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VALUATION OF OPTIONS IN A HIGH VOLATILE REGIME SWITCHING MARKET

Tasnim Mazen Sharif Alhamad

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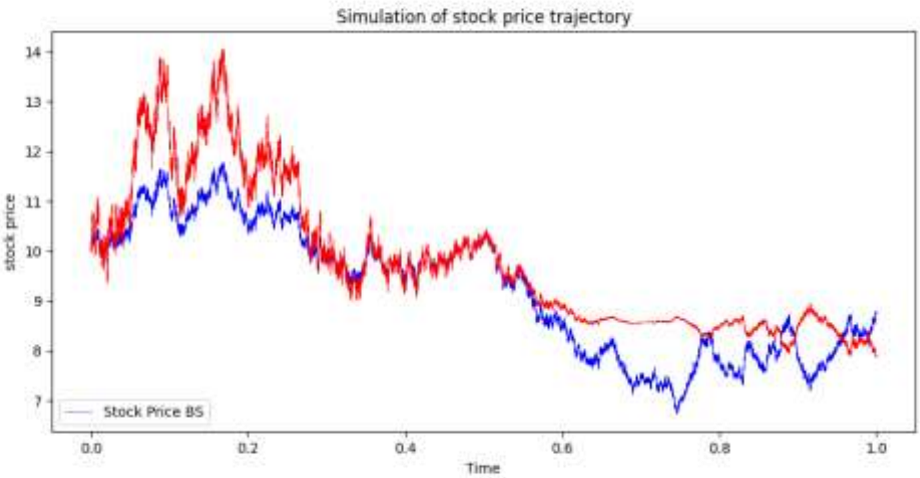

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This thesis is submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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Cover: Continuous Time Markov Chain (CTMC) regime switching Model Simulation generated via Python. High volatile regime-switching model.

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Declaration of Original Work

I, Tasnim Mazen Sharif Alhamad, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled *"Valuation of Options in A high Volatile Regime Switching Market"*, hereby, solemnly declare that this thesis is my own original research work that has been done and prepared by me under the supervision of Prof. Youssef El Khatib, in the College of Science at UAEU. This work has not previously been presented or published or formed the basis for the award of any academic degree, diploma, or similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my thesis have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this thesis.

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Abstract

Financial modeling by Stochastic Differential Equations-SDEs with regime-switching has been utilized to allow moving from one economic state to another. The aim of this thesis is to tackle the pricing of European options under a regime-switching model where the volatility is augmented. Regime-switching models are more realistic since they encompass the effect of an external event on the underlying asset prices. But they are challenging, considering in addition increased volatility in the model will for sure make the option pricing problem more complicated and its solution may not exist analytically. Numerical methods for finance could be very helpful in this case. This study proposes a new SDE for the prices of the underlying financial asset under regime-switching with a high volatility model. The suggested model covers the crisis model of [3] and [4] for highly volatile situations and the regime-switching model of [5]. Under these settings, the valuation of European options is investigated based on the works of [5], [7], [8], and [9]. As an application, a study of two states is developed: state "1" when the economy is going well, and state "2" when the economy is under stress.

Keywords: Variance swaps, Regime switching, Brownian motion, Ito formula, Continuous-time Markov chain, Financial assets.

Title and Abstract (in Arabic)

تقييم الخيارات في سوق تبديل الأنظمة عالية التقلب

الملخص

تستخدم المعادلات التفاضلية العشوائية على نطاق واسع لنمذجة الكميات المالية المختلفة. في العقود الأخيرة، تم استخدام النمذجة المالية من خلال تحويل الأنظمة للسماح بالانتقال من حالة اقتصادية إلى أخرى. إن الهدف من هذا العمل البحثي هو معالجة تسعير المقايضات المتباينة في الأسواق المتغيرة وفق نموذج تبديل النظام. يعد استخدام المعادلات التفاضلية العشوائية أكثر واقعية من الناحية العلمية ولكن ايجاد حل مناسب لها يعد أمرا معقدا وفي بعض الأحيان لا يمكن ايجاده من الناحية التحليلية ولهذا يتم ايجادها من الناحية العددية. سيتم وضع معادلات تفاضلية جديدة لنماذج عالية التقلب من الأصول المالية الأساسية. سيغطي النموذج المقترح نماذج [٣] و [٤] للحالات شديدة التقلب والجزء المغير للنظام كما في [٥]. بعد ذلك، سيتم دراسة تقييم مقايضة التباين بناءً على [٥] و [٧] و [٨] و [٩]. سيتم دراسة حالتين مختلفتين: عندما يكون الاقتصاد جيد وعندما يكون الاقتصاد سيء.

مفاهيم البحث الرئيسية : مقايضة التباين، تبديل النظام، نظام برونيان ، معادلات إيتو، سلسلة ماركوف، الأصول المالية.

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I would like to thank my beloved family, beginning with my parents, who without their love and encouragement, I would have never been where I am today.
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Dedication

To my beloved family and teachers

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Chapter 1: Introduction and Financial Derivatives

1.1 Introduction

This master thesis deals with the pricing of European Options under a regime-switching model that takes into consideration crisis situations where the volatility is increased.

An option is a financial derivative whose value depends on another underlying financial asset, the buyer has the option and is not obliged to exercise their agreement to buy or sell. Options constitute a very important tool in risk management. The Foundation For Secure Markets (OCC) announced that options were the world's largest traded derivative on October 2022 with a volume of 910.0 million contracts, which is over 10.2% compared to October 2021. Options can be traded in different styles such as:

- Commodities: Daily consumed such as grains, gold, and oil.
- Currencies: buy or sell a certain amount of an available currency at a fixed exchange rate for a fixed time period.
- Stocks: (equities), cash flow producing entities like United Arab Emirates Stock Market (ADX General).

They have also several advantages such as providing increased cost-efficiency and being less risky than equities. It is then not surprising that one of the key problematic in stochastic finance is the valuation of financial derivatives products and specially options. The price of the financial derivative $C = (C_t)_{t\in[0,T]}$ is depending on the underlying asset with price $S = (S_t)_{t\in[0,T]}$. The persistence of studying financial derivative is to find a solution to the pricing problem, which is to evaluate their fair prices C. Consequently, this problem comprises foremost on finding a model for the underlying asset prices $S = (S_t)_{t\in[0,T]}$ (using generally SDEs) then calculating the price of the derivative $C_t = f(t, S_t)$ for any $t \in [0,T]$. At large, C_T is known, it is equal to the so-called payoff. The chief matter is to determine the premium C_0 that signifies the worth of the derivative at the initial time. The pricing problem is contingent on the model chosen for

$S = (S_t)_{t\in[0,T]}$. Thus, two main questions to be raised in order to price a European option.

1. How to predict the underlying asset price trajectories from 0 to T?

2. How to find the premium, which is the value of the option at $t = 0$?

Some important related works on pricing financial derivatives are Black and Scholes [1] and Heston [2] models. One of the most important parameters in stochastic modeling is the so-called volatility that guesses the extent to which a quantity tends to alter. But, volatility of a price is not noticeable from the market. In Black and Scholes, the asset price process is ensuing a geometric Brownian motion. To account for very high variations in the price, the model considered in [3] add a new parameter to increase the volatility. On the other hand, the regime-switching model of [4] deal with the impact of external events that can change the price of the underlying asset and thus the value of the option. The thesis objective is to come up with a model for the underlying asset price that merge the augmented volatility and regime switching. Then to investigate a solution for the SDE, and discuss the solution of the price of the European option. The thesis intends to:

- Suggest a model for high volatile situations with the impact of external events.

- Solve the Stochastic Differential Equation (SDE) of the underlying asset price of the suggested model.

- Compare the option price of the regime-switching high volatility model to the price of the option under the regime-switching model.

- Employ numerical methods to simulate the continuous time Markov chain process with 2 states and the prices of the underlying asset prices

The thesis is organized as follows, in this Chapter 1, an introduction to financial derivatives, options, and their types and importance is provided. Chapter 2 is dealing with the stochastic tools needed to conduct this study. In Chapter 3, the Markov chain and continuous-time Markov chain are discussed. The main contributions of this thesis are given in Chapter 4. After introducing the model, the solution for the SDE of the underlying asset price is investigated. Then the option price under the regime-switching with high volatility is compared to the price of the option with the same settings except that the volatility is not augmented. Numerical simulations are conducted and several illustrations for the underlying asset prices are provided in this chapter. We conclude the thesis with some remarks.

1.2 Financial Derivatives

In this section, several definitions and concepts from finance needed in our project are provided. We refer the reader to the book of [5] for an extensive study on financial products, their pricing, and hedging issues.

Definition 1. *Financial derivatives are financial instruments that are linked to a specific financial tool, and using it specific financial risks can bought or sold in financial markets in their own right. Transactions in financial derivatives should be treated as separate transactions rather than as integral parts of the value of underlying transactions to which they may be linked.*

Notice that financial derivatives are used for a number of purposes including risk management, hedging, arbitrage between markets, and speculation.

1.2.1 Main Types of Financial Derivatives

There are four important derivatives contracts: options, futures, forwards, and swaps.

Definition 2. *A forward or futures contract is an agreement to buy or sell a specified quantity of an asset at a specified price with delivery at a specified date in the future.*

Definition 3. *Options are contracts can be either standardized or customized. There are two types of options: call and put options.*

Definition 4. *A swap is an agreement between two counterparties to exchange a series of cash payments for a stated period of time.*

In general, a financial derivative product is based on another financial instrument called an underlying asset.

Definition 5. *An Underlying asset is an asset that influences the performance or value of a derivative security. For example stocks, currencies, and commodities.*

Studying financial derivatives consists in general in treating two main problems:

- Pricing: How to find a fair price for a financial derivative contract accepted by the two parties involved in the contract? This is called the pricing problem.
- Hedging: Hedging is an investment that is made with the intention of reducing the risk of adverse price movements in an asset. Normally, a hedge consists of taking an offsetting or opposite position in a related security.

In the next section, some essential definitions and properties on options needed for our study are exposed.

1.3 Options

As is defined earlier options are financial derivatives contracts that give the owner a right to buy (if it's a call option) or sell (if it's a put option) an underlying asset at a pre-determined price (called the strike price) before the contract expires. The holder is not obliged to exercise, he has the choice and not the obligation. Investors should have a good understanding of the factors determining the value of an option. Understanding how to value that premium is crucial for trading options and essentially rests on the probability that the right or obligation to buy or sell a stock will end up being profitable at expiration.

Options contracts are mainly affected by primary parameters that are actually the drivers of the price of an option. The primary parameters are the:

- strike price: is a pre-determined price.

- spot price: the current underlying asset market price.
- maturity time: Expiration date to exercise the option's contract.

All the previous parameters have a role in deciding the price of each option. Therefore, option parties should concentrate their efforts on determining the ideal strike and maturity time for their purposes. A mathematical model used to compute the theoretical value of an option is usually named an "option pricing model". An option's value is an estimation of how much this option should be worth. Solving the option pricing problem aims at providing us with a fair price for the option. Knowing the estimate of the fair value of an option is important to traders since this helps them to adjust their portfolios. Thus, an accurate options pricing model is a strong tool for investors. There are several benefits of options. For example, a trader can take an option position but at lower costs than taking similar stock positions. For finance professionals, equities have more risk than options since they involve more financial commitment than options, and they generate higher risk. Options suggest a number of important alternatives, and no calculations are required to determine if the trader spends less money and makes a similar benefit, simply, investors have a bigger percentage return. Another major advantage of options is they offer more investment alternatives. Options are a very flexible tool. There are many ways to use options to recreate other positions. We call these positions synthetics. Pricing options properly will be evident why options are considered the main of attention in financial markets nowadays. Online brokerages provide direct access to the options markets and insanely low commission costs. Options are traded both on exchanges and in the over-the-counter market. Options can be calls or puts.

Calls give the buyer the right but not the obligation to buy a certain number of an asset named as the underlying asset, at an agreed price on a given date in the future called maturity.
The exercise could be also before the maturity in the case of American options.

Puts offer the holder the right, and not the obligation to sell a certain quantity of a given asset at an agreed price on or before maturity.

In summary, we can state the following definition "an option is a financial derivative product that gives the owner the right to buy (sell) a given number of an asset at a predetermined price on or at maturity." Option markets depend on several parameters listed below.

- Buy = Call option. Sell = Put option
- On/before: American. Only on: European
- Specified price = Strike or exercise price
- Specified date = Maturity or expiration date
- Buyer = holder = long position
- Seller = writer = short position

Note that the buyer and the seller are the two parties of either call or put options. Options can be exercised in different types (styles) of options, usually defined by the expiration dates. The most popular types of options are European, Asian, or American options. Here in below, I will define each option's type briefly;

- American Option: An American option is a style of option that its owner buys or sells at any time before and including the expiration date. An American option allows investors to take benefits as early as the stock price change to their advantage. American options are frequently exercised before an ex-dividend date allowing investors to own shares and get the next dividend payment.
- European Option: It is a version of an options contract that limits rights exercise to only the day of expiration. Although American options can be exercised early, it comes at a price since their premiums are often higher than European options. Investors can sell a European option contract back to the market before expiry and receive the net difference between the premiums earned and paid initially. Investors usually don't have a choice of buying either the American or the European option and most indexes use European options. The Black-Scholes model is used to value European options in general.
- Asian Option: An Asian option is an option type where the payoff depends on the average price of the underlying asset over a certain period of time as opposed to standard options (American and European) where the payoff depends on the price of the underlying asset

at a specific point in time (maturity). These options allow the buyer to sell or to buy an underlying asset at the average of all its prices instead of its spot price. Asian options are also known as average options.

1.3.1 Why Do We Use Options?

After the appearance of the pioneer Black-Scholes Formula [1] in 1973, options started to be traded in financial markets. Despite that they can be seen as being risky investments, they are also useful to an investor. Actually, there are several advantages to options. The first one is that they may provide increased cost-efficiency. Options could be a big leveraging power. Another advantage of trading options is that they may be less risky than equities. Options can be less risky for investors because they require less financial commitment than equities, and they can also be less risky due to their relative imperviousness to the potentially catastrophic effects of gap openings. Moreover, an additional possible benefit that can be obtained by investing in options is that they have the potential to deliver higher percentage returns. You can spend less money for making a similar profit, you'll get a bigger percentage return. In addition, various methods exist for utilizing options to reproduce other positions. These positions are named synthetics. The next example illustrates more on option advantages.

Example 6. *To buy 100 shares of a* 70 *underlying asset, an investor has to pay out* $7,000$*. But, if he buys one* 20 *call contract representing 100 shares, the total outlay would be only* $2,000$*. (1 contract x 100 shares x* 20 *price). The investor would then have an additional* $5,000$ *to use as he prefers.*

1.4 Pricing Options

There are several options pricing models that use options parameters to determine the fair market value of an option. The first widely used mathematical method to calculate the theoretical value of an option contract has been developed in 1973 and called (the Black-Scholes model), using current stock prices, expected dividends, the option's strike price, expected interest rates, time to expiration, and expected volatility. The black-Scholes model is the continuous-time analog of the binomial model. In 1973, the model was applied with certain assumptions.

- The main assumption of the Black-Scholes model pertains to the evolution of the stock price.
- This price is taken to evolve according to a geometric Brownian motion.
- Returns on the stock have a lognormal distribution with constant volatility.
- Stock prices cannot jump (the market cannot "gap").

The asset's price at time t is (X_t) where (X_t) is determined by the stochastic differential equation

$$dX_t = X_t(\mu dt + \sigma dWt) \tag{1.1}$$

with $W_t, t \geq 0$ being a standard Brownian motion and $\sigma > 0, \mu$ are constants.

1.4.1 Lower and Upper Bounds for the Price of the Option

Some results on upper and lower bounds for option prices are given in this part. These bounds do not depend on any particular assumptions about the options preliminaries mentioned earlier. Herein I will provide examples of the lower bound and specifically the European lower bound.

1.4.1.1 Lower Bound for Calls on Non-Dividend-Paying Stocks

Considering a call option built on a stock with prices determined by the Equation (1.1) then a lower bound for the price of the option is

$$c \geq X_0 - ke^{-rT}, \quad \text{and} \quad c \geq max(X_0 - ke^{-rT}, 0). \tag{1.2}$$

The following notations were used:

- K: strike price.
- X_0: is the current asset price.

- r: risk-free interest rate.
- T: Maturity.
- c: Value of European call option to buy one share.

This makes option pricing the best method for investors, as per the assumption no loss will occur, the premium $c \geq 0$.

Example 7. *Consider a European call option on a non-divided-paying stock with a value of* 40 *AED, the interest rate is* $r = 12\%$ *per year, the strike is* 38 *AED, and the expiration is in 6 months. Thus, we assume that* $X_0 = 40, K = 38, T = 0.5,$ *and* $r = 12\%$ *from Equation (1.2), a lower bound for the option price is* $X_0 - ke^{-rT}$ *or*$40 - 38e^{-0.12(0.5)} = 4.21.$

There is also a lower bound for the put option as we can see in the coming subsection.

1.4.1.2 Lower Bound for Puts on Non-Dividend-Paying Stocks

Again, the price of the stock is modeled by Equation (1.1). For a European put option on the non-dividend-paying stock with price dynamics X, a lower bound for the price is

$$p \geq ke^{-rT} - X_0, \quad \text{and} \quad p \geq max(ke^{-rT} - X_0-, 0). \tag{1.3}$$

where p is the European put option price to sell one asset, then the worst that can happen to a put option is that it expires worthless, its value cannot be negative.

Example 8. *Assume there is a European put option stock when the asset price is* 38 *AED, the Strike value is* 40 *AED, the maturity is one semester, and the risk-free rate is* 10% *per year. In this case* $X_0 = 38, K = 40, T = 0.25,$ *and* $r = 0.10.$ *From Equation (1.3), a lower bound for the put option price is* $ke^{-rT} - X_0$ *or* $40e^{-0.10(0.25)-38} = 1.01.$

Notice that there is a relationship that links the price of a call and the price of a put if they are built on the same underlying asset and strike. The next subsection provides this relationship and it is called "Parity Call-Put".

1.4.2 Parity Call-Put

Let's consider two European options one calls with price c and the other is put whose value is denoted by p. Both options have the same underlying asset X given by the Equation (1.1), strike K, and maturity T. The values c and p are $max(X_0 - K, 0)$ and $max(K - X_0, 0)$ at maturity of the options. Because the options are European, they cannot be exercised before the maturity date. Therefore,

$$c + ke^{-rT} = p + X_0 \tag{1.4}$$

This relationship is known as put-call parity. It shows that the value of a European call with a certain strike price and exercise date can be deduced from the value of a European put with the same strike price and exercise date, and vice versa.

Chapter 2: Preliminaries from Stochastic Calculus

In this chapter, some tools from stochastic calculus are provided. Let T be a subset of the interval $[0,\infty)$. A family of random variables $(X_t)_{t\in T}$, indexed by T, is called a stochastic (or random) process, when $T = N$, and $(X_t)_{t\in T}$, is said to be a discrete-time process, and when $T = [0,\infty)$, it is called a process with continuous time.

2.1 Some Tools from Probability

A probability space has three components (Ω, F, P), respectively the sample space, event space, and probability function. The set Ω is the sample space, which is the set of the outcomes of an experiment. An event F is a subset of Ω.

Finally, a probability function P assigns a number ("probability") to each event in F. Simply, we can define the stochastic process as follows: A stochastic process is a family of random variables $(X_t)_{t\in T}$, indexed by time, defined on a probability space (Ω, F, P), and assuming values in R.

2.2 Stochastic Process

This section is dedicated to elementary stochastic calculus where some definitions and tools are provided.

Definition 9. *(Ω, F, P) is a probability space. A filtration on (Ω, F, P) is an increasing collection $(F_t)_{t\geq 0}$ of sub $\sigma-$ algebras of F. In other words, for each t, F_t is a $\sigma-$ algebra included in F and if $s \leq t, F_s \subset F_t$.*

Definition 10. *A probability space (Ω, F, P) endowed with a filtration $(F_t)_{t\geq 0}$ is called a filtered probability space $(\Omega, F, (F_t)_{t_0\leq t\leq T}, P)$*

Another definition for a stochastic process as follows:

Definition 11. *A stochastic process is a parameterized collection of random variables* $(X_t)_{t \in T}$*, defined on a probability space* (Ω, F, P)*, and assuming values in* R.

One of the most important stochastic process is the Brownian motion known also as the *Wiener process*.

Definition 12. *A Brownian motion is a stochastic process* $(B_t)_0 \leq t \leq T$ *with values in* R *defined for* $t \in [0, \infty)$ *such that it satisfies the following four conditions:*

1. $B_0 = 0$.
2. *If* $0 < s < t$*, then* $(B_t) - (B_s)$ *has a normal distribution* $N \sim (0, t-s)$ *with mean 0 and variance* $(t-s)$.
3. *If* $0 \leq s \leq t \leq u \leq v$ *(i.e., the two intervals* $[s,t]$ *and* $[u,v]$ *do not overlap) then* $B_t - B_s$ *and* $B_v - B_u$ *are independent random variables.*
4. *The sample paths* $t \to (B_t)$ *are almost surely continuous.*

Definition 13. *Let* (F_t) *be a filtration on* (Ω, F, P)*. A stochastic process* (X_t) *is* F_t*-adapted if* $\forall t \geq 0 X_t$ *is* F_t*-martingale.*

Definition 14. *(Martingale)*

Let $(F_t)_{t \geq 0}$ *be a filtration on* (Ω, F, P)*. A stochastic process* $(M_t)_{t \geq 0}$ *is called* F_t*-martingale if the following three properties are satisfied:*

(i) (M_t) *is* F_t*-adapted.*

(ii) $E||M_t|| < \infty, \quad \forall t \geq 0$.

(iii) $E(M_t / F_s) = M_s, \quad \forall 0 \leq s \leq t$.

The next remark provides some interesting information that can be obtained from the above definition.

Remark. 1.

(a) If the condition (iii) of the previous definition is replaced by $E(M_t/F_s) \geq M_s, \quad \forall \quad 0 \leq s \leq t,$ then (M_t) is called submartingale.

(b) If the condition (iii) of the previous definition is replaced by $E(M_t/F_s) \leq M_s, \quad \forall \quad 0 \leq s \leq t,$ then (M_t) is called supermartingale.

(c) A positive submartingale is a submartingale $(X_t)_{t\geq 0}$ satisfying $X_t \geq 0 \quad \forall \quad t \geq 0.$

Definition 15. *(Predictable process)*

Let $(F_t)_{t\geq 0}$ be a filtration on (Ω, F, P). A stochastic process $(X_t)_{t\geq 0}$ is called F_t-predictable process when $t > 0, X_t$ is measurable with respect to the σ-algebra generated by $\{X_s, s < t\}$ for any $t > 0$.

2.3 Stochastic Integration

Definition 16. *Let $\mathbf{M}^p([0,T],\mathbf{R})$ be the subspace of $\mathbf{L}^p([0,T],\mathbf{R})$ such that for any process $(X_t) \in \mathbf{M}^p([0,T],\mathbf{R})$ we have*

$$E\left(\int_0^T |X(t)|^p dt\right) < \infty.$$

Consider a Brownian motion B and a stochastic process (X_t) both adapted to a given filtration (F_t). We will define the following expression called stochastic integral

$$I_t(X) = \int_0^t X_s dB_t.$$

Now, let's use the stochastic integral of a simple process and then will define some of its properties.

Definition 17. *(Elementary process or simple process)*

A process $(X_t)_{t\in\mathbf{R}} \in \mathbf{L}^p([0,T],\mathbf{R})$ *is called a simple or elementary process if there exists a partition* $0 = t_0 < t_1 < \cdots < t_n = T$ *such that*

$$X_s(\omega) = \sum_{j=0}^{n} 1_{[t_j,t_{j+1}]}\theta_j(\omega),$$

where θ_j *is a bounded* F_{t_j}*-measurable random variable.*

Definition 18. *(Ito's integral) The Itô's Integral of the simple process* $(X_t)_{t\in\mathbf{R}} \in \mathbf{L}^2([0,T],\mathbf{R})$ *is define by*

$$I_t(X) = \int_0^t X_s dB_s := \sum_{j=0}^{n-1} \theta_j(B_{t_{j+1}} - B_{t_j}).$$

Lemma 19. *If* f *is an elementary function in* $\mathbf{L}^2([a,b],\mathbf{R})$ *and* B_t *a Brownian motion, then*

1. $E\left(\int_a^b f(t)dB_t\right) = 0.$
2. $E\left(\int_a^b f(t)dB_t\right)^2 = \int_a^b E(f^2(t))dt.$

Proof. 1. We have by definition

$$\int_a^b f(t)dB_t = \sum_{j=0}^{n-1} f_j(B_{t_{j+1}} - B_{t_j}).$$

By taking expectations from both sides, we get

$$E\left[\int_a^b f(t)dB_t\right] = \sum_{j=0}^{n-1} E(f_j)E(B_{t_{j+1}} - B_{t_j}) = 0,$$

since $B_{t_{j+1}} - B_{t_j}$ is a normal distribution with mean 0 and standard deviation $\sqrt{t_{j+1} - t_j}$.

2.

$$\left(\int_a^b f(t)dB_t\right)^2 = \left[\sum_{j=0}^{n-1} f_j(B_{t_{j+1}} - B_{t_j})\right]^2$$

$$= \sum_{j=0}^{n-1}(f_j)^2(B_{t_{j+1}} - B_{t_j})^2 + \sum_{l=0}^{n-1}\sum_{k=0,k\neq l}^{n-1} f_l f_k(B_{t_{l+1}} - B_{t_l})(B_{t_{k+1}} - B_{t_k}).$$

Taking expectations from both sides and using the independence of the increments of Brownian motion, we obtain

$$\begin{aligned} E\left(\int_a^b f(t)dB_t\right)^2 &= \sum_{j=0}^{n-1} E(f_j)^2 E(B_{t_{j+1}} - B_{t_j})^2 \\ &= \sum_{j=0}^{n-1} E(f_j)^2 E(t_{j+1} - t_j) \\ &= \int_a^b E(f^2(t))dt. \end{aligned}$$

□

Proposition 20. *[6] When $X = (X_t)_{t\geq 0} \in M^2([0,T],\mathbf{R})$, such that $E|X_t|^2 < \infty, \quad \forall t \geq 0$, then there is a sequence $(f_t^{(n)})_{t\geq 0}$ of simple process such that $E|f_t^{(n)}|^2 < \infty$ and*

$$\lim_{n\to\infty} E\left[\int_0^t |X_s - f_s^{(n)}|^2 ds\right] = 0.$$

Definition 21. *If $X = (X_t)_{t\geq 0} \in M^2([0,T],\mathbf{R})$, a stochastic integral of the process X is written as*

$$\int_0^t X_s dB_s := \lim_{n\to\infty}\int_0^t f_s^{(n)} dB_s,$$

where $f_t^{(n)}$ is the sequence of simple process converging almost surely to X according to the previous proposition. Moreover, using Ito isometry for elementary functions one can prove that the limit on this definition does not depend on the actual choice of $(f^{(n)})$.

Proposition 22. *(Properties of Itô integral) For any process $X = (X_t)_{t\geq 0} \in M^2([0,T],\mathbf{R})$, such that $E|X_t|^2 < \infty$, for any functions f, g in the set $M^2([0,T],\mathbf{R})$ and $0 \leq S < U < T$, the following holds:*

1. $\int_S^T f dB_t = \int_S^U f dB_t + \int_U^T f dB_t$ *almost surely.*

2. $\int_S^T (cf+g) dB_t = c\int_S^T f dB_t + \int_S^T g dB_t,\quad$ *for any constant c.*

3. $\int_S^T f dB_t$ *is $\mathbf{F_T}$-measurable.*

4. $E\left(\int_0^t X_s dB_t\right) = 0.$

5. $E\left(\int_0^t X_s dB_t\right)^2 = \int_0^t E(X_s^2) ds.$

You can find the proof in [6].

Proposition 23. *[6] For any elementary function $f^{(n)}\mathbf{F_t}$-adapted, the integral*

$$I_n(t,\omega) = \int_0^t f^{(n)} dB_r$$

is a $\mathbf{F_t}$-martingale.

Proposition 24. *(Generalization)*

Let $f(t,\omega) \in M^2([0,T],\mathbf{R})$ for all t. Then the integral

$$M_t(\omega) = \int_0^t f(s,\omega) dB_t$$

is a $\mathbf{F}$*-martingale and* $P\left[\sup_{0\le t\le T}|M_t|\ge\lambda\right]\le\frac{1}{\lambda^2}E\left[\int_0^T f^2(s,\omega)ds\right],\forall\lambda>0.$

The proof is in [6].

2.4 One Dimensional Ito Formula

Definition 25. *(1-dimensional Ito process) Assume that* B_t *is a Brownian motion on* $(\Omega,\mathbf{F},P)$*. An Itô process (or Stochastic integral) is any stochastic process* X_t *of the form*

$$X_t = X_0+\int_0^t u(s,\omega)ds+\int_0^t v(s,\omega)dB_t, \tag{2.2}$$

where $u\in L^1([0,T],\mathbf{R})$ *and* $v\in L^2([0,T],\mathbf{R})$.

Proposition 26. *(first 1- dimensional Ito formula) Let* $(\Omega,\mathbf{F},P)$ *be a complete probability space,* $(B_t)_{t\in\mathbf{R}_+}$ *a one-dimensional Brownian motion and* $f:\mathbf{R}\to\mathbf{R}$ *such that* f *is twice continuously differentiable. If* (X_t) *is any process of the form (2.2), then* $f(X_t)$ *is an Ito processes and*

$$f(X_t)=f(X_0)+\int_0^t f^{'}(X_s)u_s ds+\frac{1}{2}\int_0^t f^{''}(X_s)v_s^2 ds+\int_0^t f^{'}(X_s)v_s dB_s.$$

You can read the proof in [7]. One of the important notations for the stochastic integral defined over a Brownian motion is that it follows the normal distribution. It means that:

$$\int_0^t f(s)dB_s\sim N\left(0,\int_0^t|f(s)|^2ds\right). \tag{2.3}$$

Now, if we use the property of independent increments for Brownian motion, its second moment is given by

$$\begin{aligned}
E\left[\int_0^t f(s)dB_s\right]^2 &= \sum_{i,j=1}^{n} f_{i1} f_{j-1} E[(B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})] \\
&= \sum_{j=1}^{n} |f_{j-1}|^2 E[(B_{t_j} - B_{t_{j-1}})^2] \\
&= \sum_{j=1}^{n} |f_{j-1}|^2 (t_j - t_{j-1})
\end{aligned}$$

This gives

$$E\left|\int_0^1 f(s)dB_s\right|^2 = \sum_{j=1}^{n} |f_{j-1}|^2 (t_j - t_{j-1}) = \int_0^1 |f(s)|^2 ds. \tag{2.4}$$

2.5 Stochastic Differential Equations

In this section, several "famous" stochastic differential equations are used in the literature. For instance, the geometric Brownian motion is

$$dS_t = \mu S_t dt + \sigma S_t dB_t.$$

The below SDE

$$dr_t = (\alpha - \beta r_t)dt + \sigma\sqrt{r_t}dB_t \tag{2.5}$$

is the Cox–Ingerson–Ross model (CIR). $dY_t = \kappa(\theta - Y_t)dt + \gamma dB_t$, which is an Ornstein-Uhlenbeck process,

$$dX_t = a(X_t)dt + b(X_t)dB_t \tag{2.6}$$

which is the general form of SDE. The Itô formula is utilized often to find the solution of the SDEs and to price the financial derivatives products. Consider (2.6). Let $f : [a,b] \to R$ be a twice continuously differentiable function $(f \in C^2[a,b])$. Then the process $f(X)$ is a continuous and

$$f(X_t) - f(X_0) = \int_0^t f'(X_t) dB_t + \frac{1}{2} \int_0^t f''(X_t) d\langle W \rangle_t. \tag{2.7}$$

and the formula with two variables $f(t, x_t)$ is given by:

$$f(t,x_t) = f(0,x_0) + \int_0^t \frac{\partial f}{\partial t} dt + \int_0^t \frac{\partial f}{\partial x} dx_t + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2} d\langle x, x \rangle_t. \tag{2.8}$$

Note that, to determine the quadratic variation $d\langle B \rangle$, we will use the following properties:

$$\begin{aligned} \langle dB_t, dB_t \rangle &= dt \\ \langle dB_t, dt \rangle &= 0 \\ \langle dt, dB_t \rangle &= 0 \\ \langle dt, dt \rangle &= 0 \end{aligned}$$

2.6 Some Applications of Ito's Formula

2.6.1 Geometric Brownian Motion

Our aim in this section is to solve the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t. \tag{2.9}$$

that will define the price (S_t) of a risky asset at time t, where $\mu \in R$ and $\sigma > 0$. Then

$$S_t = S_0 + \mu \int_0^T S_s ds + \sigma \int_0^T S_s dB_s, \quad t \in R_+.$$

It can be solved by applying Ito's formula (2.7-2.8) to $f(S_t) = \ln S_t$ with $f(x) = \ln x$, which shows that

$$\begin{aligned} df(S_t) &= \mu S_t f^{'}(S_t)dt + \sigma S_t f^{'}(S_t)dB_t + (1/2)(\sigma)^2(S_t)^2 f^{''}(S_t)dt \\ &= \mu dt + \sigma dB_t - (1/2)(\sigma)^2 dt, \end{aligned}$$

hence

$$\ln S_t - \ln S_0 = \int_0^t d\ln S_u = \left(\mu - (1/2)(\sigma)^2 \int_0^t\right) du + \sigma \int_0^t dB_u.$$

Then

$$\ln S_t = \ln S_0 + \left(\left(\mu - (1/2)(\sigma)^2\right)t + \sigma B_t\right).$$

Then the solution of the Equation (2.9) is given by

$$S_t = S_0 \exp((\mu - \frac{1}{2}\sigma^2)t + \sigma\beta_t), \quad t \in R_+$$

2.6.2 Cox Ingerson Ross Model (CIR)

CIR is a mathematical formula used to model interest rate movements. It is used as a method to forecast interest rates and is based on a stochastic differential equation. In mathematical finance, the Cox–Ingersoll–Ross (CIR) model describes the evolution of interest rates. It is a type of "one factor model" (short rate model) as it describes interest rate movements as driven by only one source of market risk. Consider the equation

$$dr_t = (\alpha - \beta r_t)dt + \sigma\sqrt{r_t}dB_t \tag{2.10}$$

modeling the variations of a short term interest rate process r_t, where α, β, σ and r_0 are positive parameters.

2.6.3 Ornstein-Uhlenbeck Process

Example 27. *Apply Ito formula to express the process $(X_t)_{0\leq t\leq T}$ given by the following SDE:*

$$
\begin{aligned}
dX_t &= rX_t dt + \sigma dB_t. \\
X_0 &> 0. \qquad (given)
\end{aligned}
$$

Solution: Assume that $f(t,x) = e^{-rt}x$. We can rewrite equation (2.9) as:

$$f(T,X_T) = f(0,X_0) + \int_0^T f_t dt + \int_0^T f_x dX_t + \frac{1}{2}\int_0^T f_{xx} d\langle X,X\rangle_t. \tag{2.11}$$

where $f_t = -re^{-rt}x, f_x = e^{-rt}$ and $f_{xx} = 0$.

Now, using (2.11) , we have:

$$
\begin{aligned}
e^{-rt}X_T &= X_0 - r\int_0^T e^{-rt}X_t dt + \int_0^T e^{-rt}(rX_t dt + \sigma dB_t) \\
&= X_0 + \sigma\int_0^T e^{-rt} dW_t \\
X_T &= e^{rT}X_0 + \sigma\int_0^T e^{r(T-t)} dB_t.
\end{aligned}
$$

This process is named as Ornstein-Uhlenbeck process.

Now, let us compute the expected value and the variance of X_T. The Expected value can be computed by:

$$
\begin{aligned}
E[X_T] &= e^{rT}X_0 + \sigma E[\int_0^T e^{r(T-t)} dB_t] \\
&= X_0 e^{rT}.
\end{aligned}
$$

While the variance can be calculated as:

$$\begin{aligned}
Var[X_T] &= E[(X_T - E(X_T))^2] \\
&= E[(e^{rT}X_0 + \sigma \int_0^T e^{r(T-t)} dW_t - X_0 e^{rT})^2] \\
&= \sigma^2 E[(\int_0^T e^{r(T-t)} dW_t)^2] \\
&= \sigma^2 . \int_0^T (e^{r(T-t)})^2 dt \\
&= \sigma^2 . \int_0^T (e^{2r(T-t)} dt \\
&= \sigma^2 . e^{2rT} \int_0^T e^{-2rt} dt \\
&= -\frac{\sigma^2}{2r}[1 - e^{2rT}].
\end{aligned}$$

Generally, the Ornstein–Uhlenbeck process is defined by the following stochastic differential equation:

$$dx_t = -\theta\, x_t\, dt + \sigma\, dB_t$$

where, $\theta > 0$ and $\sigma > 0$ are parameters and B_t denotes a Wiener process.

An additional drift term is sometimes added $dx_t = \theta(\mu - x_t)dt + \sigma\, dW_t$ where μ is a constant.

Chapter 3: Continuous Time Markov Chain

Markov chains are mathematical systems that switch from a set of values, a situation, or a state to another. They are applied in many sciences and known to be advantageous to modeling real-life problems arising from finance such as stock price movement. Our interest in this thesis is to utilize Continuous Time Markov Chain (CTMC) processes. A brief introduction to CTMC processes is given in this chapter. The works of [8], [9], and [10] offer more details about this topic.

3.1 Markov Chains in Continuous Time

The Markov property for continuous-time processes can be stated as follows. For a continuous-time stochastic process $(X_t)_{t\geq 0}$ with state space S, we say it has the Markov property if

$$P\left(X(t)=j \mid X(s)=i, X\left(t_{n-1}\right)=i_{n-1}, \ldots, X\left(t_1\right)=i_1\right)=P(X(t)=j \mid X(s)=i).$$

The memoryless property of the exponential distribution implies that the process satisfies the Markov property, the future, $X(s+t): t \geq 0$ given the present state, X_s,is independent of the past, $X(u): 0 \leq u < s$.

The formal definition is given by

Definition 28. *A continuous-time stochastic process $(X_t)_{t\in[0,T]}$ is called a continuous-time Markov chain if it has the Markov property.*

If $P_{ij}(t)$ represents the probability that the chain will be in state j, t time units from now, given it is in state i now. It moves according to a transition matrix $P = P_{ij}(t)$. Thus a CTMC can be defined by a transition matrix $P = P_{ij}$.

3.2 Two States CTMC

In this section, we are going to consider a continuous Markov process with only two states. Let $(x_t)_{0\leq t\leq T}$ be a continuous Markov process with:

1. State space $S=\{0,1\}$ i.e. 2-state space.
2. Transition semi-group: $p(t)=[p_{i,j}(t)]_{i,j\in S}$, where $p_{i,j}(t)=p[x_{t+s}=j|x_s=i]$.

Example 29. *Let* $S=\{0,1\}$, *we have:* $p(t)=\begin{bmatrix} p_{0,0}(t) & p_{0,1}(t) \\ p_{1,0}(t) & p_{1,1}(t) \end{bmatrix}$

The Markov property can be written as

- $p(x_t=j|x_s=i_n, x_{s-1}=i_{n-1},\dots,x_{s_0}=i_0)=p(x_t=j|x_s=i_n)$. where $0<s_0<s_1<\cdots<s_{n-1}<s<t$.
- $p_{i,j}(t+s)=\sum_{l\in s} p_{i,l}(s)p_{i,l}(t)$.

The infinitesimal generator Q of $(x_t)_{t\in\mathbf{R}_+}$ is $Q:=p'(0)=\lim_{s\to 0}\frac{p(s)-p(0)}{s}$.

For two states: $Q=\begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}$ where $\alpha,\beta>0$. The Forward Kolmogorov equation is given by:

$$\begin{aligned} p'(t) &= p(t)Q \\ &= p(t)\begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix} \end{aligned}$$

The solution is given by:

$$p(t)=\frac{1}{\alpha+\beta}\left\{\begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix}+\begin{bmatrix} \alpha & -\alpha \\ -\beta & \beta \end{bmatrix}e^{-t(\alpha+\beta)}\right\}$$

See [8].

3.3 Simulation of Continuous-Time Markov Chain with Two States

Consider a two state Markov process $(x_t)_{0 \le t \le T}$ with $x_t = 0$ or $x_t = 1$. These states represent the economy by either good economy $(x_t = 0)$ or poor economy $(x_t = 1)$. Let our time period is $t \in [0, T]$. We will always assume that the process starts with good economy $x_t = 0$. The process will continue at this state until time $\tau \in [0, T]$, then it will change to the other state $x_t = 1$. The purpose of the section is to find the probability of changing from one state to other.

3.3.1 Python Code

```
# This Project is Simulation of the Continuous
Time Markov Chain( CTMC )
import math
import numpy as np
import matplotlib.pyplot as plt
S0 = 10
T = 1  # Maturity date
r = 0.04  # interest rate
sigma = 0.35  # Volatility of the stock price
N = 1000000  # Number of discretization
delta = T / N  # Step size
Pii = 3.14
Xtime = []
S0 = 10
X0 = 1              #CTMC at t = 0
X = np.zeros(N)   # CTMC
```

```
for l in range(N - 1):
    X[l]=X0
#### SIMULATION OF THE CTMC
Rate1 = 3.5
Rate2 = 2.8
timet = 0.
curstate = 1
while timet < T:
 # generate holding time
    HoldTime = np.random.exponential
    (scale=1./Rate1, size=None)
    timet = timet + HoldTime
    if timet < T:
        if curstate == 1:
          curstate = 2
        else:
          curstate = 1
        k = int(timet * N / T)
        for l in range(k, N - 1):
           X[l] = curstate

for i in np.linspace(0, T, N):  # Vector of time
    Xtime.append(i)
```

3.3.2 Illustrations

This subsection provides some Illustrations of the CTMC process.

After several runs of the simulations, we get the below illustrations.

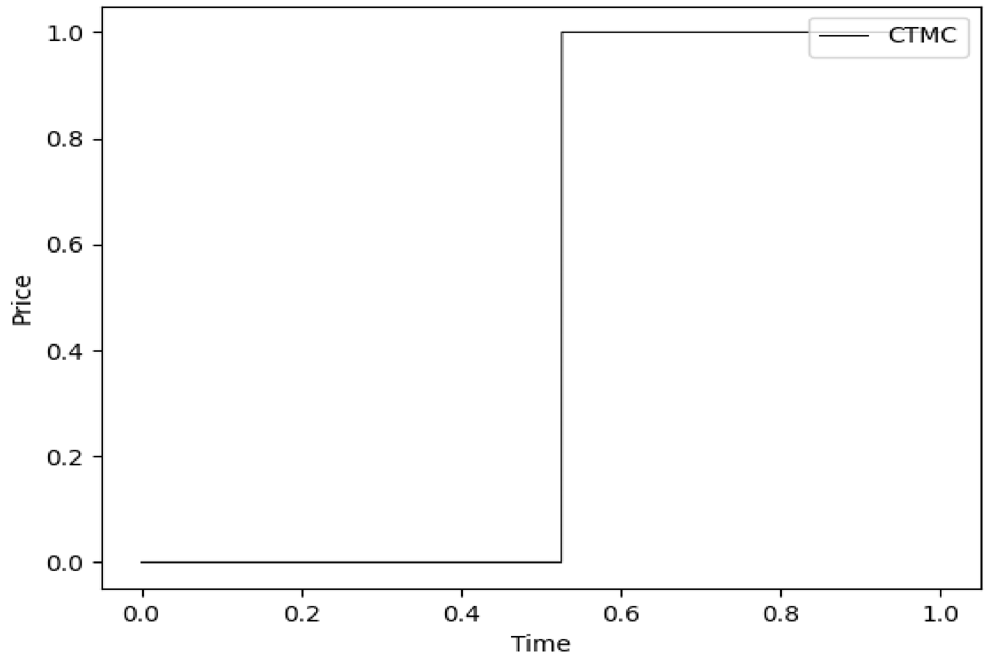

Figure 3.1: Realizations of the CTMC. First run of the simulations.

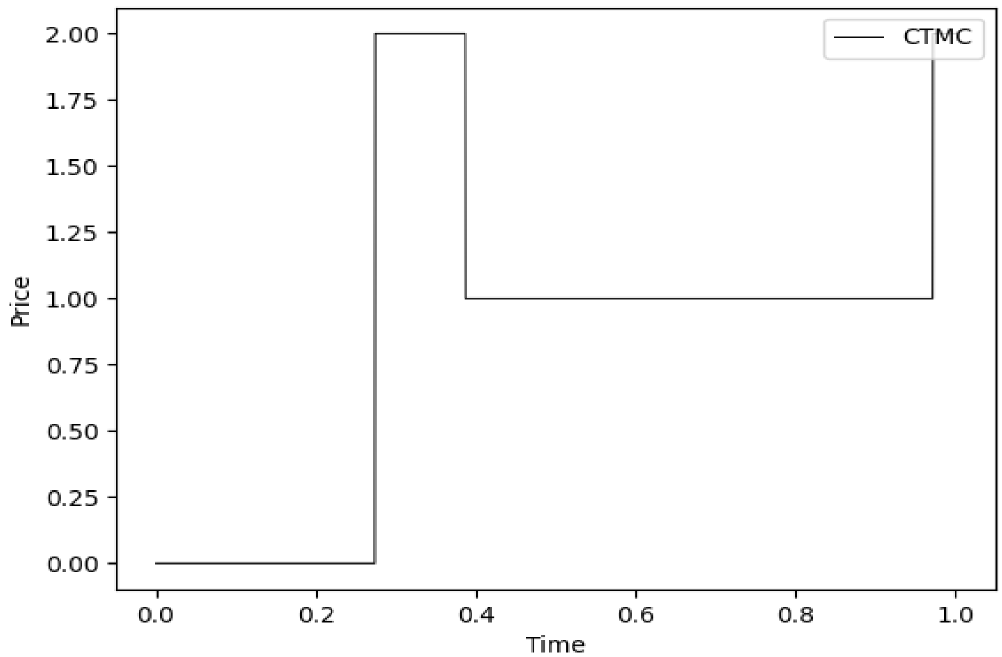

Figure 3.2: Realizations of the CTMC. A second run of the simulations.

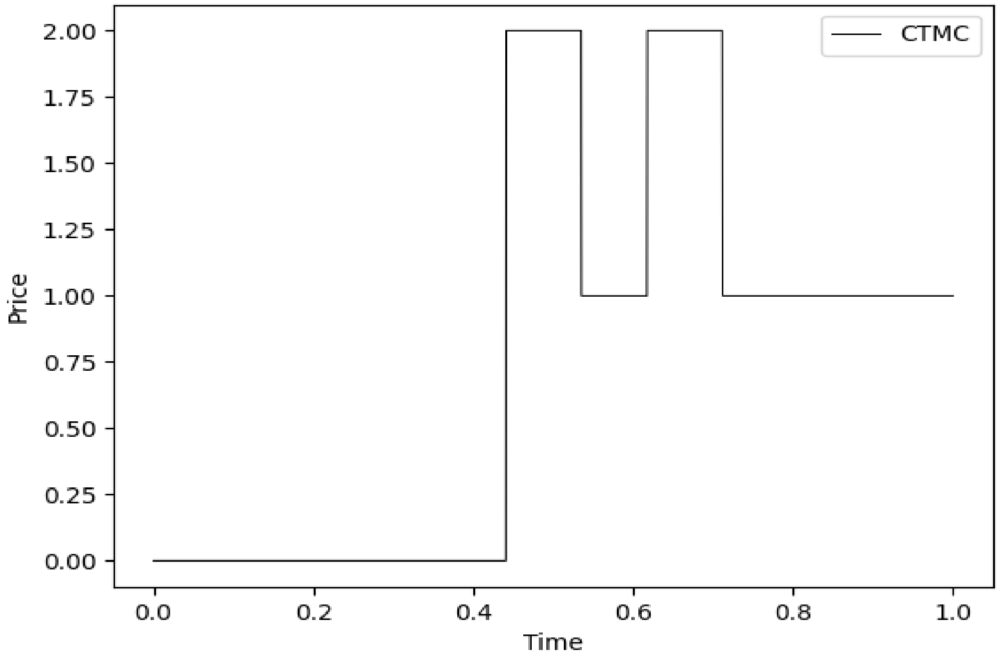

Figure 3.3: Realizations of the CTMC. A third run of the simulations.

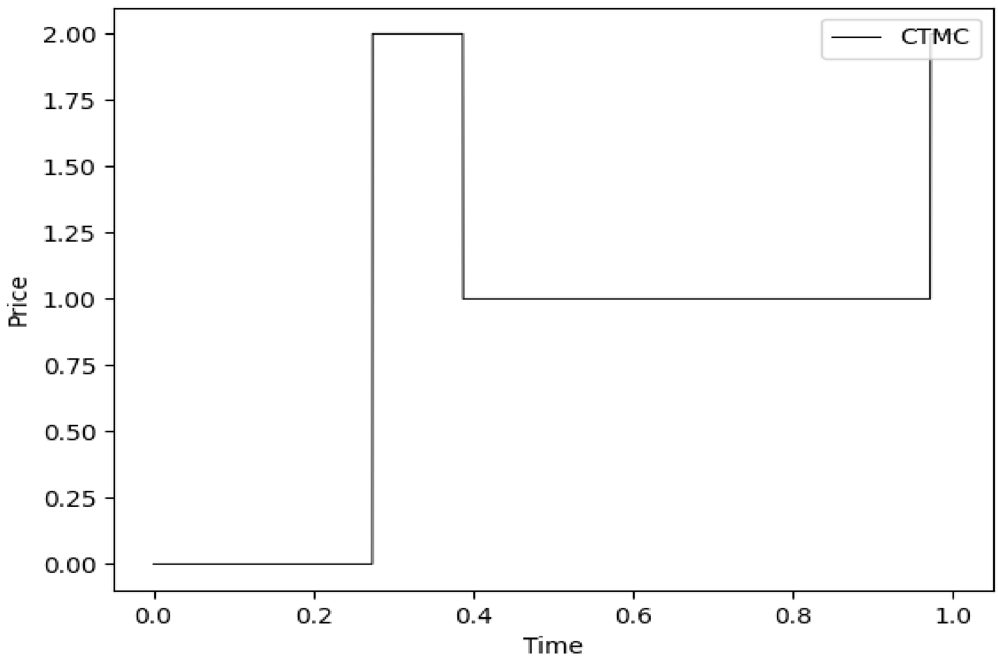

Figure 3.4: Realizations of the CTMC. A fourth run of the simulations.

Chapter 4: High Volatile Regime Switching Market

Regime-switching models are well-known models nowadays, see [11] for example. Many research works exist in the literature on regime-switching models. The studies of [4] [12], [13], [14], [15], and [16], are some samples of these investigations. In this chapter, we provide our main contribution. The previous research papers were on finance, but there also exist studies on modeling with regime switching for other purposes as in [17], and [18] for instance. A model combining the high volatility model of [3] and a regime-switching model is considered. Pricing European options is the main goal. To this end, consider a European option call with the following parameters:

- $(S_t)_{(0 \leq t \leq T)}$ is the price of the underlying asset from $t = 0$ to maturity $t = T$
- S_0: current spot price (value of the underlying asset at $t = 0$)
- r: interest rate
- T: maturity time, agreed time for exercise
- K: strike, represents the agreed price of the asset in the contract at maturity
- σ: volatility of the asset price, it is a measure of how much an asset has a tendency to variate.

The premium price of the call option denoted by C can be expressed as:

$$C = E[f(S_T)]e^{-rT},$$

and in the case of a European call option, the premium is

$$C = E[(S_T - K)^+]e^{-rT}$$

Where $(S_T - K)^+ = \max(S_T - K, 0)$. This is thanks to the relationship between future and the present value of the money and the Payoff $(S_T - K)^+$.

4.1 The Research Questions

Knowing that the premium is

$$C = E[(S_T - K)^+]e^{-rT}$$

- Question 1: How to predict S_T which is the price of the underlying assets at maturity?
- Question 2: How to evaluate the expected value?

4.1.1 Answers

- Answer 1- Prediction model: using Stochastic Differential Equations(SDE's) to predict trajectories for the assets price $(S_t)_{0\leq t\leq T}$
- Answer 2- Using probability measures theory tools if possible or numerical techniques for finance.

The next section is dedicated to the presentation of our model.

4.2 Modeling the Underlying Asset Prices

A model that describes perfectly the behavior of a financial asset price does not exist. Stochastic models are suggested to take into consideration stylized facts observed in the Markets. One of the most popular models is the Black-Scholes model introduced by [1] in 1973. Its popularity comes from the fact that it is the first model providing a closed-form solution for the option pricing problem. However, the model has some shortcomings such as constant volatility. In the next subsection, we provide brief information about the Black-Scholes model.

4.2.1 Black and Scholes Model

The Black Scholes model The market is consisting of two assets, a riskless asset given by

$$dA_t = rA_t dt, \quad t \in [0,T], \quad A_0 = 1,$$

and stock with prices

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \quad t \in [0,T], \quad S_0 = x > 0,$$

The below assumptions are adopted in the Black-Shcoles model of [1]:

- no arbitrage (no profit without taking risk)
- every contingent claim is attainable
- no transaction costs
- volatility is constant
- no jumps

Under the above assumptions, the option price at the initial time (the premium) is

$$C(S_0, r, \sigma, T) \quad = \quad e^{-rT} E\left[\max(S_T - K, 0)\right]$$

can be explicitly calculated and obtain the famous Black-Scholes formula

$$\begin{aligned} C &= N(d_1) S_0 - N(d_2) K e^{-rT} \\ d_1 &= \frac{1}{\sigma\sqrt{T}} \left[\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T \right] \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

where $N(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$ denotes the standard normal cumulative distribution function. Notice that the volatility σ cannot be observed from the market.

4.3 Our Alternative Model

A model for a financial asset price $(S_t)_{t\in[0,T]}$ has in general this form

$$\frac{dS_t}{S_t} = \mu(t)dt + \sigma(t)dY_t, \quad t \in [0,T], \quad S_0 > 0,$$

where here μ is the expected rate of return (called the drift): deterministic or constant for most of the models in the literature.

The volatility σ is a measure of the uncertainty that can be deterministic or constant in general, but when stochastic the model is said to be a "stochastic volatility model". The process $(Y_t)_{t\in[0,T]}$ is a stochastic process usually the Brownian motion. Other possibilities: Poisson process (pure jumps), a combination of Brownian and Poisson (jump diffusion, Lévy). In [3] a stochastic model for a market suffering from a financial crisis is studied. The underlying asset price is driven by the following stochastic differential equation

$$dS_t = \mu S_t dt + (\sigma S_t + \beta g(t))dW_t, \quad S_0 > 0. \tag{4.1}$$

The above model was introduced in [19] and later pricing and hedging solutions were obtained in [3] for the case $g(t) = e^{-rt}$. The model is based on empirical research papers (see for instance [20]). In contrast with the Black-Scholes model which assumes that the volatility is constant. The above model can be seen as a generalization of the Black-Scholes model since it allows the volatility to be stochastic. Moreover, the volatility is inversely proportional to the underlying asset price. This feature of the volatility models the leverage effect stylized fact. However, the model (4.1) does not deal with the jumps observed in the underlying asset prices because of an event external to (or independent from) the underlying asset itself. This stylized fact was encompassed in the model of [4] where the dynamics of asset price process by a Markov-modulated Geometric Brownian Motion as below

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \quad S_0 > 0, \tag{4.2}$$

where here $(X_t)_{(t\in[0,T])}$ denotes a continuous-time hidden Markov Chain process with a finite state space $X := (x_1, x_2, ..., x_N)$. We suppose that X and W are independent. Moreover, it is assumed that

$$\begin{aligned}
\mu_t := \mu(t, X_t) &= <\mu, X_t>, \quad \mu = (\mu_1, \dots, \mu_N) \\
\sigma_t := \sigma(t, X_t) &= <\sigma, X_t> \quad \sigma = (\sigma_1, \dots, \sigma_N) \\
r_t := r(t, X_t) &= <r, X_t> \quad r = (r_1, \dots, r_N).
\end{aligned} \tag{4.3}$$

In this work, we consider a model which includes the features of the crisis model (4.1) and the regime-switching model (4.2-4.3). Our suggested model can be seen as a hybrid regime-switching with augmented volatility.

Consider a filtered probability space $(\Omega, {F^W}_T, ({F^W}_t)_{t\in[0,T]}, P)$.
The stochastic process $W := (W(t))_{t\in[0,T]}$ represents a 1-dimensional Brownian motion and the filtration
$({F^W}_t)_{t\in[0,T]}$ is the natural filtration generated by the Brownian motion W. The market is composed of two assets, risky-free asset $(A_t)_{t\in[0,T]}$ and a risky asset (underlying asset) with price process $(S_t)_{t\in[0,T]}$. Moreover, a continuous time Markov chain process models the external events that impact the prices of the stock. We assume that there is a European option built on the asset S. The risk-free asset values over time are given by the differential equation

$$dA_t = rA_t dt, \quad t \in [0, T], \tag{4.4}$$

where r here denotes the free interest rate.

The stochastic differential equation representing the evolution of the underlying asset S is

$$dS_t = r(X_t)S_t dt + (\sigma(X_t)S_t + \alpha(X_t)e^{\int_0^T r(X_s)ds})dW_t, \quad t \in [0, T], \quad S_0 > 0. \tag{4.5}$$

The above model as mentioned before includes the models of [3] and [4]. It is assumed that the values of σ, α, and r are all affected by the values of the CTMC X_t. As in (4.3), the below equation determines their relationships:

$$\begin{aligned} \sigma(X_t) &= <\sigma, X_t> \quad \text{where} \quad \sigma = (\sigma_1, \ldots, \sigma_N) \\ r(X_t) &= <r, X_t> \quad \text{where} \quad r = (r_1, \ldots, r_N) \\ \alpha(X_t) &= <\alpha, X_t> \quad \text{where} \quad \alpha = (\alpha_{1t}, \ldots, \alpha_{Nt}) \end{aligned} \tag{4.6}$$

The above model is assumed to be under an equivalent martingale measure P where the discounted prices are P-martingales. In addition, the market is subject to N different economic situations or states. In our work here, we will assume that the economy of the market is subject only to two states. In this case, the state space of the process X_t is $\{1,2\}$.
These states represent two different economic situations:
bad for $X_t = s_1 = 1$, and normal $X_t = s_2 = 2$.

The two situations can be described below:
First case: $X_t = 1$ or the market is facing a bad economic situation

- $\sigma(X_t) = \sigma_1$ volatility is very high.
- $\alpha(X_t)$ this parameter will increase the volatility as a positive constant.

Second case: $X_t = 2$ the market is in a normal economic situation

- $\sigma(X_t) = \sigma_2 > 0, \sigma_2 < \sigma_1$
- $\alpha(X_t) = 0$

The model has the following assumptions

- the risk free rate, r is depending on the CTMC X_t
- no dividends are paid.
- costs of transactions are not considered.
- there is a possibility of short selling.

- the variance of the original asset can increase across the time span and it is also dependent on the CTMC X_t.

4.4 Option Pricing

In this section, we investigate a solution of the SDE in our model (4.5-4.6), then we study the option pricing problem under our regime-switching model.

4.4.1 Solution of the SDE in the High Volatile Model

The solution of the underlying asset price is important in order to obtain a solution for the premium of the option. The next proposition provides a solution to the stochastic differential equation in (4.5). The proof is based on the work of [3].

Proposition 30. *The solution of the stochastic differential Equation (4.5) is given by*

$$S_T = S_0 \zeta_T - \int_0^T \alpha_t e^{\int_0^t r_s ds} \zeta_{T-t} [\sigma_t dt - dB_t]. \tag{4.7}$$

where

$$\zeta_T := \exp\left[\int_o^T \left(r_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^T \sigma_s dW_s\right] \tag{4.8}$$

Proof. We follow [3] the process ζ_t satisfies $d\zeta_t = r_t \zeta_t dt + \sigma_t \zeta_t dW_t,\ \zeta_0 = 1$. Using Ito formula, one can find out that $S_0 \zeta_t$ is the solution of (4.5) when α is equal to 0.

Now the variation of the constant method is utilized by searching for a solution of (4.5) in the form of $S_t = Y_t \zeta_t$, with $Y_0 = S_0 = x$, is employed. The Itô formula is utilized again to obtain a solution of Y_t. Finally, the solution of S_t is obtained as $S_t = Y_t \zeta_t$, and it is given by (4.7). □

4.4.2 On the Valuation of the Premium

In this subsection, we discuss the value of the option in the regime-switching highly volatile model as presented in (4.5). The below proposition provides a relationship between the premium of the regime-switching with constant volatility and the premium in our model.

Proposition 31. *Let $C^{RS}_{HV}(S_T, K)$ be the premium of our model where the dynamic of the price process, S_T, is given by (4.5-4.6) and $C^{RS}(S_0\zeta_T, K)$ be the premium of a European call option with the same parameters except that $\alpha = 0$, in other words, without an increase in the volatility, then we have the relation*

$$C^{RS}_{HV}(S_T, K) = C^{RS}(S_0\zeta_T, K^{'}), \tag{4.9}$$

where $K^{'} = S_0\zeta_T - \int_0^T \alpha_t e^{\int_0^t r_s ds}\zeta_{T-t}[\sigma_t dt - dB_t + K$

Proof. We use the fact that

$$C^{RS}_{HV}(S_T, K) = E[e^{\int_0^T -r_s ds}(S_T - K)^+] \tag{4.10}$$

and the solution of S_T given in (4.7). □

4.5 Numerical Simulations

The main focus of the thesis is to study the price of a European option under a highly volatile regime-switching model. In this section, simulating the underlying asset prices and the price of European options are investigated using numerical methods for finance. The book of [21] provides numerical techniques and methods for finance and stochastic calculus needed in our study.

4.5.1 Asset Prices

The simulation of the underlying asset price trajectories is essential in the pricing of the option itself. We first present the methodology used to simulate a stochastic differential equation. Assume there is a stochastic process $Z := (Z_t)_{t\in[0,T]}$ driven by the following SDE

$$dZ_t = a(t, Z_t, X_t)dt + b(t, Z_t, X_t)dW_t, \tag{4.11}$$

where Z_0 is a given constant real number. The above process is dependent also on the CTMC X_t. The algorithm to simulate the Equation (4.11) is to simulate the Brownian motion and the CTMC X_t over a continuous time period independently, then generate a path of the underlying asset price. Assume that there are discrete versions $\tilde{W}$, $\tilde{X}$, $\tilde{Z}$ of the processes W, X, and Z with values over a finite number of times denoted by N. Here N is also the number of time steps and let $\Delta t := \frac{T}{N}$ is the time step size. The denotations $\tilde{W}_k$, $\tilde{X}_k$, and $\tilde{Z}_k$ are employed instead of $\tilde{W}_{(k\Delta t)}$, $\tilde{X}_{(k\Delta t)}$, and $\tilde{Z}_{(k\Delta t)}$. The discretization of (4.11) can be written as

$$\tilde{Z}_1, \ldots, \tilde{Z}_k, \ldots, \tilde{Z}_N$$

There are various methods to discretize a stochastic differential equation among others Euler-Maruyama, Milstein, or Runge-Kutta. In this thesis, we use the Euler-Maruyama scheme to get a discretized trajectory of Z to form the SDE (4.11).

The algorithm utilized in this thesis can be summarized in the following steps

1. simulate $\tilde{X}_k$ the discretized version of the continuous-time Markov chain process as explained in the previous chapter

2. simulate $\Delta\tilde{W}_k$ as normally distributed random variable $N(0, \Delta t)$

3. set $\tilde{Z}_0 := Z_0 = z$ and evaluate $\tilde{Z}_{k+1}$ using

$$\tilde{Z}_{k+1} = \tilde{Z}_k + a(k\Delta t, \tilde{Z}_k, \tilde{X}_k)\Delta t + b(k\Delta t, \tilde{Z}_k, \tilde{X}_k)\Delta\tilde{W}_k, \tag{4.12}$$

for $k = 0, \ldots, N-1$. Notice that $\Delta\tilde{W}_k = \tilde{W}_{k+1} - \tilde{W}_k$. We will omit, from now on, the use of the symbol ˜ for discretized version of a given SDE.

4.5.2 Option Price

This subsection is dedicated to the simulation of the premium of a European call option given by (4.10) where the underlying asset price is governed by the SDE (4.5). To reduce the computational time the antithetic variable technique. This variance reduction approach enhances the simulations results. Let t_n be a fixed time in $[0,T]$, we simulate two outputs $S_{t_n}^{+}$ and $S_{t_n}^{-}$, the first one utilizing Z_n and the second is obtained via $-Z_n$. Totally, $2N$ values are obtained for S but with only N trials and the simulations are performed M times. A total of M paths are produced for S

$$S_{m,t_1}^{+}, \ldots, S_{m,t_N}^{+}, S_{m,t_1}^{-}, \ldots, S_{m,t_N}^{-}, \qquad m = 1, \ldots, M.$$

The payoff is computed for each path

$$f_m(S_{m,t_N}) = \frac{\max\{S_{m,t_N}^{+} - K, 0\} + \max\{S_{m,t_N}^{-} - K, 0\}}{2}, \qquad m = 1, \ldots, M.$$

Monte Carlo is employed to calculate the price of the call as below

$$C = \frac{e^{-\int_0^T r_s ds}}{M} \sum_{m=1}^{M} f_m(S_{m,t_N}).$$

Below are the values of the parameters selected for our simulations $S_0 = 20, S_{min} := 2$, $S_{max} = 80, r = 0.06, g(t) = 0, \alpha = 100, T = 1, N = 10000$, and $\sigma = 1.5$ on the interval $[0,1]$.

4.5.3 Python Code

Below, is the code program of our simulations

```
# This is a sample Python script.

# Press R to execute it or replace it with your code.

# Press Double  to search everywhere for classes

, files, tool windows, actions, and settings.

import math

import numpy as np

import matplotlib.pyplot as plt

#### SIMULATION OF THE CTMC

Rate1 = 3.5

Rate2 = 2.8

timet =0.

curstate =1

while timet < T:

# generate holding time

HoldTime = np.random.exponential(scale=1.

 / Rate1, size=None)

timet = timet+HoldTime

if timet<T:

        if curstate == 1 :

             curstate =2

        else: curstate =1

        k = int(timet*N/T)
```

```
        for l in range(k, N - 1):
            X[l] = curstate
for i in np.linspace(0, T, N):
    Xtime.append(i)
BM = np.zeros(N)  # Brownian motion

SS = np.zeros(N)  # Stock price

BM[0] = 0.

SS0] = S0
for l in range(N - 1):
BM[l + 1] = BM[l] + np.random.randn()
* math.sqrt(deltat)
SS[l + 1] = SS[j] *
(1 + mu * deltat +
sigma * (BM[l + 1] - BM[l]))
fig, ax = plt.subplots()
ax.plot(Xtime, SS, color='red', lw=0.4)
ax.legend(['Stock Price BS'], loc=3)
plt.title("Simulation of stock price trajectory")
plt.xlabel('Time')
plt.ylabel('CTMC')
plt.show()
```

4.5.4 Illustrations

This subsection provides some illustrations of the CTMC process.

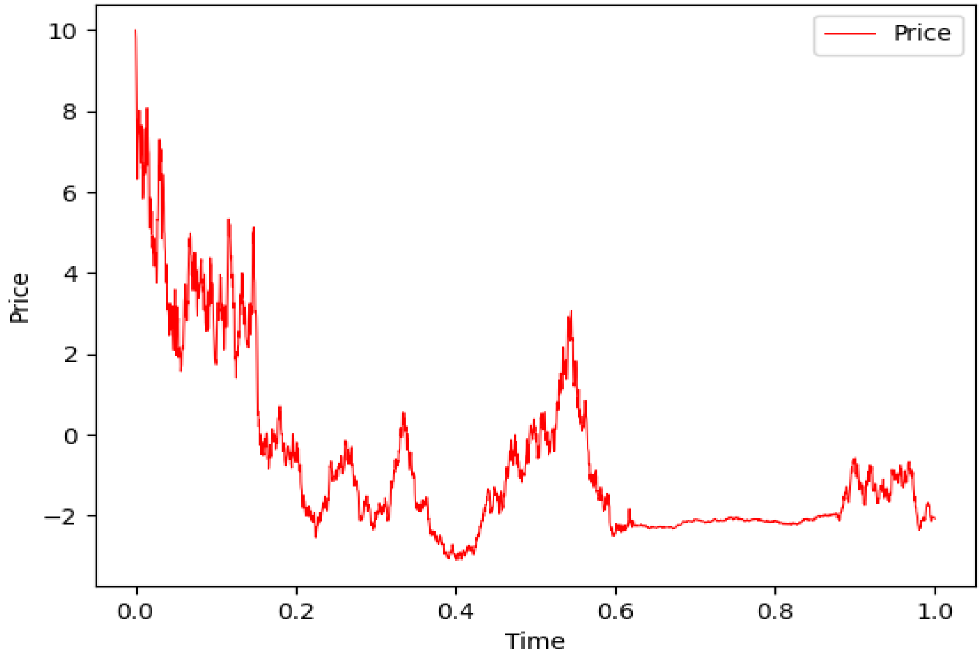

Figure 4.1: Realizations of the asset.The First run the simulations.

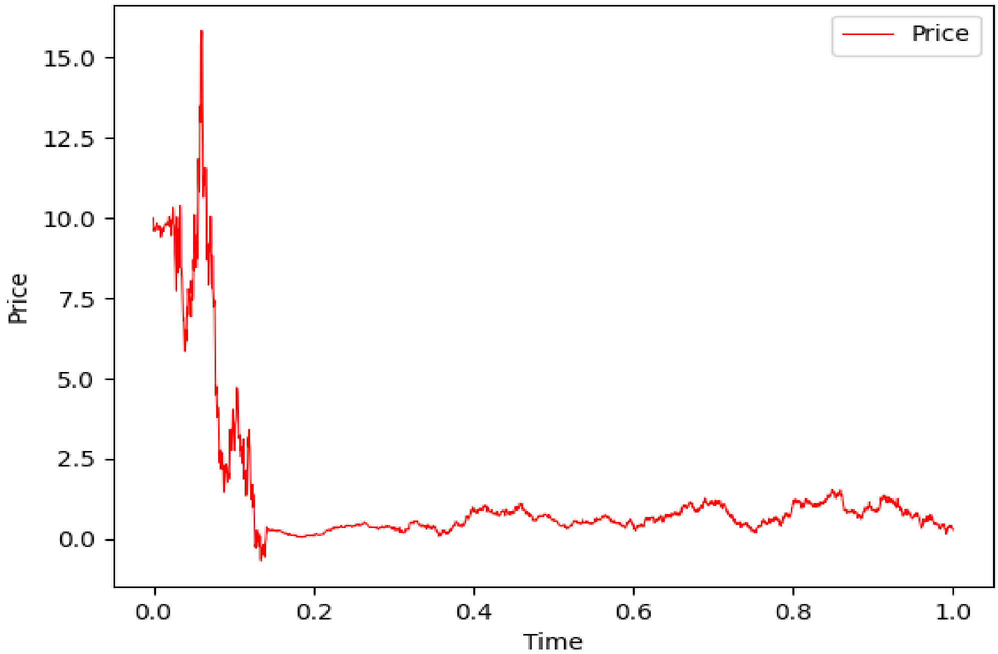

Figure 4.2: Realizations of the asset. The second run of the simulations.

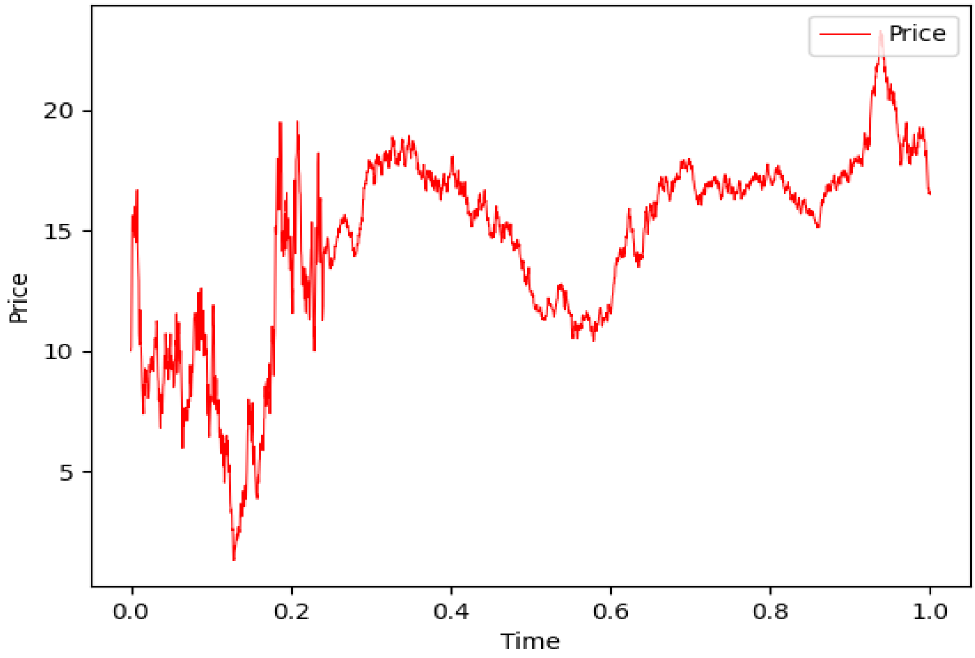

Figure 4.3: Realizations of the asset. The third run of the simulations.

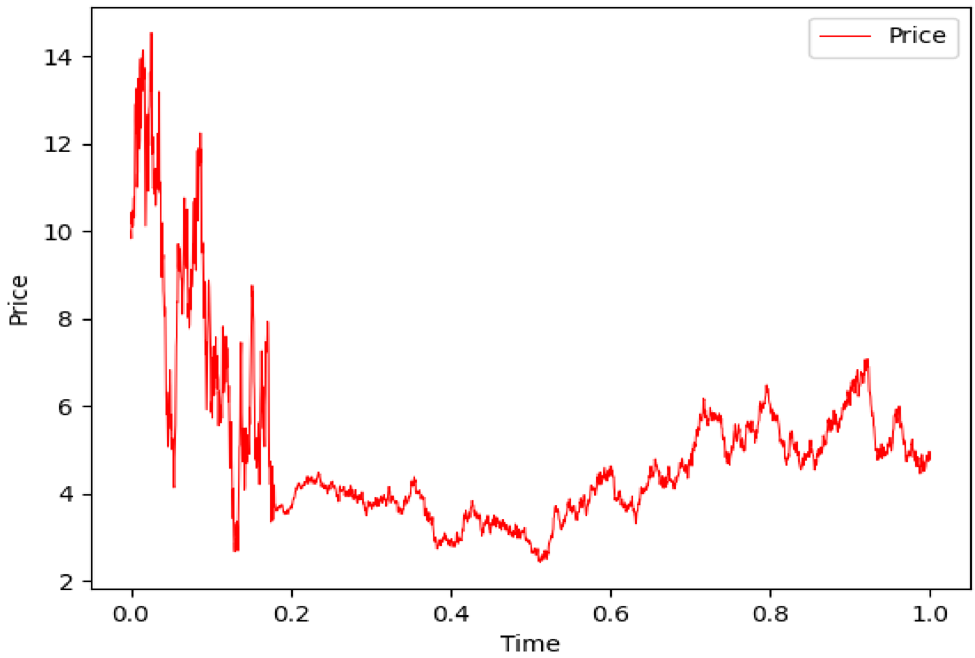

Figure 4.4: Realizations of the asset. The fourth run of the simulations.

Chapter 5: Conclusion

Many real-life phenomena such as stock market conditions, weather changes, or oil prices can be seen as stochastic processes that are depending on hidden variables. These quantities can be investigated utilizing regime-switching models with a hidden process. In this thesis, a regime-switching model in a highly volatile market is built. More precisely, the suggested model combines two important stylized facts observed in financial markets. The first is the impact of a crisis on the volatility of an underlying asset. The second property considered in this model is the effect of an event external to the market represented by a continuous-time Markov chain. We investigated a solution for the underlying asset price SDE. Then pricing problem and evaluating the expected value of the payoff and thus the price of the options are studied. Numerical methods for finance have been utilized to show the goodness of the model. Coding in python is performed and several illustrations are offered. The suggested model overcomes two shortcomings of the Black and Scholes model by adding regime switching and a parameter to increase the volatility.

References

[1] F. Black and M. Scholes, "The pricing of options and corporate liabilities," *Journal of Political Economy*, vol. 81, pp. 637–654, 1973.

[2] S. L. Heston, "A closed-form solution for options with stochastic volatility with applications to bond and currency options," *The review of financial studies*, vol. 6, no. 2, pp. 327–343, 1993.

[3] Y. El-Khatib and A. J. Hatemi, "Option valuation and hedging in markets with a crunch," *Journal of Economic Studies*, vol. 44, no. 5, pp. 801–815, 2017.

[4] R. J. Elliott, L. Chan, and T. K. Siu, "Option pricing and esscher transform under regime switching," *Annals of Finance*, vol. 1, no. 4, pp. 423–432, 2005.

[5] J. C. Hull, *Options futures and other derivatives*. Pearson Education India, 2003.

[6] B. Oksendal, *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media, 2013.

[7] B. Øksendal and A. Sulem, *Stochastic Control of jump diffusions*. Springer, 2005.

[8] N. Privault, "Understanding markov chains," *Examples and Applications, Publisher Springer-Verlag Singapore*, vol. 357, p. 358, 2013.

[9] S. Brooks, A. Gelman, G. Jones, and X.-L. Meng, *Handbook of markov chain monte carlo*. CRC press, 2011.

[10] A. D. Martin, K. M. Quinn, and J. H. Park, "Mcmcpack: Markov chain monte carlo in r," 2011.

[11] J. D. Hamilton, "Regime switching models," in *Macroeconometrics and time series analysis*, pp. 202–209, Springer, 2010.

[12] N. P. Bollen, “Valuing options in regime-switching models,” *Journal of Derivatives*, vol. 6, pp. 38–50, 1998.

[13] J. Barthélemy and M. Marx, “Solving endogenous regime switching models,” *Journal of Economic Dynamics and Control*, vol. 77, pp. 1–25, 2017.

[14] D. D. Yao, Q. Zhang, and X. Y. Zhou, “A regime-switching model for european options,” in *Stochastic processes, optimization, and control theory: applications in financial engineering, queueing networks, and manufacturing systems*, pp. 281–300, Springer, 2006.

[15] R. Liu, Q. Zhang, and G. Yin, “Option pricing in a regime-switching model using the fast fourier transform,” *International Journal of Stochastic Analysis*, vol. 2006, 2006.

[16] J. Buffington and R. J. Elliott, “Regime switching and european options,” in *Stochastic Theory and Control*, pp. 73–82, Springer, 2002.

[17] J. T. Lim, B. S. Dickens, S. Haoyang, N. L. Ching, and A. R. Cook, “Inference on dengue epidemics with bayesian regime switching models,” *PLoS computational biology*, vol. 16, no. 5, p. e1007839, 2020.

[18] Q. Liu, “The threshold of a stochastic susceptible–infective epidemic model under regime switching,” *Nonlinear Analysis: Hybrid Systems*, vol. 21, pp. 49–58, 2016.

[19] G. Dibeh and H. Harmanani, “Option pricing during post-crash relaxation times,” *Physica A*, vol. 380, pp. 357–365, 2007.

[20] D. Sornette, *Why Stock Markets Crash: Critical Events in Complex Financial Markets*. Princeton University Press, Princeton, NJ, 2003.

[21] A. Hirsa, *Computational methods in finance*. CRC Press Boca Raton, FL, 2013.

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Financial modeling by SDEs under regime-switching have been utilized to allow moving from an economic state to another. The aim of this research work is to tackle the pricing of European options in an augmented volatility market under regime switching model. SDEs under regime-switching models are more realistic but the solution is more complicated and may not exist analytically. Therefore, numerical methods for finance are explored. The study will propose a new SDE under regime-switching with high volatility model for the prices of the underlying financial asset. The suggested model will cover the models of option pricing during post-crash relaxation times and option valuation and hedging in markets with a crunch for high volatile situations and the regime-switching part of the model. Then, under these settings, the valuation of European options will be investigated based on the Black-Scholes model and the Crisis model.

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