\{\mathbf{h},\mathbf{e}^+,\mathbf{e}^-\}c^{-1}$. Then

$$\begin{aligned}\mathbf{k} &= i(\mathbf{e}^- - \mathbf{e}^+),\\ \mathbf{n}^+ &= -\frac{1}{2}(-i\mathbf{h}+\mathbf{e}^+ + \mathbf{e}^-),\\ \mathbf{n}^- &= -\frac{1}{2}(i\mathbf{h}+\mathbf{e}^+ + \mathbf{e}^-).\end{aligned}$$

One checks easily that $\{\mathbf{k},\mathbf{n}^+,\mathbf{n}^-\}$ is an $\mathfrak{sl}_2$-triple. Using the relations between the two standard basis $\{\mathbf{k},\mathbf{n}^+,\mathbf{n}^-\}$ and $\{\mathbf{h},\mathbf{e}^+,\mathbf{e}^-\}$ via the Cayley transform, we obtain

$$\begin{aligned}\omega_k(\mathbf{k}) &= \omega_k(i(\mathbf{e}^- - \mathbf{e}^+))\\ &= \|x\| - \|x\|\Delta_k := \widetilde{\mathbb{H}_k},\\ \omega_k(\mathbf{n}^+) &= \omega_k(-\frac{1}{2}(-i\mathbf{h}+\mathbf{e}^+ + \mathbf{e}^-))\\ &= \frac{i}{2}\Big(2E+(n+2\gamma_k-1)-\|x\|\Delta_k-\|x\|\Big) := \widetilde{\mathbb{E}_k}^+,\\ \omega_k(\mathbf{n}^-) &= \omega_k(-\frac{1}{2}(i\mathbf{h}+\mathbf{e}^+ + \mathbf{e}^-))\\ &= -\frac{i}{2}\Big(2E+(n+2\gamma_k-1)+\|x\|\Delta_k+\|x\|\Big) := \widetilde{\mathbb{E}_k}^-.\end{aligned}$$

where $E=\sum_{i=1}^n x_i\partial_{x_i}$ is the Euler operator.

For $m\in\mathbb{N}$, we set

$$\lambda_{k,m} := 2m+2\langle k\rangle+n-2.$$

The following lemma is needed for later use.

Lemma 6.1. *For all $f\in C^\infty(\mathbb{R}_+)$ and $p\in\mathcal{H}_m(\mathbb{R}^n)$, we have*

$$\begin{aligned}\mathbb{H}_k\Big(p(x)f(\|x\|)\Big) &= \Big\{(\lambda_{k,m}+1)f(\|x\|)+2\|x\|f'(\|x\|)\Big\}p(x),\\ \Delta_k\Big(p(x)f(\|x\|)\Big) &= \Big\{(\lambda_{k,m}+1)\|x\|^{-1}f'(\|x\|)+f''(\|x\|)\Big\}p(x).\end{aligned}$$

Proof. *The first statement is straight forward because the Euler operator E is of the form $r\frac{\partial}{\partial r}$ in the polar coordinates $x=r\omega$.*

To see the second statement, we will proceed as following: Take an arbitrary C^∞ function f on $\mathbb{R}^n$. Assume that g is a G-invariant smooth function.We have

$$T_j(k)g = \partial_j g, \qquad (6.1)$$
$$T_j(k)(fg) = (T_j(k)f)g + f(\partial_j g),$$

Take an arbitrary C^∞ function ψ on $\mathbb{R}^+$. Clearly we have

$$T_j\left[\psi\left(\|x\|\right)\right] = x_j\|x\|^{-1}\psi'\left(\|x\|\right).$$

Further,

$$\begin{aligned} T_j\left[f(x)\psi\left(\|x\|\right)\right] &= T_j(f(x))\psi\left(\|x\|\right) + f(x)T_j\left(\psi\left(\|x\|\right)\right) \\ &= T_j(f)\psi\left(\|x\|\right) + f(x)x_j\|x\|^{-1}\psi'\left(\|x\|\right) \end{aligned}$$

Apply the Dunkl operator T_j again to the above we get

$$\begin{aligned} & T_j^2\left[f(x)\psi\left(\|x\|\right)\right] \\ = \;& T_j^2(f)\psi\left(\|x\|\right) + T_j(f)T_j\psi\left(\|x\|\right) + T_j\left\{f(x)x_j\|x\|^{-1}\;\psi'\left(\|x\|\right)\right\} \\ = \;& T_j^2(f)\psi\left(\|x\|\right) + T_j(f)T_j\psi\left(\|x\|\right) + T_j(x_jf(x))\|x\|^{-1}\Psi'\left(\|x\|\right) \\ & +x_jf(x)T_j\left(\|x\|^{-1}\psi'(x)\right) \\ = \;& T_j^2(f)\psi\left(\|x\|\right) + \|x\|^{-1}\psi'\left(\|x\|\right)x_jT_j(f) + \|x\|^{-1}\psi'\left(\|x\|\right)T_j(x_jf) \\ & +x_jf(x)\partial_j\left\{\|x\|^{-1}\psi'\left(\|x\|\right)\right\} \end{aligned}$$

Taking the summation over j, we arrive at

$$\begin{aligned} & \Delta_k\left(f(x)\psi\left(\|x\|\right)\right) \\ = \;& \left(\Delta_k f(x)\right)\psi\left(\|x\|\right) + \|x\|^{-1}\psi'\left(\|x\|\right)\left(2E+n+2\langle k\rangle\right)f(x) \\ & +f(x)E\left(\|x\|^{-1}\psi'\left(\|x\|\right)\right). \end{aligned}$$

If we assume that $f = p$ is a homogeneous harmonic polynomial of degree m. Then

$$\begin{aligned}
& \Delta_k \left(p(x)\psi\left(\|x\|\right)\right) \\
= & \left(\Delta_k p(x)\right)\psi\left(\|x\|\right) + \|x\|^{-1}\psi'\left(\|x\|\right)\left(2E + n + 2\langle k\rangle\right)p(x) \\
& + p(x)E\left(\|x\|^{-1}\psi'\left(\|x\|\right)\right) \\
= & \|x\|^{-1}\psi'\left(\|x\|\right)\left(2m + n + 2\langle k\rangle\right)p(x) \\
& + p(x)E\left(\|x\|^{-1}\psi'\left(\|x\|\right)\right)
\end{aligned}$$

To finish the proof, one may use the following observation: in the polar coordinate $x = r\omega$, the Euler operator E amounts to $r(\partial/\partial r)$, and

$$r(d/dr)\left(r^{-1}\psi'(r)\right) = -r^{-1}\psi'(r) + \psi''(r).$$

We consider the following linear operator:

$$\alpha_m : \mathcal{H}_m(\mathbb{R}^n) \otimes C^\infty(\mathbb{R}_+) \to C^\infty(\mathbb{R}^n \setminus \{0\})$$

defined by

$$\alpha_m(p \otimes f) = p(x)f(\|x\|).$$

Using Lemma 6.1 and the definition of the operators $\mathbb{H}_k$, $\mathbb{E}_k^+$, and $\mathbb{E}_k^-$, we can prove that the opera $\mathbb{H}_k$, $\mathbb{E}_k^+$, and $\mathbb{E}_k^-$ act only on the radial part f when applied to those functions $p(x)f(\|x\|)$ for $p \in \mathcal{H}_m(\mathbb{R}^n)$.

Lemma 6.2. *The operators $\mathbb{H}_k$, $\mathbb{E}_k^+$, and $\mathbb{E}_k^-$ take the following forms on $\mathcal{H}_m(\mathbb{R}^n) \otimes C^\infty(\mathbb{R}_+)$:*

$$\mathbb{H}_k \circ \alpha_m = \alpha_m \circ \left(\mathrm{id} \otimes \left(2r\frac{d}{dr} + (\lambda_{k,m} + 1)\right)\right) \tag{6.2 a}$$

$$\mathbb{E}_k^+ \circ \alpha_m = \alpha_m \circ \left(\mathrm{id} \otimes ir\right) \tag{6.2 b}$$

$$\mathbb{E}_k^- \circ \alpha_m = \alpha_m \circ \left(\mathrm{id} \otimes i\left(r\frac{d^2}{dr^2} + (\lambda_{k,m} + 1)\frac{d}{dr}\right)\right) \tag{6.2 c}$$

We define an endomorphism of $C^\infty(\mathbb{R}_+)$ by

$$M : C^\infty(\mathbb{R}_+) \xrightarrow{\sim} C^\infty(\mathbb{R}_+),\ g(t) \mapsto (Mg)(r) := \exp(-r)g(2r).$$

Composing with α_m, we define the following linear operator β_m by

$$\beta_m := \alpha_m \circ (\mathrm{id} \otimes M).$$

That is, $\beta_m : \mathcal{H}_m(\mathbb{R}^n) \otimes C^\infty(\mathbb{R}_+) \to C^\infty(\mathbb{R}^n \setminus \{0\})$ is given by

$$\beta_m(p \otimes g)(x) := p(x) \exp\Big(-\|x\|\Big) g\Big(2\|x\|\Big). \tag{6.3}$$

In view of the above lemma, we get the following forms of $\mathbb{H}_k$, $\mathbb{E}_k^+$, and $\mathbb{E}_k^-$ on $\mathcal{H}_m(\mathbb{R}^n) \otimes C^\infty(\mathbb{R}_+)$.

Lemma 6.3. *Let*

$$P_t := t\frac{d^2}{dt^2} + (\lambda_{k,m} + 1 - t)\frac{d}{dt}. \tag{6.4}$$

Then,

$$\begin{aligned}
\mathbb{H}_k \circ \beta_m &= \beta_m \circ \Big(\mathrm{id} \otimes \Big(2t\frac{d}{dt} + (\lambda_{k,m} + 1 - t)\Big)\Big),\\
\mathbb{E}_k^+ \circ \beta_m &= \beta_m \circ \Big(\mathrm{id} \otimes \frac{i}{2}t\Big),\\
\mathbb{E}_k^- \circ \beta_m &= \beta_m \circ \Big(\mathrm{id} \otimes i\Big(2P_t + \frac{t}{2} - \lambda_{k,a,m} - 1\Big)\Big).
\end{aligned}$$

Proof. *Immediate from the above Lemma 6.2 and the following relations:*

$$\begin{aligned}
\frac{d}{dr} \circ M &= M \circ \left(2\frac{d}{dt} - 1\right),\\
r \circ M &= M \circ \frac{t}{2}.
\end{aligned}$$

Using the relations between $\{\mathbf{k}, \mathbf{n}^+, \mathbf{n}^-\}$ and $\{\mathbf{h}, \mathbf{e}^+, \mathbf{e}^-\}$, and therefore the relations between the operators $\mathbb{H}_k$, $\mathbb{E}_k^+$ and $\mathbb{E}_k^-$ and $\widetilde{\mathbb{H}_k}$, $\widetilde{\mathbb{E}_k}^+$ and $\widetilde{\mathbb{E}_k}^-$, we obtain the following actions:

Lemma 6.4. *Through the linear map β_m, the operators $\widetilde{\mathbb{H}_k}$, $\widetilde{\mathbb{E}_k^+}$ and $\widetilde{\mathbb{E}_k^-}$ take the following forms on $\mathcal{H}_m(\mathbb{R}^n) \otimes C^\infty(\mathbb{R}_+)$,*

$$\begin{aligned} \widetilde{\mathbb{H}_k} \circ \beta_m &= \beta_m \circ \Big(\mathrm{id} \otimes \Big(-2P_t + \lambda_{k,m} + 1\Big)\Big), \\ \widetilde{\mathbb{E}_k}^+ \circ \beta_m &= \beta_m \circ \Big(\mathrm{id} \otimes \Big(-i\Big(P_t - t\frac{d}{dt} + t - \lambda_{k,m} - 1\Big)\Big)\Big), \\ \widetilde{\mathbb{E}_k}^- \circ \beta_m &= \beta_m \circ \Big(\mathrm{id} \otimes \Big(-i\Big(P_t + t\frac{d}{dt}\Big)\Big)\Big). \end{aligned}$$

Now we are ready to exhibit the action of the $\mathfrak{sl}_2$-triple $\{\mathbf{k}, \mathbf{n}^+, \mathbf{n}^-\}$ on the functions $\Phi_\ell(p,\cdot)$ from the previous chapter. We recall from above that

$$\omega_k(\mathbf{k}) = \widetilde{\mathbb{H}_k}, \quad \omega_k(\mathbf{n}^+) = \widetilde{\mathbb{E}_k}^+, \quad \omega_k(\mathbf{n}^-) = \widetilde{\mathbb{E}_k}^-.$$

Recall also that for $\ell, m \in \mathbb{N}$ and $p \in \mathcal{H}_m(\mathbb{R}^n)$, we have the following functions on $\mathbb{R}^n$,

$$\Phi_\ell(p,x) := p(x) L_\ell^{(\lambda_{k,m})}\Big(2\|x\|\Big) \exp\Big(-\|x\|\Big). \tag{6.5}$$

Here, $\lambda_{k,m} = 2m + 2\gamma_k + n - 2$, and $L_\ell^{(\lambda)}(t)$ is the Laguerre polynomial. Hence, for $x = r\omega \in \mathbb{R}^n$ ($r > 0$, $\omega \in S^{n-1}$), we have

$$\Phi_\ell(p,x) = p(\omega) r^m L_\ell^{(\lambda_{k,m})}\Big(2r\Big) \exp\Big(-r\Big).$$

Finally, recall from the previous chapter the vector space of functions on $\mathbb{R}^n$ defined by

$$D_k(\mathbb{R}^n) := \{\Phi_\ell(p,\cdot) \mid \ell \in \mathbb{N}, m \in \mathbb{N}, p \in \mathcal{H}_m(\mathbb{R}^n)\}. \tag{6.6}$$

Theorem 6.2. *The space $D_k(\mathbb{R}^n)$ is stable under the action of $\mathfrak{sl}(2,\mathbb{C})$. More precisely, for each fixed $p \in \mathcal{H}_m(\mathbb{R}^n)$, the action ω_k is given as follows:*

$$\begin{aligned} \omega_k(\mathbf{k})\Phi_\ell(p,x) &= (2\ell + \lambda_{k,m} + 1)\Phi_\ell(p,x), \\ \omega_k(\mathbf{n}^+)\Phi_\ell(p,x) &= i(\ell + 1)\Phi_{\ell+1}(p,x), \\ \omega_k(\mathbf{n}^-)\Phi_\ell(p,x) &= i(\ell + \lambda_{k,m})\Phi_{\ell-1}(p,x). \end{aligned}$$

Proof. *Let* $P_t = t\frac{d^2}{dt^2} + (\lambda_{k,m}+1-t)\frac{d}{dt}$. *By the formula* $\Phi_\ell(p,\cdot) = \beta_m(p \otimes L_\ell^{(\lambda_{k,m})})$ *(see* (6.5)*) and by Lemma 6.4, it is enough to prove the following identities*

$$
\begin{aligned}
(-2P_t + (\lambda_{k,m}+1))L_\ell^{(\lambda_{k,m})} &= (2\ell + \lambda_{k,m}+1)L_\ell^{(\lambda_{k,m})},\\
(-i(P_t - t\frac{d}{dt} + t - \lambda_{k,m} - 1))L_\ell^{(\lambda_{k,m})} &= i(\ell+1)L_{\ell+1}^{(\lambda_{k,m})},\\
-i(P_t + t\frac{d}{dt})L_\ell^{(\lambda_{k,m})} &= i(\ell+\lambda_{k,m})L_{\ell-1}^{(\lambda_{k,m})}.
\end{aligned}
$$

However, since the Laguerre polynomial $L_\ell^{(\lambda_{k,m})}(t)$ *satisfies the Laguerre differential equation*

$$P_t L_\ell^{(\lambda_{k,m})}(t) = -\ell L_\ell^{(\lambda_{k,m})}(t),$$

the first assertion is now clear. The remaining identities and are reduced to the recurrence relations in 5.3 , respectively.

By using the orthonormal basis $\{f_{\ell,m}(r)\}$, we may normalize the functions $\Phi_\ell(p,x)$ as following

$$
\begin{aligned}
\widetilde{\Phi}_\ell(p,x) &:= f_{\ell,m}(r)p(\omega)\\
&= \Big(\frac{2^{\lambda_{k,m}+1}\Gamma(\ell+1)}{\Gamma(\lambda_{k,m}+\ell+1)}\Big)^{\frac{1}{2}}\Phi_\ell(p,x)
\end{aligned}
$$

for $x = r\omega$ ($r>0$, $\omega \in S^{n-1}$). Then, Theorem 6.2 is reformulated as follows:

Theorem 6.3. *For any* $p \in \mathcal{H}_m(\mathbb{R}^n)$, *we have*

$$
\begin{aligned}
\omega_k(\mathbf{k})\widetilde{\Phi}_\ell(p,x) &= (2\ell+\lambda_{k,m}+1)\widetilde{\Phi}_\ell(p,x),\\
\omega_k(\mathbf{n}^+)\widetilde{\Phi}_\ell(p,x) &= i\sqrt{(\ell+1)(\lambda_{k,m}+\ell+1)}\,\widetilde{\Phi}_{\ell+1}(p,x),\\
\omega_k(\mathbf{n}^-)\widetilde{\Phi}_\ell(p,x) &= i\sqrt{(\lambda_{k,m}+\ell)\ell}\,\widetilde{\Phi}_{\ell-1}(p,x).
\end{aligned}
$$

For $f,g \in L^2\big(\mathbb{R}^n, \vartheta_k(x)dx\big)$, we write its inner product as

$$\langle\langle f,g\rangle\rangle_k := \int_{\mathbb{R}^n} f(x)\overline{g(x)}\,\vartheta_k(x)dx.$$

Proposition 6.1. *The representation ω_k of $\mathfrak{sl}(2,\mathbb{R})$ on $D_k(\mathbb{R}^n)$ is infinitesimally unitary with respect to the inner product $\langle\langle\ ,\ \rangle\rangle_k$, namely,*

$$\langle\langle \omega_k(X)f, g\rangle\rangle_k = -\langle\langle f, \omega_k(X)g\rangle\rangle_k$$

for any $X \in \mathfrak{sl}(2,\mathbb{R})$ and $f, g \in D_k(\mathbb{R}^n)$.

Proof. *We already knew that the Dunkl operators are skew-symmetric with respect to the measure $\prod_{\alpha\in\mathcal{R}^+} |\langle\alpha, x\rangle|^{2k_\alpha} dx$. In view of the definitions $\mathbb{E}_k^- = i\|x\|\Delta_k$, we see that $\mathbb{E}_k^-$ is a skew-symmetric operator with respect to the inner product $\langle\langle\cdot,\cdot\rangle\rangle_k$. Likewise for $\mathbb{E}_k^+$. Further, the commutation relation $\mathbb{H}_k = [\mathbb{E}_k^+, \mathbb{E}_k^-]$ shows that $\mathbb{H}_k$ is also skew-symmetric. Thus, for all $X \in \mathfrak{sl}(2,\mathbb{R})$, $\omega_k(X)$ is skew-symmetric.*

Recall that an operator $\mathcal{O}$ is called essentially self-adjoint, if it is symmetric and its closure is a self-adjoint operator. Let $\mathcal{O}$ be a symmetric operator on a Hilbert space $\mathcal{H}$ with domain $\mathbb{D}(\mathcal{O})$, and let $\{f_n\}_n$ be a complete orthogonal set in $\mathcal{H}$. If each $f_n \in \mathbb{D}(\mathcal{O})$ and there exists $\mu_n \in \mathbb{R}$ such that $\mathcal{O}f_n = \mu_n f_n$, for every n, then $\mathcal{O}$ is essentially self-adjoint.

Let G be the simply connected covering Lie group with Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$. Now we are ready to apply Nelson's theorem to the representation ω_k.

Theorem 6.4. *The infinitesimal representation ω_k exponentiates to define a unique unitary representation of G on $L^2(\mathbb{R}^n, \vartheta_k(x)dx)$.*

Proof. *Let $\mathbf{u_1} = \mathbf{e}^+ - \mathbf{e}^-$, $\mathbf{u_2} = \mathbf{e}^+ + \mathbf{e}^-$ and $\mathbf{u_3} = \mathbf{h}$. Since $\{\mathbf{e}^+, \mathbf{e}^-, \mathbf{h}\}$ is a basis for the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$, it follows that $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ is also a basis for $\mathfrak{sl}(2,\mathbb{R})$. Now,*

$$-\mathbf{u_1}^2 + \mathbf{u_2}^2 + \mathbf{u_3}^2 = \mathbf{h}^2 + 2\mathbf{e}^+\mathbf{e}^- + 2\mathbf{e}^-\mathbf{e}^+ = \mathbf{k}^2 + 2\mathbf{n}^+\mathbf{n}^- + 2\mathbf{n}^+\mathbf{n}^-$$

and

$$\mathbf{u_1}^2 = -\mathbf{k}^2.$$

Therefore

$$\mathbf{u_1}^2 + \mathbf{u_2}^2 + \mathbf{u_3}^2 = -\mathbf{k}^2 + 2\mathbf{n}^+\mathbf{n}^- + 2\mathbf{n}^+\mathbf{n}^-.$$

By Theorem 6.3,the orthonormal basis $\left\{\widetilde{\Phi}_{\ell,m,j} \mid \ell \in \mathbb{N}, m \in \mathbb{N}, j \in J_m\right\}$ *of the space* $L^2(\mathbb{R}^n, \vartheta_k(x)dx)$ *are eigenvectors for the action*

$$\omega_k(\mathbf{u_1}^2 + \mathbf{u_2}^2 + \mathbf{u_3}^2) = \omega_k(-\mathbf{k}^2 + 2\mathbf{n}^+\mathbf{n}^- + 2\mathbf{n}^+\mathbf{n}^-)$$

with real eigenvalues. More precisely,

$$\begin{aligned} & \omega_k(\mathbf{u_1}^2 + \mathbf{u_2}^2 + \mathbf{u_3}^2)\widetilde{\Phi}_{\ell,m,j}(x) \\ = \ & \omega_k(-\mathbf{k}^2 + 2\mathbf{n}^+\mathbf{n}^- + 2\mathbf{n}^+\mathbf{n}^-)\widetilde{\Phi}_{\ell,m,j}(x) \\ = \ & \Big\{ -(2\ell + \lambda_{k,a,m} + 1)^2 - 2(\lambda_{k,a,m} + \ell)\ell - 2(\ell+1)(\lambda_{k,a,m} + \ell + 1) \Big\} \widetilde{\Phi}_{\ell,m,j}(x). \end{aligned}$$

Thus, the symmetric operator $\omega_k(\mathbf{u_1}^2 + \mathbf{u_2}^2 + \mathbf{u_3}^2)$ *on the Hilbert space* $L^2(\mathbb{R}^n, \vartheta_k(x)dx)$ *is indeed essentially self-adjoint. Moreover, since* $\omega_k(X)$ *is skew-symmetric for every* $X \in \mathfrak{sl}(2,\mathbb{R})$, *it follows from Nelson's theorem that* ω_k *exponentiates to define on* $L^2(\mathbb{R}^n, \vartheta_k(x)dx)$ *a unique unitary representation of the simply connected Lie group G with Lie algebra* $\mathfrak{sl}(2,\mathbb{R})$.

The above theorem finishes the question asked in this chapter.

Chapter 7: Conclusion

One of the most intriguing aspects of the theory of partial differential equations is the wave equation $\Delta u(x,t) - \partial_{tt} u(x,t) = 0$. A remarkable fact about wave equations, called Huygens' principle, states that the solution $u(x,t)$ is supported in the set $\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \|x\| = |t|\}$ if and only if the dimension n is odd and greater than 3. The classification of all second order differential equations satisfying the Huygens' principal is still an open problem, known under the name of Hadamard's conjecture. In this thesis we proved that this conjecture is far from being solved. More specifically, we considered the wave equation $2\|x\|\Delta_k u_k(x,t) - \partial_{tt} u_k(x,t) = 0$, with $u_k(x,0) = f(x)$ and $\partial_t u_k(x,0) = g(x)$. Here Δ_k is the Dunkl Laplacian operator. The first main result of the thesis consists of showing that our wave equation satisfies the Huygens' principle under some conditions. More precisely, the support of the unique solution $u_k(x,t)$ is contained in the set $\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : 2\|x\|^2 = |t|^2\}$ if and only if $2n + 2\sum_{\alpha \in \mathcal{R}} k_\alpha - 5$ is an even integer. Our approach is essentially based on the representation theory of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ by the construction of the infinitesimal representation

$$\omega_k(\mathbf{h}) = n + \sum_{\alpha \in \mathcal{R}} k_\alpha - 1 + 2\sum_{i=1}^{N} x_i \partial_i, \quad \omega_k\left(\mathbf{e}^+\right) = i\|x\|, \quad \omega_k\left(\mathbf{e}^-\right) = i\|x\|\Delta_k,$$

where $\{\mathbf{h}, \mathbf{e}^+, \mathbf{e}^-\}$ is the canonical basis of $\mathfrak{sl}(2,\mathbb{R})$.

The second main result of the thesis consists of the integrability of the representation ω_k. The integrability fact is not obvious since in infinite dimensions, the existence of a Lie group representation is not guaranteed from the existence of a Lie algebra representation. However, we proved that ω_k integrates to a unique unitary representation of the universal covering Lie group of $SL(2,\mathbb{R})$.

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In this thesis, a non-trivial wave equation is introduced. The main aim is to find under which conditions the unique solution to the deformed wave equation is supported on the light cone.The approach is based on the representation theory of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, where a new representation ω_k of $\mathfrak{sl}(2,\mathbb{R})$ is constructed. Further, we prove that ω_k lifts to give raise to a unitary representation of a simply connected Lie group with Lie algebra $\mathfrak{sl}(2,\mathbb{R})$.

Aysha Alshamsi received her Master of Science in Mathematics from the Department of Mathematics, College of Science and her Bachelor of Science in Mathematics from the College of Science, UAE University

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