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FLOW OF QUANTUM GENETIC LOTKA-VOLTERRA ALGEBRAS ON $M_2(\mathbb{C})$

Sondos Muhammed Syam

This thesis is submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

Under the Supervision of Professor Farrukh Mukhamedov

December 2021

Declaration of Original Work

I, Sondos Muhammed Syam, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled *"Flow of Quantum Genetic Lotka-Volterra Algebras on* $M_2(\mathbb{C})$*"*, hereby, solemnly declare that this thesis is my own original research work that has been done and prepared by me under the supervision of Professor Farrukh Mukhamedov, in the College of Science at UAEU. This work has not previously been presented or published, or formed the basis for the award of any academic degree, diploma or a similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my thesis have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this thesis.

Student's Signature S Syam Date 12/12/2021

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Approval of the Master Thesis

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Abstract

In this thesis, a class of flow quantum Lotka-Volterra genetic algebras (FQLVG-A) is investigated and its structure is studied. Moreover, the necessary and sufficient conditions for the associativity and alternatively of FQGLV-A are derived. In addition, idempotent elements in FQGLV-A are found. Also, derivations of a class of FQLVG-A are described. Also, the automorphisms of a class of FQLVG-A and their positivity are examined.

Keywords: Flow quantum Lotka-Volterra genetic algebras, Jordan algebra, associativity, idempotent, derivation, automorphism.

Title and Abstract (in Arabic)

الكم المتدفق من الجبر الوراثي للوتكا فولتيرا على $M2(\mathbb{C})$

الملخص

افي هذه الأطروحة، تم البحث في مجموعة من الكم المتدفق من الجبر الوراثي للوتكا فولتيرا ودراسة هيكلها. علاوة على ذلك، تم اشتقاق الشروط الضرورية والكافية لخواص التجميع والتبديل الى مجموعات الكم المتدفق من الجبر الوراثي للوتكا فولتيرا. بالإضافة إلى ذلك، تم إيجاد العناصر المثالية في لتك المجموعات. أيضا، تم وصف مشتقات مجموعات الكم المتدفق من الجبر الوراثي للوتكا فولتيرا. أيضاً، تم فحص الأشكال التلقائية الى مجموعات الكم المتدفق من الجبر الوراثي للوتكا فولتيرا ومتى تكون موجبة.

مفاهيم البحث الرئيسية: الكم المتدفق من الجبر الوراثي للوتكا فولتيرا، جبر جوردن، التجمبع، عنصر مثالي، المشتقة، شكل تلقائي.

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Dedication

To my beloved parents and teachers

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List of Abbreviations

QSO	Quadratic Stochastic Operator
FQLVG-A	Flow Of Quantum Lotka-Volterra Genetic Algebra
KCE	Kolmogorov-Chapman Equation
QGA	Quantum Genetic Algebra

Chapter 1: Introduction

1.1 Literature review

The history of the quadratic stochastic operators can be traced back to the work of Bernshtein in 1924 [1]. Many physical systems were investigated by reducing them to Markov processes which connected with these systems. Quadratic dynamical systems have been proved to be a rich source of analysis for the investigation of dynamical properties and modeling in different domains, such as population dynamics [1-3], physics [4, 5], economy [6], mathematics [7-9]. However, there are systems that were not described by Markov processes. Many of these problems involve quadratic stochastic operators. One of such systems is given by Quadratic Stochastic Operators (QSO), which is related to population genetics [1]. The problem of studying the behavior of trajectories of quadratic stochastic operators was stated in [9]. Let us describe this model.

Consider a biological population, that is, a community of organisms closed with respect to reproduction according to Bernstein [1]. Assume that every individual in this population belongs to precisely one of the species $1,2,...,n$. The scale of species is such that the species of parents i and j unambiguously determine the probability of every species k for the first generation of direct descendants. The probability (the heredity coefficient) is denoted by $p_{ij,k}$. Then,

$$p_{ij,k} \geq 0$$
$$\sum_{i=1}^{n} p_{ij,k} = 1, \forall i, j, k.$$

Assume that the population is so large that frequency fluctuation can be neglected. Then, the state of the population can be described by the tuple $\mathbf{x} = (x_1, x_2, ..., x_n)$ of species probabilities, that is, x_i is the fraction of the species i in the population. In the case of random interbreeding, the parents pairs i and j arise for a fixed state

$\mathbf{x} = (x_1, x_2, ..., x_n)$ with probability $x_i x_j$. Hence,

$$x'_k = \sum_{i,j=1}^{n} p_{ij,k} x_i x_j \tag{1.1}$$

is the total probability of the species k in the first generation of direct descendants. The set:

$$S^{n-1} = \left\{ \mathbf{x} = (x_1, x_2, ..., x_n) | x_i \geq 0, \sum_{j=1}^{n} x_j = 1 \right\}$$

is an $(n-1)$-dimensional simplex. Since $x'_j \geq 0$ and $\sum_{j=1}^{n} x'_j = 1$, the quadratic stochastic operator is defined by formula (1.1) maps S^{n-1} into itself. The concept of quadratic stochastic operator was introduced by Bernstein [1]. The problem of investigation the trajectories of quadratic stochastic operator was posted in Ulam [9]. Complicated and bulky recurrences made it impossible to develop analytical methods. The study of concrete quadratic operator involved a lot of calculations, which did not stimulate interest in this problem.

Let V, W, and X be three vector spaces over the same field F. A bilinear map is a function:

$$B : V \times W \to X$$

such that for all $w \in W$, the map B_w:

$$v \to B(v, w)$$

is a linear map from V to X, and for all $v \in V$, the map B_v:

$$w \to B(v, w)$$

is a linear map from W to X.

For example, if a vector space V over the real numbers $\mathbb{R}$ carries an inner product, then the inner product is a bilinear map $V \times V \to \mathbb{R}$.

An algebra is a vector space equipped with a bilinear product. The multiplication in an algebra may or may not-associative, leading to the notations of associative algebra or non-associative algebras. An algebra is unital or unitary if it has an identity element with respect to the multiplication. For example, complex numbers is associative and commutative with the bilinear operator:

$$(a+ib)\cdot(c+id), a,b,c,d \in \mathbb{R}.$$

Another example is $\mathbb{R}^3$ with bilinear operator is the cross product of vectors. Then, it is neither associative nor commutative. The third example is $\mathbb{R}^4$ with Hamilton product as bilinear operator:

$$\begin{aligned}
&(a_1+b_1\overrightarrow{i}+c_1\overrightarrow{j}+d_1\overrightarrow{k})(a_2+b_2\overrightarrow{i}+c_2\overrightarrow{j}+d_2\overrightarrow{k})\\
&=(a_1a_2-b_1b_2-c1c_2-d_1d_2)\\
&+(a_1c_2-b_1d_2+c_1a_2+d_1b_2)\overrightarrow{i}\\
&+(a_1b_2+b_1a_2+c_1d_2-d_1c_2)\overrightarrow{j}\\
&+(a_1d_2+b_1c_2-c_1b_2+d_1a_2)\overrightarrow{k}
\end{aligned}$$

is associative but not commutative.

In mathematical genetics, genetic algebras are (possibly non-associative) used to model inheritance in genetic. In application of genetic, this algebra often has a basis corresponding to genetically different gametes, and the structure constant of the algebra encode the probabilities of producing offspring of various types. There exist several classes of non-associative algebras (baric, evolution, Bernstein, train, stochastic, etc.), whose investigations have provided a number of significant contributions to theoretical population genetics. Such classes have been defined different times by several authors, and all algebras belonging to these classes are generally called genetic. In recent years many authors have tried to investigate the difficult problem of classification of these algebras. The most comprehensive references for the mathematical research done in this area are [8, 10-13].

Let A be an algebra over a field K. Assume that A admits a basis $\{e_1, \ldots, e_n\}$ such that the multiplication constants $p_{ij,k}$ with respect to this basis, are given by

$$e_i \circ e_j = \sum_{k=1}^{n} p_{ij,k} e_k.$$

We say that A is a genetic algebra if the multiplication constants $P_{ij,k}$ satisfy

1. $p_{ij,k} \geq 0$,

2. $\sum_{k=1}^{n} p_{ij,k} = 1$.

General properties of genetic algebras were investigated in [8]. One of the important algebras is Lotka-Volterra genetic algebras which emerge in connection with biological problems and Lotka Volterra systems for the interactions of neighboring individuals. Lotka-Volterra algebras over the real numbers were introduced in 1981 by Y. Itoh [14]. These algebras are associated to quadratic differential equations [15, 16] and they give descriptions of solutions and singularities. These algebras present many connections with other mathematical fields including graph theory, Markov chains, dynamical systems and theory of population genetics, [17, 8]. P. Holgate showed how a derivation of a genetic algebra can be interpreted in biological terms [18]. In [19-24], a classification of the derivations of Lotka-Volterra algebras up to dimension 3 has been given and also the paper [25] gives some examples of derivations of Lotka-Volterra algebras with dimension 4. However, the classification in dimension 4 is not complete. More details can be found in [26-28].

A system which has a rule that gives a description of the time dependence of the state is called a dynamical system. In this thesis, we study a dynamical system in which at a specific time it has a state that is a finite-dimensional algebra. A Kolmogorov-Chapman equation that describes what the states follow for the present algebra is called the evolution rule of this dynamical system. One can see that any finite-dimensional algebra can generate a cubic matrix of structural constants. This matrix produces an evolution quadratic operator. For this reason, dynamical systems produced by quadratic operators have significant attention and become one of the main factors to make researchers study the dynamical properties and modeling in many domains, such as population dynamics [1], physics [5], economy [6] or mathematics [29].

In recent work in [30, 31, 32, 33, 34], several chains of evolution algebras were given and investigated. In each of these papers, the matrices of structural constants (depending on pair of time (s, t)) are square or rectangular and satisfy the Kolmogorov Chapman Equation. This means a chain of evolution algebras is a continuous-time dynamical system which in any fixed time is an evolution algebra. It is well-known that any matrix satisfying the Kolmogorov–Chapman Equation is stochastic which generates a Markov process. Hanggi and Thomas [35] studied time evolution of non-Markov processes as they occur in coarse-grained description of open and closed systems. Several properties of the theory are given for the two-state process and Gauss process. In [29], they generalized the notion of chain of evolution algebras to a notion of flow of arbitrary finite-dimensional algebras and their matrices of structural constants are cubic matrices. Due to the general form of the matrix of structural constants in each flow of algebras, the non-Markov processes of [35] can be derived from structural constant matrices in chains or flows of algebras. Therefore, they can be applied to biology and physics.

The purpose of this thesis is to investigate a quantum analogues of genetic algebras, and to discuss in detail many properties of a Flow Of Quantum Lotka-Volterra Genetic Algebras (FQLVG-A). It is worth mentioning that such types of algebras are first appeared in this thesis.

A flow of algebras is a particular case of a continuous-time dynamical system whose states are algebras, the matrix of structural constant of which depending on time and satisfy an analogue of the Kolmogorov-Chapman Equation (KCE), see [40]. Since there are several kinds of multiplications between cubic matrices, the multiplication in this thesis is fixed and then one can study the KCE for this fix multiplication. The existence of the solution of KCE provides the existence of a flow algebra. The aim of this thesis is to construct the flow algebras with respect to the given multiplication. Moreover, some time-dependant behavior properties of such flow algebras are given.

1.2 Objectives

The followings are main objective of this thesis.

1. To construct a class of Quantum Genetic Algebras (QGA) depending on parameter t.
2. To investigate structures of a class of FQLVGA.
3. To describe derivations of a class of FQLVGA.
4. To describe automorphisms of a class of FQLVGA.

1.3 Overview

This thesis consists of six chapters. Chapter 2 contains preliminary facts and necessary definitions of positive maps. Moreover, positive, trace preserving and unital operators on $M_2(\mathbb{C})$ are described. The quadratic stochastic operators are defined as well at the end of this chapter, some properties of quantum quadratic stochastic operators on $M_2(\mathbb{C})$ are presented. Chapter 3 is divided into three sections. In the first section, symmetric commutative q.q.o.s on the commutative algebra $DM_2(\mathbb{C})$ are described. In the second one, symmetric quasi q.q.o. on $DM_2(\mathbb{C})$ are studied. In the third section, a quantum analogue of Lotka-Volterra operators on $M_2(\mathbb{C})$ is defined. Also, some properties of these operators are presented. In chapter 4, a flow of quantum genetic Lotka-Volterra algebras are defined. Moreover, the necessary and sufficient conditions for the associativity and alternatively of FQGLV-A are derived. Also, the idempotent elements in FQGLV-A are found. Chapter 5 is devoted to the derivations of FQGLV-A. In Chapter 6, ten types of automorphisms are derived and necessary conditions are obtained.

Chapter 2: Preliminaries

2.1 Positive elements in $M_n(\mathbb{C}\ )$

Let $\mathscr{H}$ be the $n-$dimensional Hilbert space $\mathbb{C}^n$. The inner product between two vectors **x** and **y** is written as $\langle \mathbf{x}, \mathbf{y} \rangle$, where

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i \overline{y_i},$$

$\mathbf{x} = (x_1, x_2, ..., x_n), \mathbf{y} = (y_1, y_2, ..., y_n)$. It is noted that inner product is linear in the first variable and conjugate linear in the second. $\mathscr{L}(\mathscr{H})$ is denoted to be the space of all linear operators on $\mathscr{H}$, and by $M_n(\mathbb{C})$ or simply M_n to be the space of $n \times n$ matrices with complex entries. Every element A of $\mathscr{L}(\mathscr{H})$ can be identified with its matrix with respect to the standard basis $\{e_j\}$ of $\mathbb{C}^n$. The symbol A is used for this matrix as well. In what fellows, $\mathbb{1}$ is denoted the identity operator in $\mathscr{H}$ (i.e., $\mathbb{1}x = x, x \in \mathscr{H}$). A matrix A is called positive if

$$\langle \mathbf{x}, A\mathbf{x} \rangle \geq 0 \tag{2.1}$$

for all **x** in $\mathscr{H}$. The notation $A \geq 0$ is used to mean that A is positive. There are several conditions that characterize positivity of matrices. Some of them are listed below.

Theorem 2.1.1 *A matrix A is positive if and only if one of the following conditions holds*

(i) A is Hermitian $(A = A^*)$ *and all its eigenvalues are nonnegative.*

(ii) A is Hermitian and all its principal minors are nonnegative.

(iii) $A = B^*B$ *for some matrix B.*

(iv) $A = B^2$ *for some positive matrix B. Such B is unique. In this case,* $B = A^{1/2}$ *and*

call it the (positive) square root of A.

If A, B are Hermitian, it is said that $A \geq B$ if $A - B \geq 0$. Some notations will be fixed. The polar decomposition of A is written as $A = UP$. If the factor U is unitary and P is positive, then $P = (A^*A)^{1/2}$. This is called the positive part or the absolute value of A and is written as $|A|$. This implies that $A^* = PU^*$, and:

$$|A^*| = (AA^*)^{1/2} = (UP^2U^*)^{1/2} = UPU^*.$$

A is said to be normal if $AA^* = A^*A$. This condition is equivalent to $UP = PU$; and to the condition $|A| = |A^*|$. The singular value decomposition of A is written as $A = USV$. Here U and V are unitary and S is diagonal with nonnegative diagonal entries $s_1(A) \geq ... \geq s_n(A)$. These are the singular values of A. The symbol $||A||$ will always denote the norm of A as a linear operator on the Hilbert space $\mathscr{H}$, i.e.,

$$||A|| = \sup_{||\mathbf{x}||=1} ||A\mathbf{x}|| = \sup_{||\mathbf{x}||\leq 1} ||A\mathbf{x}||.$$

It is easy to see that $||A|| = s_1(A)$. Among the important properties of this norm are the following:

$$\begin{aligned} ||AB|| &\leq ||A||||B||, \\ ||A|| &= ||A^*||, \\ ||A|| &= ||UAV||, \end{aligned} \tag{2.2}$$

for all unitary U, V. This last property is called unitary invariance. Finally,

$$||A^*A|| = ||A||^2. \tag{2.3}$$

There are several other norms on $M_n(\mathbb{C})$ that share the three properties (2.2). It is the condition (2.3) that makes the operator norm $||\cdot||$ very special. A matrix A is called contractive, or A is a contraction, if $\|A\| \leq 1$. Next are some known results.

Proposition 1 *[41] The operator A is contractive if and only if the operator*

$$\begin{pmatrix} \mathbb{1} & A \\ A^* & \mathbb{1} \end{pmatrix}$$

is positive.

Proposition 2 *[41] Let A, B be positive. Then the matrix*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$$

is positive if and only if $X = A^{1/2}KB^{1/2}$ for some contraction K.

Theorem 2.1.2 *[41] Let A, B be positive matrices. Then the block matrix*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$$

is positive if and only if $A \geq XB^{-1}X^*$.

Lemma 2.1.3 *[41] The matrix A is positive if and only if*

$$\begin{pmatrix} A & A \\ A & A \end{pmatrix}$$

is positive.

Proof. From

$$\begin{pmatrix} A & A \\ A & A \end{pmatrix} = \begin{pmatrix} A^{1/2} & 0 \\ A^{1/2} & 0 \end{pmatrix} \begin{pmatrix} A^{1/2} & A^{1/2} \\ 0 & 0 \end{pmatrix},$$

one gets the desired assertion. □

Corollary 2.1.4 *Let A be any matrix. Then the matrix:*

$$\begin{pmatrix} |A| & A^* \\ A & |A^*| \end{pmatrix}$$

is positive.

Proof. Use the polar decomposition $A = UP$ to write

$$\begin{pmatrix} |A| & A^* \\ A & |A^*| \end{pmatrix} = \begin{pmatrix} P & PU^* \\ UP & UPU^* \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} P & P \\ P & P \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & U^* \end{pmatrix},$$

and then use the Lemma 2.1.2. □

Corollary 2.1.5 *If A is normal, then*

$$\begin{pmatrix} |A| & A^* \\ A & |A| \end{pmatrix}$$

is positive.

2.2 Positive mappings

A mapping $\Phi : M_m(\mathbb{C}) \to M_k(\mathbb{C})$ is called linear if for any $A, B \in M_m(\mathbb{C})$ and $\lambda \in \mathbb{C}$ one has

$$\Phi(\lambda A + B) = \lambda \Phi(A) + \Phi(B).$$

The symbol Φ is used for a linear map from $M_m(\mathbb{C})$ into $M_k(\mathbb{C})$. When $k = 1$ such a map is called a linear functional, and the lower-case symbol φ is used for it. The norm of Φ is

$$\|\Phi\| = \sup_{\|A\|=1} \|\Phi(A)\| = \sup_{\|A\|\leq 1} \|\Phi(A)\|.$$

Definition 2.2.1 A linear mapping Φ: $M_m(\mathbb{C}) \to M_k(\mathbb{C})$ is called

(i) unital, if $\Phi(\mathbb{1}_{M_m(\mathbb{C})}) = \mathbb{1}_{M_k(\mathbb{C})}$.

(ii) positive, if $\Phi(x) \geq 0$, whenever $x \geq 0$.

Definition 2.2.2 Let $A \in M_m(\mathbb{C})$ be a matrix. Then, trace of A is denoted by $tr(A)$ and it is defined by the sum of diagonal elements of A.

Definition 2.2.3 Let $\varphi : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ be a linear map. Then, φ is called trace preserving if and only if $tr(\varphi(A)) = tr(A)$ for all $A \in M_m(\mathbb{C})$.

Example 2.2.1 (i) $\varphi : M_m(\mathbb{C}) \to \mathbb{R}, \varphi(A) = trA$ is a positive linear functional; $\varphi(A) = \frac{1}{m} trA$ is positive and unital.

(ii) Every linear functional $\varphi : M_m(\mathbb{C}) \to \mathbb{R}$ on $M_m(\mathbb{C})$ has the form $\Phi(A) = trAX$ for some $X \in M_m(\mathbb{C})$. It is easy to see that Φ is positive if and only if X is a positive matrix; φ is unital if $trX = 1$.

(iii) The map $\Phi(A) = \frac{trA}{m}\mathbb{1}$ is a positive map of $M_m(\mathbb{C})$ into itself. (Its range consists of scalar matrices.)

(iv) Let A^{tr} denote the transpose of A. Then the map $\Phi(A) = A^{tr}$ is positive.

(v) Let X be an $m \times k$ matrix. Then $\Phi(A) = X^*AX$ is a positive map from $M_m(\mathbb{C})$ into $M_k(\mathbb{C})$.

Lemma 2.2.1 *[41] Every positive linear map is adjoint-preserving; i.e.,* $\Phi(T^*) =$

$\Phi(T)^*$ *for all* T.

Proof. First, $\Phi(A)$ is Hermitian if A is Hermitian is needed to prove. Every Hermitian matrix A has a Jordan decomposition:

$$A = A_+ - A_-$$

where $A_\pm \geq 0$. So,

$$\Phi(A) = \Phi(A_+) - \Phi(A_-)$$

is the difference of two positive matrices, and is therefore Hermitian. Every matrix T has a Cartesian decomposition:

$$T = A + iB$$

where A, B are Hermitian. So,

$$\Phi(T)^* = \Phi(A) - i\Phi(B) = \Phi(A - iB) = \Phi(T^*).$$

□

Theorem 2.2.2 *[41] Let* Φ *be positive and unital. Then for every Hermitian A,*

$$\Phi(A)^2 \leq \Phi(A^2). \tag{2.4}$$

Remark The inequality (2.4) may not be true if Φ is not unital [37].

Theorem 2.2.3 *[41] If* Φ *is positive and unital, then* $\|\Phi\| = 1$.

Corollary 2.2.4 *[41] Let* Φ *be a positive linear map. Then* $\|\Phi\| = \|\Phi(\mathbb{1})\|$.

2.3 Pauli matrices and their properties

In this section, the algebra $M_2(\mathbb{C})$ is considered. In the sequel, $\mathbb{1}$ is meant an identity matrix:

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and τ is denoted to be a normalized trace, i.e.

$$\tau \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \frac{x_{11} + x_{22}}{2}.$$

It is known that the positivity of a matrix:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is equivalent to the conditions: $a_{11} \geq 0$ and $a_{11}a_{22} - |a_{12}|^2 \geq 0$. The Pauli matrices are denoted by $\sigma_1, \sigma_2, \sigma_3$ which are defined as follows:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.5}$$

It is noted that the identity and Pauli matrices, i.e. $\{\mathbb{1}, \sigma_1, \sigma_2, \sigma_3\}$ form a basis for $M_2(\mathbb{C})$. Namely, every matrix $A \in M_2(\mathbb{C})$ can be written in this basis as $A = w_0\mathbb{1} + \mathbf{w} \cdot \sigma$ with $w_0 \in \mathbb{C}, \mathbf{w} = (w_1, w_2, w_3) \in \mathbb{C}^3$, here by $\mathbf{w} \cdot \sigma$ is meant the following:

$$\mathbf{w} \cdot \sigma = w_1\sigma_1 + w_2\sigma_2 + w_3\sigma_3.$$

For the sake of completeness, let us demonstrate how the coefficients w_0, w_1, w_2, w_3 are represented by the matrix entries. Assume that:

$$A = w_0 \mathbb{1} + w_1 \sigma_1 + w_2 \sigma_2 + w_3 \sigma_3, \tag{2.6}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Thus, equations (2.5) and (2.6) imply that:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} w_0 + w_3 & w_1 - iw_2 \\ w_1 + iw_2 & w_0 - w_3 \end{pmatrix}$$

which means

$$\begin{cases} w_0 + w_3 = a_{11}, \\ w_1 - iw_2 = a_{12}, \\ w_1 + iw_2 = a_{21}, \\ w_0 - w_3 = a_{22}. \end{cases} \tag{2.7}$$

Solving System (2.7) one finds:

$$w_0 = \frac{a_{11} + a_{22}}{2}, \quad w_1 = \frac{a_{12} + a_{21}}{2}, \tag{2.8}$$

$$w_2 = \frac{a_{21} - a_{12}}{2i}, \quad w_3 = \frac{a_{11} - a_{22}}{2}. \tag{2.9}$$

Lemma 2.3.1 *The following assertions hold true:*

(i) A is self-adjoint if and only if $w_0, \mathbf{w}$ *are reals;*

(ii) $A \geq 0$ if and only if $w_0, \mathbf{w}$ are reals and $\|\mathbf{w}\| \leq w_0$, where

$$\|\mathbf{w}\| = \sqrt{|w_1|^2 + |w_2|^2 + |w_3|^2}. \tag{2.10}$$

Proof. (i). Assume $A \in M_2(\mathbb{C})$ is self adjoint, then A is represented as follows

$$A = w_0 \mathbb{1} + w_1 \sigma_1 + w_2 \sigma_2 + w_3 \sigma_3 \tag{2.11}$$

so,

$$A^* = \overline{w_0} \mathbb{1} + \overline{w_1} \sigma_1 + \overline{w_2} \sigma_2 + \overline{w_3} \sigma_3.$$

Self adjointness of A implies that:

$$w_0 \mathbb{1} + w_1 \sigma_1 + w_2 \sigma_2 + w_3 \sigma_3 = \overline{w_0} \mathbb{1} + \overline{w_1} \sigma_1 + \overline{w_2} \sigma_2 + \overline{w_3} \sigma_3$$

hence,

$$(w_0 - \overline{w_0}) \mathbb{1} + (w_1 - \overline{w_1}) \sigma_1 + (w_2 - \overline{w_2}) \sigma_2 + (w_3 - \overline{w_3}) \sigma_3 = 0.$$

Linearly independence of $\mathbb{1}, \sigma_1, \sigma_2, \sigma_3$ yields:

$$w_0 = \overline{w_0}, \;\; w_1 = \overline{w_1}, \;\; w_2 = \overline{w_2}, \;\; w_3 = \overline{w_3}.$$

Thus, w_0 and $\mathbf{w}$ are real. Conversely, if w_0, w_1, w_2, w_3 are real numbers, then:

$$A^* = \overline{w_0} \mathbb{1} + \overline{w_1} \sigma_1 + \overline{w_2} \sigma_2 + \overline{w_3} \sigma_3$$

$$= w_0 \mathbb{1} + w_1 \sigma_1 + w_2 \sigma_2 + w_3 \sigma_3 = A.$$

Thus, A is self adjoint.

(ii) Assume that a matrix $\mathbf{A} \in M_2(\mathbb{C})$ is positive. This means that:

$$a_{11} \geq 0, \quad a_{11}a_{22} \geq |a_{12}|^2. \tag{2.12}$$

According to Equations (2.8), (2.9), one gets:

$$\mathbf{w} = \left(\frac{a_{12}+a_{21}}{2}, \frac{a_{21}-a_{12}}{2i}, \frac{a_{11}-a_{22}}{2} \right).$$

Then,

$$\begin{aligned}
\|\mathbf{w}\| &= \sqrt{\left|\frac{a_{12}+a_{21}}{2}\right|^2 + \left|\frac{a_{21}-a_{12}}{2i}\right|^2 + \left|\frac{a_{11}-a_{22}}{2}\right|^2} \\
&= \sqrt{\frac{(a_{12}+a_{21})(\overline{a_{12}+a_{21}})}{4} + \frac{(a_{21}-a_{12})(\overline{a_{21}-a_{12}})}{4} + \frac{(a_{11}-a_{22})(\overline{a_{11}-a_{22}})}{4}} \\
&= \sqrt{\frac{(a_{12}+a_{21})(a_{21}+a_{12})}{4} + \frac{(a_{21}-a_{12})(a_{12}-a_{21})}{4} + \frac{(a_{11}-a_{22})(a_{11}-a_{22})}{4}} \\
&= \frac{1}{2}\sqrt{a_{12}^2 + 2a_{12}a_{21} + a_{21}^2 - (a_{21}^2 - 2a_{12}a_{21} + a_{12}^2) + a_{11}^2 - 2a_{11}a_{22} + a_{22}^2} \\
&= \frac{1}{2}\sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22} - |a_{12}|^2)} \\
&\leq \frac{1}{2}\sqrt{(a_{11}+a_{22})^2} = \frac{a_{11}+a_{22}}{2} = w_0.
\end{aligned}$$

Conversely, assume $\|\mathbf{w}\| \leq w_0$ holds. Since $w_0, \mathbf{w}$ are real numbers, A is self adjoint. This means $w_0 \geq 0$, which with (2.8) yields:

$$\frac{a_{11}+a_{22}}{2} \geq 0. \tag{2.13}$$

Above calculations imply that:

$$\sqrt{\frac{(a_{11}+a_{22})^2}{4}-(a_{11}a_{22}-|a_{12}|^2)} \leq \frac{a_{11}+a_{22}}{2}.$$

Taking square for both sides of last inequality, one has:

$$\left(\frac{a_{11}+a_{22}}{2}\right)^2-(a_{11}a_{22}-|a_{12}|^2) \leq \left(\frac{a_{11}+a_{22}}{2}\right)^2$$

hence,

$$a_{11}a_{22}-|a_{12}|^2 \geq 0$$

which implies:

$$a_{11}a_{22} \geq |a_{12}|^2.$$

This means a_{11}, a_{22} have the same sign. Due to Equation (2.13), one concludes $a_{11} \geq 0$. The proof is completed. □

Recall that a functional $f : M_2(\mathbb{C}) \to \mathbb{C}$ is called a linear functional if for any $A, B \in M_2(\mathbb{C})$ and $\lambda \in \mathbb{C}$, one has:

$$f(A+\lambda B) = f(A)+\lambda f(B). \tag{2.14}$$

A linear functional f is called positive if $f(A) \geq 0$ whenever $A \geq 0$. A positive linear functional f is called a state if $f(\mathbb{1}) = 1$. $S(M_2(\mathbb{C}))$ denotes the set of all states defined on $M_2(\mathbb{C})$. Let φ be a linear functional. Then,

$$\begin{aligned}\varphi(x) &= \varphi(w_0\mathbb{1} + w_1\sigma_1 + w_2\sigma_2 + w_3\sigma_3)\\ &= w_0\varphi(\mathbb{1}) + w_1\varphi(\sigma_1) + w_2\varphi(\sigma_2) + w_3\varphi(\sigma_3)\\ &= w_0 f_0 + w_1 f_1 + w_2 f_2 + w_3 f_3,\end{aligned}$$

where $f_i = \varphi(\sigma_i), i = 1,2,3$. Hence, any linear functional φ on $M_2(\mathbb{C})$ can be represented by

$$\varphi(w_0\mathbb{1} + \mathbf{w}\sigma) = w_0 f_0 + \langle \mathbf{w}, \bar{\mathbf{f}} \rangle \tag{2.15}$$

where $\langle \cdot,\cdot \rangle$ stands for the standard scalar product in $\mathbb{C}^3$, i.e. if $\mathbf{p} = (p_1, p_2, p_3), \mathbf{q} = (q_1, q_2, q_3) \in \mathbb{C}^3$,

$$\langle \mathbf{p}, \mathbf{q} \rangle = p_1\overline{q_1} + p_2\overline{q_2} + p_3\overline{q_3}.$$

Let $x \in M_2(\mathbb{C})$, then

$$x = \frac{x + x^*}{2} + i\frac{x - x^*}{2i}.$$

Denote

$$x_1 = \frac{x + x^*}{2}, \;\; x_2 = \frac{x - x^*}{2i}.$$

Note that x_1, x_2 are self adjoint. Indeed,

$$\begin{aligned}x_1^* &= \left(\frac{x + x^*}{2}\right)^* = \frac{x^* + x}{2} = \frac{x + x^*}{2} = x_1,\\ x_2^* &= \left(\frac{x - x^*}{2i}\right)^* = \frac{x^* - x}{-2i} = \frac{x - x^*}{2i} = x_2.\end{aligned}$$

It is known that any self adjoint matrix $x \in M_2(\mathbb{C})$ can be represented as follows:

$$x = x_+ - x_-, \tag{2.16}$$

where x_+, x_- are positive elements of $M_2(\mathbb{C})$. Hence, arbitrary $x \in M_2(\mathbb{C})$ has the following form:

$$x = x_{1,+} - x_{1,-} + i(x_{2,+} - x_{2,-})$$

where $x_{1,+}, x_{1,-}, x_{2,+}, x_{2,-}$ are positive elements of $M_2(\mathbb{C})$.

Lemma 2.3.2 *A linear function φ is a state on $M_2(\mathbb{C})$ if and only if*

$$\varphi(w_0 \mathbb{1} + \mathbf{w}\sigma) = w_0 + \langle \mathbf{w}, \mathbf{f} \rangle, \tag{2.17}$$

where $\mathbf{f} = (f_1, f_2, f_3), \mathbf{f} \in \mathbb{R}^3, \|\mathbf{f}\| \leq 1.$

Proof. 'only if' part. Assume $x \in M_2(\mathbb{C})$ is self adjoint, then from Equation (2.16), one finds

$$\varphi(x) = \varphi(x_+ - x_-) = \varphi(x_+) - \varphi(x_-).$$

It is known that φ is a positive functional. Therefore, $\varphi(x_+), \varphi(x_-)$ are positive numbers. Hence, $\varphi(x)$ is a real number. So, $f_i = \varphi(\sigma_i)$ are real numbers, i.e. $\mathbf{f} \in \mathbb{R}^3$. Since φ is a state then $f_0 = 1 (= \varphi(\mathbb{1}))$ in Equation (2.12). Thus, Equation (2.17) is derived. Let $x \in M_2(\mathbb{C})$ be a positive i.e. $x = w_0 \mathbb{1} + \mathbf{w}\sigma, \|\mathbf{w}\| \leq w_0, w_0 \geq 0$. Putting $\mathbf{v} = \frac{1}{w_0}\mathbf{w}$, one finds

$$\|\mathbf{v}\| = \frac{1}{w_0}\|\mathbf{w}\| \leq \frac{w_0}{w_0} = 1.$$

So, $x = w_0(\mathbb{1} + \mathbf{v}\sigma)$, with $\|\mathbf{v}\| \leq 1$. Positivity of φ implies that $\varphi(x) \geq 0$. This means $\varphi(w_0(\mathbb{1} + \mathbf{v}\sigma)) = w_0 \varphi(\mathbb{1} + \mathbf{v}\sigma) \geq 0$. From $w_0 \geq 0$, it follows that:

$$\varphi(\mathbb{1} + \mathbf{v}\sigma) \geq 0 \tag{2.18}$$

for every $\mathbf{v} \in \mathbb{R}^3$ with $||\mathbf{v}|| \leq 1$. Therefore, from Equations (2.17), (2.18), one gets

$$1 + v_1 f_1 + v_2 f_2 + v_3 f_3 \geq 0, \;\; \mathbf{v} = (v_1, v_2, v_3).$$

So,

$$-\sum_{i=1}^{3} v_i f_i \leq 1$$

Now changing v_i to $-v_i$, implies that:

$$\sum_{i=1}^{3} v_i f_i \leq 1.$$

Hence,

$$\left| \sum_{i=1}^{3} v_i f_i \right| \leq 1 \;\text{ for any }\; ||\mathbf{v}|| \leq 1. \tag{2.19}$$

Put:

$$\tilde{\mathbf{v}} = \left(\tilde{v_1}, \tilde{v_2}, \tilde{v_3}\right), \tilde{v_i} = \frac{f_i}{||\mathbf{f}||}.$$

Therefore, Equation (2.19) implies that

$$\left| \sum_{i=1}^{3} \tilde{v_i} f_i \right| \leq 1.$$

This means

$$\frac{1}{||\mathbf{f}||} \sum_{i=1}^{3} f_i^2 \leq 1.$$

So,

$$\frac{1}{||\mathbf{f}||} ||\mathbf{f}||^2 \leq 1 \Rightarrow ||\mathbf{f}|| \leq 1.$$

’if’ part. Assume that Equation (2.17) is valid. The task now is to show that φ

is a state on $M_2(\mathbb{C})$. It is clear that φ is a linear functional and $\varphi(\mathbb{1}) = 1$. To show φ is positive, let x be positive. If $x = w_0\mathbb{1} + \mathbf{w}\sigma$, then $\|\mathbf{w}\| \leq w_0$. The Cauchy-Schwarz inequality implies that:

$$|\langle \mathbf{w}, \mathbf{f} \rangle| \leq \|\mathbf{w}\|\|\mathbf{f}\| \leq w_0.$$

Hence,

$$w_0 + |\langle \mathbf{w}, \mathbf{f} \rangle| \geq 0 \;\; \Rightarrow \;\; \varphi(x) = w_0 + |\langle \mathbf{w}, \mathbf{f} \rangle| \geq 0.$$

□

By $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ is denoted to be the tensor product $M_2(\mathbb{C})$ to itself. Namely,

$$M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) = \overline{\left\{ \sum_{k=1}^{n} a_k \otimes b_k \middle| a_k, b_k \in M_2(\mathbb{C}) \right\}}^{\|\cdot\|},$$

where for $A = (a_{ij}), B = (b_{ij})$ one has:

$$A \otimes B = \begin{pmatrix} b_{11}A & b_{12}A \\ b_{21}A & b_{22}A \end{pmatrix} = \begin{pmatrix} b_{11}a_{11} & b_{11}a_{12} & b_{12}a_{11} & b_{12}a_{12} \\ b_{11}a_{21} & b_{11}a_{22} & b_{12}a_{21} & b_{12}a_{22} \\ b_{21}a_{11} & b_{21}a_{12} & b_{22}a_{11} & b_{22}a_{12} \\ b_{21}a_{21} & b_{21}a_{22} & b_{22}a_{21} & b_{22}a_{22} \end{pmatrix}.$$

For a given state φ, the following linear operators are denoted by $E_\varphi : M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \to M_2(\mathbb{C})$ and $\tilde{E}_\varphi : M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \to M_2(\mathbb{C})$ by

$$E_\varphi(x \otimes y) = \varphi(x)y, \;\; \tilde{E}_\varphi(x \otimes y) = \varphi(y)x,$$

where $x, y \in M_2(\mathbb{C})$. It is known that the defined mappings are positive, see [23].

2.4 Bistochastic mappings on $M_2(\mathbb{C})$

In this section, the positive, trace preserving and unital operators on $M_2(\mathbb{C})$ will be described. Let $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be a linear mapping. Let us find a matrix form of Φ in $\{\mathbb{1}, \sigma_1, \sigma_2, \sigma_3\}$ basis, where as before, $\sigma_1, \sigma_2, \sigma_3$ denote Pauli matrices. Thus,

$$
\begin{aligned}
\Phi(\mathbb{1}) &= \lambda_1 \mathbb{1} + t_1\sigma_1 + t_2\sigma_2 + t_3\sigma_3 \\
\Phi(\sigma_1) &= \lambda_2 \mathbb{1} + a_{11}\sigma_1 + a_{21}\sigma_2 + a_{31}\sigma_3 \\
\Phi(\sigma_2) &= \lambda_3 \mathbb{1} + a_{12}\sigma_1 + a_{22}\sigma_2 + a_{32}\sigma_3 \\
\Phi(\sigma_3) &= \lambda_4 \mathbb{1} + a_{13}\sigma_1 + a_{23}\sigma_2 + a_{33}\sigma_3.
\end{aligned} \tag{2.20}
$$

Therefore, the corresponding matrix of Φ is denoted by $\mathbf{F}$, i.e.,

$$
\mathbf{F} = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ t_1 & a_{11} & a_{12} & a_{13} \\ t_2 & a_{21} & a_{22} & a_{23} \\ t_3 & a_{31} & a_{32} & a_{33} \end{pmatrix}.
$$

Lemma 2.4.1 *[41] Let $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be a linear mapping. Then Φ trace preserving ($\tau(\Phi(x)) = \tau(x)$ for all $x \in M_2(\mathbb{C})$) if and only if*

$$
\mathbf{F} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{t} & \mathbf{T} \end{pmatrix}, \tag{2.21}
$$

where $\mathbf{t} = (t_1, t_2, t_3)^T, \mathbf{T} = (a_{ij})_{ij}^3, \mathbf{0} = (0,0,0)$.

Proof. 'only if' part. Using $\tau(\Phi(x)) = \tau(x)$ and Equation (2.20), one finds:

$$\tau(\Phi(\mathbb{1})) = \lambda_1 \tau(\mathbb{1}) + t_1 \tau(\sigma_1) + t_2 \tau(\sigma_2) + t_3 \tau(\sigma_3) = \lambda_1 = \tau(\mathbb{1}) = 1, \quad (2.22)$$

$$\tau(\Phi(\sigma_1)) = \lambda_2 \tau(\mathbb{1}) + a_{11} \tau(\sigma_1) + a_{21} \tau(\sigma_2) + a_{31} \tau(\sigma_3) = \lambda_2 = \tau(\sigma_1) = 0, \quad (2.23)$$

$$\tau(\Phi(\sigma_2)) = \lambda_3 \tau(\mathbb{1}) + a_{12} \tau(\sigma_1) + a_{22} \tau(\sigma_2) + a_{32} \tau(\sigma_3) = \lambda_3 = \tau(\sigma_2) = 0, \quad (2.24)$$

$$\tau(\Phi(\sigma_3)) = \lambda_4 \tau(\mathbb{1}) + a_{13} \tau(\sigma_1) + a_{23} \tau(\sigma_2) + a_{33} \tau(\sigma_3) = \lambda_4 = \tau(\sigma_3) = 0. \quad (2.25)$$

The reverse implication immediately comes from Equation (2.21) □

Lemma 2.4.2 *[41] Let $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be a trace preserving linear mapping. Then one has:*

$$\Phi(w_0 \mathbb{1} + \mathbf{w}\sigma) = w_0 \mathbb{1} + (w_0 \mathbf{t} + \mathbf{T}\mathbf{w})\sigma. \quad (2.26)$$

Proof. According to the matrix representation of Φ, $\mathbf{F}$ will be used instead of Φ. Therefore, one gets:

$$\Phi(\mathbf{w}\sigma) = \mathbf{F}(\mathbf{w}\sigma) = \mathbf{F}(w_1\sigma_1 + w_2\sigma_2 + w_3\sigma_3) = w_1\mathbf{F}\sigma_1 + w_2\mathbf{F}\sigma_2 + w_3\mathbf{F}\sigma_3. \quad (2.27)$$

Note, the matrices $\mathbb{1}, \sigma_1, \sigma_2, \sigma_3$ can be represented in a vector form as follows (in $\mathbb{1}, \sigma_1, \sigma_2, \sigma_3$ bases):

$$\mathbb{1} = (1,0,0,0), \;\; \sigma_1 = (0,1,0,0), \;\; \sigma_2 = (0,0,1,0), \;\; \sigma_3 = (0,0,0,1).$$

Hence,

$$w_0\mathbf{\Phi}(\mathbb{1}) = w_0\mathbf{F}(\mathbb{1}) = w_0 \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & a_{11} & a_{12} & a_{13} \\ t_2 & a_{21} & a_{22} & a_{23} \\ t_3 & a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} w_0 \\ w_0t_1 \\ w_0t_2 \\ w_0t_3 \end{pmatrix}$$

$$w_1\mathbf{\Phi}(\sigma_1) = w_1\mathbf{F}(\sigma_1) = w_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & a_{11} & a_{12} & a_{13} \\ t_2 & a_{21} & a_{22} & a_{23} \\ t_3 & a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ w_1a_{11} \\ w_1a_{21} \\ w_1a_{31} \end{pmatrix}$$

$$w_2\mathbf{\Phi}(\sigma_2) = w_2\mathbf{F}(\sigma_2) = w_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & a_{11} & a_{12} & a_{13} \\ t_2 & a_{21} & a_{22} & a_{23} \\ t_3 & a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ w_2a_{12} \\ w_2a_{22} \\ w_2a_{32} \end{pmatrix}$$

$$w_3\mathbf{\Phi}(\sigma_3) = w_3\mathbf{F}(\sigma_3) = w_3 \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & a_{11} & a_{12} & a_{13} \\ t_2 & a_{21} & a_{22} & a_{23} \\ t_3 & a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ w_3a_{13} \\ w_3a_{23} \\ w_3a_{33} \end{pmatrix}$$

Consequently, one finds:

$$\Phi(\mathbf{w}\sigma) = \begin{pmatrix} 0 \\ w_1 a_{11} \\ w_1 a_{21} \\ w_1 a_{31} \end{pmatrix} + \begin{pmatrix} 0 \\ w_2 a_{12} \\ w_2 a_{22} \\ w_2 a_{32} \end{pmatrix} + \begin{pmatrix} 0 \\ w_3 a_{13} \\ w_3 a_{23} \\ w_3 a_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ w_1 a_{11} + w_2 a_{12} + w_3 a_{13} \\ w_1 a_{21} + w_2 a_{22} + w_3 a_{23} \\ w_1 a_{31} + w_2 a_{32} + w_3 a_{33} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \\ \mathbf{Tw} \end{pmatrix}.$$

The last one implies:

$$\Phi(w_0\mathbb{1} + \mathbf{w}\sigma) = w_0\Phi(\mathbb{1}) + \Phi(\mathbf{w}\sigma) = \begin{pmatrix} w_0 \\ w_0\mathbf{T} \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{Tw} \end{pmatrix} = \begin{pmatrix} w_0 \\ w_0\mathbf{t} + \mathbf{TW} \end{pmatrix}$$

$$= w_0\mathbb{1} + (w_0\mathbf{t} + \mathbf{Tw})\sigma.$$

The proof is complete. □

Lemma 2.4.3 *[41] Let $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be a trace preserving linear mapping. Then, Φ is adjoint preserving if and only if $\mathbf{t}, \mathbf{T}$ are real.*

Proof. 'only if' part. Let $x = w_0\mathbb{1} + \mathbf{w}\sigma$ then $x^* = \overline{w_0}\mathbb{1} + \overline{\mathbf{w}}\sigma$. According to Lemma 2.4.2 one has:

$$\Phi(x) = w_0\mathbb{1} + (w_0\mathbf{t} + \mathbf{Tw})\sigma \tag{2.28}$$

$$\Phi(x^*) = \overline{w_0}\mathbb{1} + (\overline{w_0}\mathbf{t} + \mathbf{T}\overline{w})\sigma. \tag{2.29}$$

Hence,

$$\Phi(x)^* = \overline{w_0}\mathbb{1} + (\overline{w_0\mathbf{t}} + \overline{\mathbf{T}\mathbf{w}})\sigma = \overline{w_0}\Phi(\mathbb{1}) + (\overline{w_0}\mathbf{t} + \overline{\mathbf{T}}\overline{\mathbf{w}})\sigma \tag{2.30}$$

From the condition, one has $\Phi(x^*) = \Phi(\mathbf{x})^*$. Therefore,

$$\overline{w_0}\Phi(\mathbb{1}) + (\overline{w_0}\mathbf{t} + \mathbf{T}\overline{w})\sigma = \overline{w_0}\Phi(\mathbb{1}) + (\overline{\mathbf{w_0}\mathbf{t}} + \overline{\mathbf{T}\mathbf{w}})\sigma. \tag{2.31}$$

So,

$$(w_0(\mathbf{t} - \bar{\mathbf{t}})) + (\mathbf{T} - \overline{\mathbf{T}})\mathbf{w})\sigma = \mathbf{0} \tag{2.32}$$

where $w_0 \in \mathbb{R}$ and $\mathbf{w} \in \mathbb{R}^3$ and $\mathbf{0} = (0,0,0)$. Due to the linear independence of $\sigma_1, \sigma_2, \sigma_3$ one gets:

$$w_0(\mathbf{t} - \bar{\mathbf{t}}) + (\mathbf{T} - \overline{\mathbf{T}})\mathbf{w} = 0. \tag{2.33}$$

Now arbitrariness of w_0 and $\mathbf{w}$ implies:

$$\mathbf{t} - \bar{\mathbf{t}} = 0, \ \ \mathbf{T} - \overline{\mathbf{T}} = \mathbf{0}.$$

This means that **t** and **T** are real.

’if’ part. Let **t** and **T** are real, one can easily see that $\Phi(x^*) = \Phi(x)^*$. □

Corollary 2.4.4 *Let $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be trace preserving linear mapping. Then Φ is unital if and only if $\mathbf{t} = 0$ and one has:*

$$\Phi(w_0\mathbb{1} + \mathbf{w}\sigma) = w_0\mathbb{1} + (\mathbf{T}\mathbf{w})\sigma. \tag{2.34}$$

Proof. 'only if' part. From Lemma 2.4.1, one finds:

$$\Phi(\mathbb{1}) = \mathbb{1} + (\mathbf{t})\sigma.$$

Now taking into account unitality ($\Phi(\mathbb{1}) = \mathbb{1}$) of Φ, one gets $\mathbf{t} = 0$. The reverse implication is obvious. □

Proposition 2.4.5 *[41] Let $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be a unital trace preserving linear mapping. Then, Φ is positive if and only if $\mathbf{T}$ is real and $\|\mathbf{T}\| \leq 1$.*

Proof. 'only if' part. Since Φ is positive, then according to Lemma (2.4.3), then $\mathbf{T}$ is real. Take $x = w_0\mathbb{1} + \mathbf{w}\sigma$. Without loss of generality, take $w_0 = 1$. From equation (2.34), one gets:

$$\Phi(\mathbb{1} + \mathbf{w}\sigma) = \mathbb{1} + (\mathbf{T}\mathbf{w})\sigma. \tag{2.35}$$

If $x = \mathbb{1} + \mathbf{w}\sigma \geq 0$, then

$$\Phi(x) = \mathbb{1} + (\mathbf{T}\mathbf{w})\sigma \geq 0. \tag{2.36}$$

It means that $\|\mathbf{T}\mathbf{w}\| \leq 1$ for any $\|\mathbf{w}\| \leq 1$. Therefore, $\|\mathbf{T}\| \leq 1$.

'if' part. Let $\mathbf{T}$ be real and $\|\mathbf{T}\| \leq 1$. Assume $x = w_0\mathbb{1} + \mathbf{w}\sigma \geq 0$, that is $\|\mathbf{w}\| \leq |w_0|$. It enough to show that:

$$\Phi(x) = w_0\mathbb{1} + (\mathbf{T}\mathbf{w})\sigma \geq 0. \tag{2.37}$$

Since $\mathbf{T}$ is real, so $\Phi(x) = w_0\mathbb{1} + (\mathbf{T}\mathbf{w})\sigma$ is self-adjoint element. From $\|\mathbf{T}\| \leq 1$, one has $\|\mathbf{T}\mathbf{w}\| \leq \|\mathbf{w}\| \leq |w_0|$, which yields that $\Phi(x)$ is positive. □

Remark It is noted that all positive linear mappings of $M_2(\mathbb{C})$ where described in [38].

2.5 Quadratic stochastic operators

Let $I = \{1, \ldots, m\}$. $\{\mathbf{e}_i\}_{i \in I}$ is denoted to be the standard basis in $\mathbb{R}^m$, i.e. $\mathbf{e}_i = (\delta_{i1}, \ldots, \delta_{im})$, where δ is the Kronecker's Delta. Throughout this thesis, consider the simplex:

$$S^{m-1} = \{\mathbf{x} = (x_i) \in \mathbb{R}^m : x_i \geq 0, \forall i \in I, \sum_{i=1}^{m} x_i = 1\} \tag{2.38}$$

A quadratic stochastic operator (QSO) is a mapping of the simplex S^{m-1} into itself of the form

$$V : x'_k = \sum_{i,j=1}^{m} p_{ij,k} x_i x_j, \;\; k = 1, 2, \ldots, m \tag{2.39}$$

where $p_{ij,k}$ are heredity coefficients, which satisfy the following conditions:

$$p_{ij,k} \geq 0, \;\; p_{ij,k} = p_{ji,k}, \;\; \sum_{k=1}^{m} p_{ij,k} = 1, \;\; i, j, k \in \{1, 2, \ldots, m\}. \tag{2.40}$$

A QSO V defined by Equation (2.39) is called Lotka-Volterra operator [39] if

$$p_{ij,k} = 0 \;\text{ if }\; k \notin \{i, j\}, \;\; \text{for all } i, j, k \in I. \tag{2.41}$$

Equations (2.40) and (2.41) imply that:

$$p_{ii,i} = 1 \;\text{ and }\; p_{ij,i} + p_{ij,j} = 1, \;\; \text{for all } i, j \in I, \; (i \neq j). \tag{2.42}$$

Remark Note that it is obvious that the biological behavior of Condition (2.41) is that the offspring repeats one of its parents' genotype (see [26,39]).

Let V be a QSO and suppose that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ are arbitrary vectors, a multiplication rule (see [31]) on $\mathbb{R}^m$ is introduced by

$$(\mathbf{x} \circ \mathbf{y})_k = \sum_{i,j=1}^{m} p_{ij,k} x_i y_j \tag{2.43}$$

where $\mathbf{x} = (x_1, \ldots, x_m), \mathbf{y} = (y_1, \ldots, y_m) \in \mathbb{R}^m$. The pair $(\mathbb{R}^m, \circ)$ is called *genetic algebra*. It worth to mention that this algebra is commutative, i.e. $\mathbf{x} \circ \mathbf{y} = \mathbf{y} \circ \mathbf{x}$. Certain algebraic properties of such kind of algebras were investigated in [8, 14, 20]. In general, the genetic algebra is not necessarily to be associative. In [19], associativity of low dimensional genetic algebras have been studied. If V is a Lotka-Volterra QSO, then the associated genetic algebra is called *genetic Lotka-Volterra algebra*.

Remark Let A be a Lotka-Volterra algebra generated by heredity coefficients $\{p_{ij,k}\}$. Then, Equations (2.42) and (2.11) imply that

(a) for every $i, j \in I$ $(i \neq j)$, one has

$$\mathbf{e}_i \circ \mathbf{e}_j = p_{ij,i}\mathbf{e}_i + p_{ij,j}\mathbf{e}_j. \tag{2.44}$$

(b) $\mathbf{e}_i^2 = \mathbf{e}_i$ for every $i \in I$.

Theorem 2.5.1 *[30] Let A be an algebra over $\mathbb{R}$. If it has a genetic realization with respect to the natural basis $\mathbf{e}_i, \ldots, \mathbf{e}_m$, then A is a (non-associative) Banach algebra with respect to the norm $\|\mathbf{x}\| = \sum_{i=1}^{m} |x_i|$ for $\mathbf{x} = \sum_{i=1}^{m} x_i\mathbf{e}_i \in A$.*

Recall that a *derivation* on algebra $(A, \circ)$ is a linear mapping $D : A \to A$ such that $D(u \circ v) = D(u) \circ v + u \circ D(v)$ for all $u, v \in A$. It is clear that $D \equiv 0$ is also a derivation, and such derivation is called trivial one.

2.6 Quantum quadratic stochastic operators on $M_2(\mathbb{C})$

In this section, some properties of quantum quadratic stochastic operators are recalled. One can see that a basis of $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ can be formed by the system:

$$\begin{array}{llll}
\mathbb{1}\otimes\mathbb{1}, & \mathbb{1}\otimes\sigma_1, & \mathbb{1}\otimes\sigma_2, & \mathbb{1}\otimes\sigma_3,\\
\sigma_1\otimes\mathbb{1}, & \sigma_1\otimes\sigma_1, & \sigma_1\otimes\sigma_2, & \sigma_1\otimes\sigma_3,\\
\sigma_2\otimes\mathbb{1}, & \sigma_2\otimes\sigma_1, & \sigma_2\otimes\sigma_2, & \sigma_2\otimes\sigma_3,\\
\sigma_3\otimes\mathbb{1}, & \sigma_3\otimes\sigma_1, & \sigma_3\otimes\sigma_2, & \sigma_3\otimes\sigma_3.
\end{array}$$

Therefore, any unital linear operator $\Delta : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ can be represented as follows:

$$\begin{aligned}
&\Delta\mathbb{1} = \mathbb{1}\otimes\mathbb{1};\\
&\Delta(\sigma_i) = b_i(\mathbb{1}\otimes\mathbb{1}) + \sum_{j=1}^{3} b_{ji}^{(1)}(\mathbb{1}\otimes\sigma_j)\\
&+\sum_{j=1}^{3} b_{ji}^{(2)}(\sigma_j\otimes\mathbb{1}) + \sum_{m,l=1}^{3} b_{ml,i}(\sigma_m\otimes\sigma_l),
\end{aligned} \tag{2.45}$$

where $i = 1,2,3$. Due to Equation (2.45), the operator Δ has the following form:

$$\begin{aligned}
&\Delta(x) = (w_0 + \langle \mathbf{x}, \overline{\mathbf{w}}\rangle)\mathbb{1}\otimes\mathbb{1}\\
&+\mathbb{1}\otimes\mathbf{B}^{(1)}\mathbf{w}\cdot\sigma + \mathbf{B}^{(2)}\mathbf{w}\cdot\sigma\otimes\mathbb{1} + \sum_{m,l=1}^{3}\langle \mathbf{b}_{ml}, \overline{\mathbf{w}}\rangle\sigma_m\otimes\sigma_l,
\end{aligned} \tag{2.46}$$

where $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{b}_{ml} = (b_{ml,1}, b_{ml,2}, b_{ml,3})$, and $\mathbf{B}^{(k)} = (b_{ij}^{(k)})_{i,j=1}^{3}$, $k = 1,2$ are reals for every $i, j, k \in \{1,2,3\}$. Here as before $\langle\cdot,\cdot\rangle$ stands for the standard dot product in $\mathbb{C}^3$.

Definition 2.6.1 [26] A linear operator $\Delta : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ is said to be a quantum quadratic operator (q.q.o.) if it is unital and positive.

Definition 2.6.2 [26] A linear operator $\Delta : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ is called symmetric if $U\Delta = \Delta$, where $U(x \otimes y) = y \otimes x, \quad x, y \in M_2(\mathbb{C})$,

From now on, symmetric q.q.o. will be used.

Let $\Delta : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ be a linear symmetric operator. Then its dual defines an operator V_Δ given by

$$V_\Delta(\varphi) = \Delta^*(\varphi \otimes \varphi), \ \varphi \in M_2(\mathbb{C})^*. \tag{2.47}$$

This mapping is called quadratic operator. Note that this kind of operators have been introduced in [24]. Then, due to Equation (2.46), for every state $\varphi \in S(M_2(\mathbb{C}))$, the functional $\Delta^*(\varphi \otimes \varphi)$ is a state if and only if the corresponding vector:

$$\begin{aligned} \mathbf{f}_{\Delta^*(\varphi,\varphi)} = \Big(& b_1 + 2\sum_{j=1}^{3} b_{j1} f_j + \sum_{i,j=1}^{3} b_{ij,1} f_i f_j, b_2 \\ & + 2\sum_{j=1}^{3} b_{j2} f_j + \sum_{i,j=1}^{3} b_{ij,2} f_i f_j, b_3 + 2\sum_{j=1}^{3} b_{j3} f_j + \sum_{i,j=1}^{3} b_{ij,3} f_i f_j \Big) \end{aligned} \tag{2.48}$$

satisfies $\|\mathbf{f}_{\Delta^*(\varphi,\varphi)}\| \leq 1$, here the vector $\mathbf{f} = (f_1, f_2, f_3)$ corresponds to the state φ. From the last expression, one can see that:

$$V_\Delta(\varphi)(\sigma_k) = b_k + 2\sum_{j=1}^{3} b_{jk} f_j + \sum_{i,j=1}^{3} b_{ij,k} f_i f_j, \ \ \mathbf{f} \in \mathbf{B}, \tag{2.49}$$

where $\mathbf{B} = \{\mathbf{p} \in \mathbb{R}^3 : \ \|\mathbf{p}\| \leq 1\}$. This suggests to consider a nonlinear operator $V : \mathbf{B} \to \mathbb{R}^3$ defined by

$$V(\mathbf{f})_k = b_k + 2\sum_{j=1}^{3} b_{jk} f_j + \sum_{i,j=1}^{3} b_{ij,k} f_i f_j, \quad k = 1, 2, 3 \tag{2.50}$$

where $\mathbf{f} = (f_1, f_2, f_3) \in \mathbf{B}$. Hence, any unital linear operator $\Delta : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ is a quasi q.q.o. if and only if the corresponding operator V satisfies $V(\mathbf{B}) \subset \mathbf{B}$. It is noticed that dynamical behavior of quadratic operators has been investigated in

[26]. In [28], an example has been given of a quasi quadratic operator for which V_Δ has chaotic behavior. If Δ is a q.q.o. then its positivity implies that it is $*$-preserving, therefore Δ is a quasi q.q.o., but the reverse is not true. Consider the following example.

Example 2.6.1 [40] Define a mapping by

$$\begin{aligned}\Delta_\varepsilon(x) = w_0 \mathbb{1}\otimes\mathbb{1} + \varepsilon w_1 \sigma_1\otimes\sigma_1 + \varepsilon w_3\sigma_1\otimes\sigma_2 + \varepsilon w_2\sigma_1\otimes\sigma_3 \\ +\varepsilon w_3\sigma_2\otimes\sigma_1 + \varepsilon w_2\sigma_2\otimes\sigma_2 + \varepsilon w_1\sigma_2\otimes\sigma_3 \\ +\varepsilon w_2\sigma_3\otimes\sigma_1 + \varepsilon w_1\sigma_3\otimes\sigma_2 + \varepsilon w_3\sigma_3\otimes\sigma_3,\end{aligned} \tag{2.51}$$

where as before $x = w_0\mathbb{1} + \mathbf{w}\sigma$, and $\varepsilon \in \mathbb{R}$. One can find that the corresponding quadratic operator (see equation (2.50)) is given by

$$\begin{cases} V_\varepsilon(f)_1 = \varepsilon(f_1^2 + 2f_2f_3) \\ V_\varepsilon(f)_2 = \varepsilon(f_2^2 + 2f_1f_3) \\ V_\varepsilon(f)_3 = \varepsilon(f_3^2 + 2f_1f_2) \end{cases} \tag{2.52}$$

In [40] it was shown that if $1/3 < |\varepsilon| \leq 1/\sqrt{3}$, then the operator (2.52) is a quasi q.q.o., while the operator Δ_ε is not positive. Namely, Δ_ε is not q.q.o.

It is interesting to know for which class of operators Δ, its quasiness implies its positivity. Namely, when a quasi q.q.o. is a q.q.o. In the next chapter, the raised question will be discussed.

Chapter 3: Quantum Lotka-Volterra Operators

3.1 Quantum quadratic operators on $DM_2(\mathbb{C})$

In this section, symmetric commutative q.q.o.s on the commutative algebra $DM_2(\mathbb{C})$ will be described. Here $DM_2(\mathbb{C})$ is a commutative subalgebra of $M_2(\mathbb{C})$ generated by $\{\mathbb{1}, \sigma_3\}$. In this setting, every element $x \in DM_2(\mathbb{C})$ can be written as follows: $x = w_0\mathbb{1} + w_3\sigma_3$, where $w_0, w_3 \in \mathbb{C}$. Let $\Delta : DM_2(\mathbb{C}) \to DM_2(\mathbb{C}) \otimes DM_2(\mathbb{C})$ be a unital symmetric linear operator. Then, the operator Δ in terms of the basis of $DM_2(\mathbb{C}) \otimes DM_2(\mathbb{C})$ can be written as follows:

$$\Delta(w_0\mathbb{1} + w_3\sigma_3) = w_0\mathbb{1} \otimes \mathbb{1} + w_3\Delta(\sigma_3). \tag{3.1}$$

where

$$\Delta(\sigma_3) = b_1\mathbb{1} \otimes \mathbb{1} + b_2(\mathbb{1} \otimes \sigma_3 + \sigma_3 \otimes \mathbb{1}) + b_3(\sigma_3 \otimes \sigma_3).$$

Therefore, from (3.1) one gets

$$\Delta(x) = (w_0 + w_3b_1)\mathbb{1} \otimes \mathbb{1} + w_3b_2(\mathbb{1} \otimes \sigma_3 + \sigma_3 \otimes \mathbb{1}) + w_3b_3\sigma_3 \otimes \sigma_3. \tag{3.2}$$

Theorem 3.1.1 *Let $\Delta : DM_2(\mathbb{C}) \to DM_2(\mathbb{C}) \otimes DM_2(\mathbb{C})$ be a unital, symmetric linear mapping. Then Δ is a q.q.o. if and only if*

$$|b_1 \pm 2b_2 + b_3| \leq 1, \tag{3.3}$$

$$|b_1 - b_3| \leq 1. \tag{3.4}$$

Proof. Let $x = w_0\mathbb{1} + w_3\sigma_3$ be positive, i.e. $w_0 > 0$, $|w_3| \leq w_0$. Without loss of generality, assume $w_0 = 1$. Let us rewrite Δ as follows (in the standard basis):

$$\Delta(x) = \mathbb{1} \otimes \mathbb{1} + w_3 \widetilde{\Delta}$$

where

$$\widetilde{\Delta} = \begin{pmatrix} b_1 + 2b_2 + b_3 & 0 & 0 & 0 \\ 0 & b_1 - b_3 & 0 & 0 \\ 0 & 0 & b_1 - b_3 & 0 \\ 0 & 0 & 0 & b_1 - 2b_2 + b_3 \end{pmatrix}.$$

It is known that the positivity of the matrix $\Delta(x)$ is equivalent to the positivity of its eigenvalues. Its eigenvalues are given by

$$\lambda_1 = 1 + w_3(b_1 + 2b_2 + b_3),$$
$$\lambda_2 = 1 + w_3(b_1 - b_3),$$
$$\lambda_3 = 1 + w_3(b_1 - 2b_2 + b_3).$$

Using $|w_3| \leq 1$, one concludes that $\lambda_1, \lambda_2, \lambda_3$ are positive if and only if

$$|b_1 \pm 2b_2 + b_3| \leq 1$$
$$|b_1 - b_3| \leq 1.$$

This completes the proof. □

Note that any state φ on $DM_2(\mathbb{C})$ has a form:

$$\varphi(w_0 \mathbb{1} + w_3 \sigma_3) = w_0 + f_3 w_3, \quad |f_3| \leq 1. \tag{3.5}$$

By denoting $x = (1+f_3)/2$, one can rewrite the functional in Equation (3.5) as follows:

$$\varphi(w_0\mathbb{1} + w_3\sigma_3) = w_0 + (2x-1)w_3, \quad x \in [0,1]. \tag{3.6}$$

Hence, there is a one-to-one correspondence between the states of $DM_2(\mathbb{C}))$ and $[0,1]$. Then, from (3.2), one finds:

$$\begin{aligned}\varphi \otimes \varphi(\Delta(x)) &= \varphi \otimes \varphi(\Delta(w_0\mathbb{1} + w_3\sigma_3)) \\ &= w_0\mathbb{1} + w_3\varphi \otimes \varphi(\Delta(\sigma_3)) \\ &= w_0 + w_3(b_1 + b_3 f_3^2 + 2b_2 f_3). \end{aligned} \tag{3.7}$$

On the other hand, due to the correspondence (see (3.6)), one has:

$$\varphi \otimes \varphi(\Delta(x)) = w_0 + (2x'-1)w_3.$$

The last one together with (3.7) implies

$$2x' - 1 = b_1 + b_3 f_3^2 + 2b_2 f_3. \tag{3.8}$$

Keeping in mind $f_3 = 2x-1$, from (3.8) one finds:

$$x' = 2b_3x^2 + 2(b_2 - b_3)x + \frac{1 + b_1 + b_3 - 2b_2}{2}. \tag{3.9}$$

Now, the goal is to reduce the mapping (3.9) to some quadratic stochastic operator (QSO) on the simplex $S^1 = \{(x,y): \ x, y \geq 0, \ x+y = 1\}$. First, recall that any QSO on S^1, is given by

$$x' = P_{11,1}x^2 + 2P_{12,1}xy + P_{22,1}y^2, \tag{3.10}$$

$$y' = P_{11,2}x^2 + 2P_{12,2}xy + P_{22,2}y^2. \tag{3.11}$$

where

$$P_{ij,k} \geq 0,\ P_{ij,k} = P_{ji,k},\ P_{ij,1} + P_{ij,2} = 1$$

for all $i, j, k \in \{1,2\}$. Due to $x + y = 1$, it is enough to consider (3.10). So,

$$x' = P_{11,1}x^2 + 2P_{12,1}x(1-x) + P_{22,1}(1-x)^2.$$

Then

$$x' = x^2(P_{11,1} - 2P_{12,1} + P_{22,1}) + 2x(P_{12,1} - P_{22,1}) + P_{22,1}. \tag{3.12}$$

Now comparing Equations (3.12) and (3.9), one gets the following result.

Theorem 3.1.2 *Let $\Delta : DM_2(\mathbb{C}) \to DM_2(\mathbb{C}) \otimes DM_2(\mathbb{C})$ be a unital, symmetric linear operator. Then the following conditions are equivalent:*

(i) *Δ is a q.q.o.;*

(ii) *the transformation* (3.9) *is a QSO, where the corresponding coefficients are defined by*

$$P_{11,1} = \frac{b_1 + 2b_2 + b_3 + 1}{2},$$
$$P_{22,1} = \frac{b_1 - 2b_2 + b_3 + 1}{2},$$
$$P_{12,1} = \frac{b_1 - b_3 + 1}{2}.$$

Moreover, a reverse formula is given by

$$b_1 = \frac{P_{22,1} + P_{11,1} + 2P_{12,1} - 2}{2},$$
$$b_2 = \frac{P_{11,1} - P_{22,1}}{2},$$
$$b_3 = \frac{P_{11,1} - 2P_{12,1} + P_{22,1}}{2}.$$

Proof. (i) $\Rightarrow$ (ii) Equalizing the corresponding coefficients of (3.12) and (3.9), one finds

$$\begin{cases} 2b_3 = P_{11,1} - 2P_{12,1} + P_{22,1} \\ b_2 - b_3 = P_{12,1} - P_{22,1} \\ \frac{b_1 - 2b_2 + b_3 + 1}{2} = P_{22,1} \end{cases}$$

After a little calculation, one gets:

$$P_{11,1} = \frac{b_1 + 2b_2 + b_3 + 1}{2},$$
$$P_{22,1} = \frac{b_1 - 2b_2 + b_3 + 1}{2},$$
$$P_{12,1} = \frac{b_1 - b_3 + 1}{2}.$$

The positivity of Δ due to Theorem 3.1.1 yields:

$$\begin{aligned} &\mid 2P_{11,1} - 1 \mid \leq 1, \\ &\mid 2P_{22,1} - 1 \mid \leq 1, \\ &\mid 2P_{12,1} - 1 \mid \leq 1. \end{aligned} \tag{3.13}$$

The last inequalities imply that $P_{11,1}, P_{22,1}, P_{12,1} \in [0,1]$, this means that (3.9) is a QSO.

(ii) $\Rightarrow$ (i) Assume that we have a QSO is given by $\{P_{i,j,k}\}$. Let us define

$$b_1 = \frac{P_{22,1} + P_{11,1} + 2P_{12,1} - 2}{2},$$
$$b_2 = \frac{P_{11,1} - P_{22,1}}{2},$$
$$b_3 = \frac{P_{11,1} - 2P_{12,1} + P_{22,1}}{2}.$$

To show that Δ is a q.q.o. we need to check the conditions of Theorem 3.1.1.

One can see that:

$$b_1 - b_3 = 2P_{12,1} - 1,$$

$$b_1 + 2b_2 + b_3 = 2P_{11,1} - 1,$$

$$b_1 - 2b_2 + b_3 = 2P_{22,1} - 1.$$

Therefore, due to $P_{ij,1} \in [0,1]$, we obtain the required assertion. □

From this Theorem, we infer that any QSO (see (3.12)) defines a q.q.o. by the following formula:

$$\Delta(x) = \left(w_0 + \frac{P_{11,1} + 2P_{12,1} + P_{22,1} - 2}{2} \cdot w_3\right) \mathbb{1} \otimes \mathbb{1}$$
$$+ \frac{(P_{11,1} - P_{22,1})w_3}{2}\left(\mathbb{1} \otimes \sigma_3 + \sigma_3 \otimes \mathbb{1}\right)$$
$$+ \frac{(P_{11,1} - 2P_{12,1} + P_{22,1})w_3}{2}\sigma_3 \otimes \sigma_3. \quad (3.14)$$

3.2 Quasi quantum quadratic operators on $DM_2(\mathbb{C})$

In the pervious section, the conditions on the parameters (b_1, b_2, b_3) so that Δ becomes a q.q.o are found. In this section, we are going to describe symmetric quasi q.q.o. on $DM_2(\mathbb{C})$. To formulate the result, the following well-known auxiliary fact.

Lemma 3.2.1 *Let $f(x) = ax^2 + bx + c$. Then the following conditions are equivalent:*

(i) $f(x) \geq 0$ *for all* $x \in [0,1]$*;*

(ii) $c \geq 0$, $a + b + c \geq 0$ *and one of the following conditions is satisfied:*

I. $a > 0$,

(1) $b > 0$;

(2) $-b > 2a$*;*

(3) $b^2 - 4ac \leq 0$*;*

II. $a < 0$.

Proof. From $f(x) \geq 0, x \in [0,1]$, we get $f(0) \geq 0,\ f(1) \geq 0$, which yield $c \geq 0, a + b + c \geq 0$, respectively. Now, two sperate cases are considered.

I) Assume that $a > 0$, then to have $f(x) \geq 0$, for all $x \in [0,1]$, there are only three possibilities:

(1) $-\frac{b}{2a} < 0$;

(2) $-\frac{b}{2a} > 1$;

(3) $b^2 - 4ac \leq 0$.

The last conditions imply the assertion .

II. Now, let $a < 0$. Then there is only possible case, which is $b^2 - 4ac \geq 0$. Due to $a < 0$, $c \geq 0$, the last condition, i.e. $b^2 - 4ac \geq 0$ holds. Therefore $f(x) \geq 0$, for all $x \in [0,1]$, $a < 0$. This completes the proof . □

Now, the main result of this section will be stated.

Theorem 3.2.2 *Let* $\Delta : DM_2(\mathbb{C}) \to DM_2(\mathbb{C}) \otimes DM_2(\mathbb{C})$ *be a linear mapping given by* (3.2). *Then* Δ *is quasi q.q.o. if and only if* $| b_1 \pm 2b_2 + b_3 | \leq 1$ *and one of the following conditions is satisfied:*

(i) $b_3(b_3 + b_2) \leq 0$ *;*

(ii) $b_3(b_3 - b_2) \leq 0$*;*

(iii) $b_2^2 - b_1 b_3 - |b_3| \leq 0$.

Proof. To establish that Δ is a quasi q.q.o. it is enough to show $\varphi\otimes\varphi\circ\Delta$ is a state for all states φ. Thus, due to (3.9), is equivalent to

$$0\leq 2b_3x^2+2(b_2-b_3)x+\frac{b_1-2b_2+b_3+1}{2}\leq 1, \quad \text{for all} \ \ x\in[0,1]$$

The last inequality is equivalent to

$$\begin{cases} 2b_3x^2+2(b_2-b_3)x+\frac{b_1-2b_2+b_3+1}{2}\geq 0, \\ -2b_3x^2-2(b_2-b_3)x-\frac{b_1-2b_2+b_3+1}{2}+1\geq 0, \end{cases} \tag{3.15}$$

for all $x\in[0,1]$. Hence, the assertion of the Theorem immediately follows by applying Lemma 3.2.1. □

Now, an example of Δ which is a quasi q.q.o., but not a q.q.o will be provided. Let us take $b_1=0.75$, $b_2=0$, and $b_3=-0.45$. One can see that $\mid b_1-b_3\mid=1.2$ which means that the last condition of Theorem 3.1.1 is not satisfied, hence Δ is not a q.q.o. Now, the condition of Theorem 3.2.2 will be checked. It is easy to see that $\mid b_1\pm 2b_2+b_3\mid=0.3$ and $b_2^2-|b_3|-b_1b_2=-0.1125$. Hence, Theorem 3.2.2 (iii) implies that Δ is a quasi q.q.o. Now, the following question is interested: does a commutative quasi q.q.o Δ coincide with a q.q.o.?
First, recall that a QSO (3.12) is called a *Lotka-Volterra operator* (see [39] for details) if $P_{ij,k}=0$ if $k\notin\{i,j\}$. This condition implies:

$$P_{11,1}=1, P_{22,1}=0, \ \text{and} \ 0\leq P_{12,1}\leq 1.$$

Therefore, according to (3.14) we obtain,

$$\Delta_V(x)=w_0\mathbb{1}\otimes\mathbb{1}+\frac{1}{2}w_3(\mathbb{1}\otimes\sigma_3+\sigma_3\otimes\mathbb{1})+\frac{2P_{12,1}-1}{2}w_3(\mathbb{1}\otimes\mathbb{1}-\sigma_3\otimes\sigma_3). \tag{3.16}$$

Denoting $a = 2P_{12,1} - 1$, we have $|a| \leq 1$ and the corresponding quadratic operator has the following form:

$$\varphi \otimes \varphi(\Delta(x)) = w_0 + \big(f_3 + \frac{a}{2}(1 - f_3^2)\big)w_3$$

where φ is a state.

Theorem 3.2.3 *Let* $\Delta : DM_2(\mathbb{C}) \to DM_2(\mathbb{C})DM_2(\mathbb{C})$ *be a linear operator given by* (3.16)*. Then, the following statements are equivalent:*

(i) $|a| \leq 1$*;*

(ii) Δ *is a q.q.o.;*

(iii) Δ *is a quasi q.q.o.*

Proof. The implication (i)⇔(ii) follows from Theorem 3.1.2. The implication (ii)⇒(iii) is obvious. It remains to establish (iii)⇒(i). For Δ given by (3.16) one finds:

$$b_1 = \frac{a}{2}, b_2 = \frac{1}{2}, b_3 = -\frac{a}{2}, \tag{3.17}$$

where $a = 2P_{12,1} - 1$. Assume that Δ is a quasi q.q.o. Then the triple $\{b_1, b_2, b_3\}$ satisfies the conditions of Theorem 3.2.2. If $b_3 \geq 0$, then $a \leq 0$. Hence, from (ii) of Theorem 3.2.2 implies $a \geq -1$. If $b_3 < 0$, then $a > 0$. From (i) of Theorem 3.2.2, one gets find $a \geq 1$. This completes the proof. □

3.3 Quantum Lotka-Volterra operators on $M_2(\mathbb{C})$

In this section, a quantum analogue of Lotka-Volterra operators on $M_2(\mathbb{C})$ is defined. A Lotka-Volterra operator on $M_2(\mathbb{C})$ is defined as follows [23]:

$$\Delta_a(w_0\mathbb{1}+\mathbf{w}\sigma) = w_0\mathbb{1}\otimes\mathbb{1}+\frac{1}{2}w_3(\mathbb{1}\otimes\sigma_3+\sigma_3\otimes\mathbb{1})+\frac{a}{2}w_3(\mathbb{1}\otimes\mathbb{1}-\sigma_3\otimes\sigma_3), \quad (3.18)$$

where $|a|\leq 1$. Let $\tilde{\mathscr{E}} : M_2(\mathbb{C}) \to DM_2(\mathbb{C})$ denote the standard projection defined by

$$\tilde{\mathscr{E}}(w_0\mathbb{1}+\mathbf{w}\sigma) = w_0\mathbb{1}+w_3\sigma_3. \quad (3.19)$$

Denote $\mathscr{E} = \tilde{\mathscr{E}}\otimes\tilde{\mathscr{E}}$.

Definition 3.3.1 A symmetric q.q.o. $\Delta : M_2(\mathbb{C}) \to M_2(\mathbb{C})\otimes M_2(\mathbb{C})$ is called *Quantum Lotka-Volterra operator*, if one has

$$\mathscr{E}\circ\Delta = \Delta_a \quad (3.20)$$

for some $a \in [-1,1]$.

Then, using (2.46), the following proposition is obtained.

Proposition 3.3.1 *Let $\Delta : M_2(\mathbb{C}) \to M_2(\mathbb{C})\otimes M_2(\mathbb{C})$ be a quantum Lotka-Volterra operator. Then, it has the following form:*

$$\begin{aligned}\Delta(w_0\mathbb{1}+\mathbf{w}\sigma) &= (w_0+\frac{a}{2}w_3)\mathbb{1}\otimes\mathbb{1} \\ &+\mathbb{1}\otimes\mathbf{Bw}\cdot\sigma+\mathbf{Bw}\cdot\sigma\otimes\mathbb{1}+\sum_{m,i=1}^{3}\langle\mathbf{b}_{ml},\overline{\mathbf{w}}\rangle\sigma_m\otimes\sigma_l \end{aligned} \quad (3.21)$$

where $\mathbf{b}_{ml} = (b_{ml,1}, b_{ml,2}, b_{ml,3})$, $\mathbf{b}_{33} = (0, 0, -a/2)$ *and*

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1/2 \end{pmatrix}.$$

The following particular case will be studied.

Theorem 3.3.2 *Let* $\Delta_{\lambda,\mu,a} : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ *be given as follows:*

$$\begin{aligned} \Delta_{\lambda,\mu,a}(w_0\mathbb{1} + \mathbf{w}\sigma) = (w_0 + \frac{a}{2}w_3)\mathbb{1}\otimes\mathbb{1} + \lambda w_1(\sigma_1\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_1) \\ +\mu w_2(\sigma_2\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_2) \\ +\frac{w_3}{2}(\sigma_3\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_3) - \frac{a}{2}w_3(\sigma_3\otimes\sigma_3), \end{aligned} \tag{3.22}$$

where $\lambda, \mu \in \mathbb{R}$ *and* $a \in [-1, 1]$. *Then the following conditions are equivalent:*

(i) one has

$$\max\{|\lambda|, |\mu|\} \leq \frac{\sqrt{1-|a|}}{2}. \tag{3.23}$$

(ii) $\Delta_{\lambda,\mu,a}$ *is a quantum Lotka-Volterra operator.*

Proof. (i)⇒(ii) Take any $x \in M_2(\mathbb{C})$ with $x \geq 0$, i.e. $x = w_0\mathbb{1} + \mathbf{w}\sigma$, $\|\mathbf{w}\| \leq w_0$. Without lost of generality, assume that $w_0 = 1$. Then, from (3.22), one finds:

$$\Delta_{\lambda,\mu,a}(x) = \begin{pmatrix} 1+U & V & V & 0 \\ \bar{V} & 1+aU & 0 & V \\ \bar{V} & 0 & 1+aU & V \\ 0 & \bar{V} & \bar{V} & 1-U \end{pmatrix} \tag{3.24}$$

where

$$U = w_3, \;\; V = \lambda w_1 - i\mu w_2. \tag{3.25}$$

Now, to verify the positivity of the matrix (3.24), the Silvester criterion will be used. It is clear that $1+U \geq 0$ and $1+aU \geq 0$, since $|w_3| \leq 1$. One can calculate that:

$$M_2 = \begin{vmatrix} 1+U & V \\ \bar{V} & 1+aU \end{vmatrix} = (1+U)(1+aU) - |V|^2,$$

$$M_3 = \begin{vmatrix} 1+U & V & V \\ \bar{V} & 1+aU & 0 \\ \bar{V} & 0 & 1+aU \end{vmatrix} = (1+aU)\big((1+U)(1+aU) - 2|V|^2\big),$$

$$M_4 = \begin{vmatrix} 1+U & V & V & 0 \\ \bar{V} & 1+aU & 0 & V \\ \bar{V} & 0 & 1+aU & V \\ 0 & \bar{V} & \bar{V} & 1-U \end{vmatrix} = (1+aU)\big((1-U^2)(1+aU) - 4|V|^2\big).$$

It is sufficient to show the positivity of M_4, since it yields the positivity of M_3.

Indeed, keeping in mind the positivity of M_4 and $0 \leq 1-U \leq 2$, one gets:

$$\begin{aligned} 0 \leq (1-U^2)(1+aU)-4|V|^2 &= (1-U)(1+U)(1+aU)-2\cdot 2|V|^2 \\ &\leq (1-U)(1+U)(1+aU)-2(1-U)|V|^2 \\ &= (1-U)\big((1+U)(1+aU)-2|V|^2\big). \end{aligned}$$

To verify the positivity of M_3, it is enough to establish $(1-U^2)(1+aU)-4|V|^2 \geq 0$ which according to (3.25) is equivalent to:

$$4(\lambda^2 w_1^2+\mu^2 w_2^2) \leq (1-w_3^2)(1+aw_3). \tag{3.26}$$

By $w_1^2+w_2^2+w_3 \leq 1$, to verify (3.26) it is sufficient to show

$$4\gamma(1-w_3^2) \leq (1-w_3^2)(1+aw_3), \tag{3.27}$$

where $\gamma = \max\{\lambda^2,\mu^2\}$. From (3.27), it follows that

$$(1-w_3^2)\big(4\gamma-1-aw_3\big) \leq 0, \ \text{ while } |w_3| \leq 1.$$

Clearly, it is true if

$$4\gamma \leq 1 \pm a \ \Leftrightarrow \ \max\{\lambda^2,\mu^2\} \leq \frac{1-|a|}{4}$$

which implies the assertion. (ii)$\Rightarrow$(i) from (3.22), for every state φ (which corresponds to the vector $\mathbf{f} = (f_1, f_2, f_3) \in \mathbb{R}^3$), one finds:

$$(V_{\Delta_{\lambda,\mu,a}}(\varphi))(x) = w_0 + 2\lambda f_1 w_1 + 2\mu f_2 w_2 + \big(f_3 + \frac{a}{2}(1 - f_3^2)\big) w_3.$$

Hence, the quasiness condition for $\Delta_{\lambda,\mu,a}$ is equivalent to:

$$(2\lambda f_1)^2 + (2\mu f_2)^2 + \big(f_3 + \frac{a}{2}(1 - f_3^2)\big)^2 \leq 1, \text{ for all } \|\mathbf{f}\| \leq 1.$$

It is clear that the last one is satisfied if

$$4\gamma(f_1^2 + f_2^2) + \frac{a^2}{4}(1 - f_3^2) + |a||f_3|(1 - f_3^2) + f_3^2 \leq 1, \text{ for all } \|\mathbf{f}\| \leq 1. \quad (3.28)$$

Now, due to $f_1^2 + f_2^3 \leq 1 - f_3^2$, from (3.28), one obtains

$$(1 - f_3^2)\left(4\gamma + \frac{a^2}{4}(1 - f_3^2)^2 + |a||f_3| - 1\right) \leq 0, \text{ for all } |f_3| \leq 1$$

which is equivalent to

$$\frac{a^2}{4} f_3^2 - |a||f_3| + 1 - 4\gamma - \frac{a^2}{4} \geq 0, \text{ for all } |f_3| \leq 1. \quad (3.29)$$

Hence, Lemma 3.2.1 implies that (3.29) is satisfied if and only if

$$4\gamma \leq 1 - |a|, \ 4\gamma \leq 1 - \frac{a^2}{4}.$$

This, due to $|a| \leq 1$, yields:

$$\gamma \leq \frac{1 - |a|}{4},$$

which together with $\gamma = \max\{\lambda^2, \mu^2\}$ implies (3.23). The proof is completed. □

By $E : M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \to M_2(\mathbb{C})$, a conditional expectation is denoted and defined by

$$E(x \otimes y) = \tau(y)x, \;\; x, y \in M_2(\mathbb{C}), \tag{3.30}$$

where τ is a normalized trace, i.e. $\tau = tr/2$. It is well-known that E is positive [27]. By means of $\Delta_{\lambda,\mu,a}$, let us define a mapping $\Phi_{\lambda,\mu,a} : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ by

$$\Phi_{\lambda,\mu,a} := E \circ \Delta_{\lambda,\mu,a}. \tag{3.31}$$

It is evident that $\Phi_{\lambda,\mu,a}$ is unital, but not trace preserving. Its positivity is given in the next result.

Theorem 3.3.3 *Let $\Delta_{\lambda,\mu,a}$ and $\Phi_{\lambda,\mu,a}$ be given by* (3.22),(3.31)*, respectively. Then the following statements hold:*

(i) if λ, μ, and a satisfy (3.23), then both maps $\Delta_{\lambda,\mu,a}$ and $\Phi_{\lambda,\mu,a}$ are positive;

(ii) if

$$\frac{\sqrt{1-|a|}}{2} < \max\{|\lambda|, |\mu|\} \le \frac{\sqrt{5 - a^2 + \sqrt{a^4 - 10a^2 + 9}}}{2\sqrt{2}} \tag{3.32}$$

then $\Delta_{\lambda,\mu,a}$ is not positive, but $\Phi_{\lambda,\mu,a}$ is positive.

Proof. The statement (i) is immediate consequence of Theorem 3.3.2. Therefore, only (ii) will be established. Again Theorem 3.3.2 implies under (3.32) the mapping $\Delta_{\lambda,\mu,a}$ is not positive. Now, the mapping $\Phi_{\lambda,\mu,a}$ will be examined. From (3.22) and (3.31) one gets:

$$\Phi_{\lambda,\mu,a}(x) = \big(w_0 + \frac{a}{2}w_3\big)\mathbb{1} + \lambda w_1\sigma_1 + \mu w_2\sigma_2 + \frac{w_3}{2}\sigma_3$$

To establish the positivity of $\Phi_{\lambda,\mu,a}$ it is enough to prove the positivity $\phi\circ\Phi_{\lambda,\mu,a}$ for all states ϕ. Therefore, one have:

$$\begin{aligned}\phi(\Phi_{\lambda,\mu,a}(x)) &= w_0 + \frac{a}{2}w_3 + \lambda w_1 f_1 + \mu w_2 f_2 + \frac{f_3}{2}w_3\\ &= w_0 + \lambda w_1 f_1 + \mu w_2 f_2 + \frac{a+f_3}{2}w_3\end{aligned}$$

here the vector $\mathbf{f} = (f_1, f_2, f_3)$ corresponds to ϕ.

The positivity of $\phi\circ\Phi_{\lambda,\mu,a}$ is equivalent to

$$(\lambda f_1)^2 + (\mu f_2)^2 + \left(\frac{a+f_3}{2}\right)^2 \le 1, \text{ for all } \|\mathbf{f}\| \le 1.$$

The last one is satisfied if

$$\gamma(f_1^2 + f_2^2) + \frac{1}{4}(a^2 + 2|a||f_3| + f_3^2) \le 1, \text{ for all } \|\mathbf{f}\| \le 1, \tag{3.33}$$

where, as before, $\gamma = \max\{\lambda^2, \mu^2\}$. Now, due to $f_1^2 + f_2^3 \le 1 - f_3^2$, from (3.33), one finds:

$$4\gamma(1 - f_3^2) + a^2 + 2|a||f_3| + f_3^2 \le 4, \text{ for all } |f_3| \le 1$$

which is equivalent to

$$(4\gamma-1)u^2-2|a|u+4-a^2-4\gamma \geq 0, \text{ for all } 0 \leq u \leq 1. \tag{3.34}$$

Now, applying Lemma 3.2.1 to (3.34). Necessary conditions of the lemma implies:

$$\gamma \leq 1-\frac{a^2}{4}, \; a^2+2|a|-3 \leq 0.$$

Due to $|a| \leq 1$, the second one is satisfied. Hence,

$$\gamma \leq 1-\frac{a^2}{4}. \tag{3.35}$$

Now, two cases (i.e. $4\gamma-1>0$ and $4\gamma-1<0$) are considered separately.

Case (I). Let $4\gamma-1>0$, i.e. $\gamma>1/4$. Since $-2|a| \leq 0$, then, due to Lemma 3.2.1, we need to analyze two possibilities:

(a) $2|a| \geq 2(4\gamma-1)$;

(b) $4a^2-4(4\gamma-1)(4-a^2-4\gamma) \leq 0$.

The case (a) yields that

$$\gamma \leq \frac{|a|+1}{4}. \tag{3.36}$$

From (b), it follows that

$$a^2-(4\gamma-1)(4-a^2-4\gamma) \leq 0$$

which implies

$$4\gamma^2 + \gamma(a^2 - 5) + 1 \leq 0.$$

This yields

$$\frac{5 - a^2 - \sqrt{a^4 - 10a^2 + 9}}{8} \leq \gamma \leq \frac{5 - a^2 + \sqrt{a^4 - 10a^2 + 9}}{8}. \tag{3.37}$$

Since

$$\frac{5 - a^2 - \sqrt{a^4 - 10a^2 + 9}}{8} \leq \frac{|a| + 1}{4}$$

and combining both cases (3.36) and (3.37), one obtains

$$\frac{1}{4} < \gamma \leq \frac{5 - a^2 + \sqrt{a^4 - 10a^2 + 9}}{8}. \tag{3.38}$$

Case (II). Let $4\gamma - 1 < 0$, i.e. $\gamma < 1/4$, then Lemma 3.2.1 implies that (3.34) is true. Hence, combining both (I) and (II) cases and

$$\frac{5 - a^2 + \sqrt{a^4 - 10a^2 + 9}}{8} \leq 1 - \frac{a^2}{4}$$

then infer that if

$$\gamma \leq \frac{5 - a^2 + \sqrt{a^4 - 10a^2 + 9}}{8}. \tag{3.39}$$

then $\Phi_{\lambda,\mu,a}$ is positive. The following inequality is pointed

$$\frac{1 - |a|}{4} < \frac{5 - a^2 + \sqrt{a^4 - 10a^2 + 9}}{8}$$

is true. Therefore, under (3.32) the assertion is obtained. This completes the proof. □

Chapter 4: Flow of Quantum Genetic Lotka-Volterra Algebras

4.1 Definition of quantum genetic Lotka-Volterra algebras

In this section, a flow of quantum genetic Lotka-Volterra algebras will be defined. Before that, some axillary preparations are needed. Recall that:

$$\Delta_{\lambda,\mu,a}(w_0\mathbb{1}+w\sigma) = (w_0+\frac{a}{2}w_3)\mathbb{1}\otimes\mathbb{1}+\lambda w_1(\sigma_1\otimes\mathbb{1})+\mathbb{1}\otimes\sigma_1)$$
$$+\mu w_2(\sigma_2\otimes\mathbb{1}+\mathbb{1}\otimes\sigma_2)+\frac{w_3}{2}(\sigma_3\otimes\mathbb{1}+\mathbb{1}\otimes\sigma_3)-\frac{a}{2}w_3(\sigma_3\otimes\sigma_3)$$

where $\lambda,\mu\in\mathbb{R}$ and $a\in[-1,1]$. Let

$$\Phi_{\lambda,\mu,a}(x) = (w_0+\frac{a}{2}w_3)\mathbb{1}+\lambda w_1\sigma_1+\mu w_2\sigma_2+\frac{w_3}{2}\sigma_3.$$

For the sake of simplicity, in what follows, $\Delta_{\lambda,\mu,a}$ and $\phi_{\lambda,\mu,,a}$ are denoted by Δ and ϕ, respectively. The calculation of ϕ^n is given in the following lemma.

Lemma 4.1.1 *For every $n\geq 1$, one has:*

$$\Phi^n(w_0\mathbb{1}+w.\sigma) = [w_0+a(1-\frac{1}{2^n})w_3]\mathbb{1}+\lambda^n w_1\sigma_1+\mu^n w_2\sigma_2+\frac{w_3}{2^n}\sigma_3. \tag{4.1}$$

Proof. Let $x=w_0\mathbb{1}+w.\sigma$. For $n=1$, the statement is obvious. Assume that it is true for $n=k$, i.e.

$$\Phi^k(x) = [w_0+a(1-\frac{1}{2^k})w_3]\mathbb{1}+\lambda^k w_1\sigma_1+\mu^k w_2\sigma_2+\frac{w_3}{2^k}\sigma_3.$$

Then, for $n=k+1$, one gets:

$$\Phi^{k+1}(x) = \Phi(\Phi^k(w_0\mathbb{1} + w.\sigma))$$
$$= \Phi([w_0 + a(1 - \frac{1}{2^k})w_3]\mathbb{1} + \lambda^k w_1\sigma_1 + \mu^k w_2\sigma_2 + \frac{w_3}{2^k}\sigma_3)$$
$$= [w_0 + a(1 - \frac{1}{2^k})w_3 + \frac{a}{2}\frac{w_3}{2^k}]\mathbb{1} + \lambda\lambda^k w_1\sigma_1 + \mu\mu^k w_2\sigma_2 + \frac{w_3}{22^k}\sigma_3$$
$$= [w_0 + a(1 - \frac{1}{2^{k+1}})w_3]\mathbb{1} + \lambda^{k+1} w_1\sigma_1 + \mu^{k+1} w_2\sigma_2 + \frac{w_3}{2^{k+1}}\sigma_3.$$

By mathematical induction, the result is true for any $n \geq 1$. This completes the proof.

□

Using Φ^n, one may define

$$\Phi^t(x) = [w_0 + a(1 - \frac{1}{2^t})w_3]\mathbb{1} + \lambda^t w_1\sigma_1 + \mu^t w_2\sigma_2 + \frac{w_3}{2^t}\sigma_3, \quad t \geq 0 \tag{4.2}$$

The positivity of Φ implies the positivity of Φ^n, correspondingly one can infer that Φ^t is also positive mapping for every $t \geq 0$ under the condition (3.23) (see (3.32)).

Theorem 4.1.2 *The family $\{\Phi^t\}$ satisfies $\Phi^{t+s} = \Phi^t \circ \Phi^s$, $\forall t, s \geq 0$.*

Proof. Let

$$\Phi^{t+s}(x) = [w_0 + a(1 - \frac{1}{2^{t+s}})w_3]\mathbb{1} + \lambda^{t+s} w_1\sigma_1 + \mu^{t+s} w_2\sigma_2 + \frac{w_3}{2^{t+s}}\sigma_3.$$

Now, simple calculations imply that:

$$
\begin{aligned}
&(\Phi^t \circ \Phi^s)(x) = \Phi^t(v_0 \mathbb{1} + v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3) \\
&= [v_0 + a(1 - \frac{1}{2^t})v_3]\mathbb{1} + \lambda^t v_1\sigma_1 + \mu^t v_2\sigma_2 + \frac{v_3}{2^t}\sigma_3 \\
&= [w_0 + a(1 - \frac{1}{2^s})w_3 + a(1 - \frac{1}{2^t})\frac{w_3}{2^s}]\mathbb{1} + \lambda^t\lambda^s w_1\sigma_1 + \mu^t\mu^s w_2\sigma_2 + \frac{w_3}{2^{t+s}}\sigma_3 \\
&= [w_0 + aw_3(1 - \frac{1}{2^s} + \frac{1}{2^s} - \frac{1}{2^{t+s}})]\mathbb{1} + \lambda^{t+s} w_1\sigma_1 + \mu^{t+s} w_2\sigma_2 + \frac{w_3}{2^{t+s}}\sigma_3 \\
&= [w_0 + a(1 - \frac{1}{2^{t+s}})w_3]\mathbb{1} + \lambda^{t+s} w_1\sigma_1 + \mu^{t+s} w_2\sigma_2 + \frac{w_3}{2^{t+s}}\sigma_3
\end{aligned}
$$

which yields $\Phi^{t+s}(x) = (\Phi^t \circ \Phi^s)(x)$. Here it was used $v_0 = w_0 + a(1 - \frac{1}{2^s})w_3, v_1 = \lambda^s w_1, v_2 = \mu^s w_2$ and $v_3 = \frac{w_3}{2^s}$. □

Let us define $\Delta_t = \Delta \circ \Phi^t$, i.e.

$$
\begin{aligned}
\Delta_t(w_0\mathbb{1} + w\cdot\sigma) &= [w_0 + a(1 - \frac{1}{2^t})w_3 + \frac{a}{2^{t+1}}w_3]\mathbb{1}\otimes\mathbb{1} + \lambda^{t+1}w_1(\sigma_1\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_1) \\
&+\mu^{t+1}w_2(\sigma_2\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_2) + \frac{w_3}{2^{t+1}}(\sigma_3\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_3) - \frac{a}{2^{t+1}}w_3(\sigma_3\otimes\sigma_3). \qquad (4.3)
\end{aligned}
$$

Let Δ_t be given by (4.3). By means of Δ_t, one may introduce the following binary operation on $M_2(\mathbb{C})$ as follows:

$$
(\mathbf{f} \circ_t \mathbf{p})(x) = (\mathbf{f}\otimes\mathbf{p})(\Delta_t(x)), \mathbf{f}, \mathbf{p} \in M_2(\mathbb{C})^*, x \in M_2(\mathbb{C})^* \qquad (4.4)
$$

where, as before $M_2(\mathbb{C})^*$ is the dual of $M_2(\mathbb{C})$.

The triple $(M_2(\mathbb{C})^*, \circ_t, \Delta_t)$ is called a flow of quantum genetic Lotka-Volterra algebras (FQGLV-A). This flow is denoted by A_t, i.e. $A_t = (M_2(\mathbb{C})^*, \circ_t, \Delta_t)$.

From (4.4) and (4.3), one immediately finds:

$$\mathbf{f}\circ_t\mathbf{p}=(f_0p_0,\lambda^{t+1}(f_1p_0+f_0p_1),\mu^{t+1}(f_2p_0+f_0p_2),$$
$$f_0p_0(a-\frac{a}{2^t}+\frac{a}{2^{t+1}})+\frac{1}{2^{t+1}}(f_3p_0+f_0p_3)-\frac{a}{2^{t+1}}(f_3p_3)) \tag{4.5}$$

where $\mathbf{f}=(f_0,f_1,f_2,f_3)$ and $\mathbf{p}=(p_0,p_1,p_2,p_3)$.

Recall that $A_t=(M_2(\mathbb{C})^*,\circ_t,\Delta_t)$ is associative if

$$(\mathbf{f}\circ_t\mathbf{p})\circ_t\mathbf{h}=\mathbf{f}\circ_t(\mathbf{p}\circ_t\mathbf{h})$$

for all $\mathbf{f},\mathbf{p},\mathbf{h}\in M_2(\mathbb{C})^*$.

For the sake of simplicity, FQGLV-A is denoted by $(A_t,\circ_t)$.

Theorem 4.1.3 *A FQGLV-A $(A_t, \circ_t)$ is associative if and only if $t \geq 0, |a| = 1$, and $\lambda = \mu = 0$.*

Proof. Due to the conditions of the associativity of $(A_t, \circ_t)$ one has to compute $(\mathbf{f} \circ_t \mathbf{p}) \circ_t \mathbf{h}$ and $\mathbf{f} \circ_t (\mathbf{p} \circ_t \mathbf{h})$ and compare the corresponding coordinates. Hence, one gets:

$$((\mathbf{f} \circ_t \mathbf{p}) \circ_t \mathbf{h})_1 = f_0 p_0 h_0,$$

$$(\mathbf{f} \circ_t (\mathbf{p} \circ_t \mathbf{h}))_1 = f_0 p_0 h_0.$$

Thus, the first component does not produce any condition. The second component implies that:

$$((\mathbf{f} \circ_t \mathbf{p}) \circ_t \mathbf{h})_2 = \lambda^{t+1}[(\lambda^{t+1}(f_1 p_0 + f_0 p_1))h_0 + f_0 p_0 h_1]$$

$$= \lambda^{2t+2} f_1 p_0 h_0 + \lambda^{2t+2} f_0 p_1 h_0 + \lambda^{t+1} f_0 p_0 h_1$$

$$(\mathbf{f} \circ_t (\mathbf{p} \circ_t \mathbf{h}))_2 = \lambda^{t+1}[(f_1 p_0 h_0) + f_0(\lambda^{t+1}(p_1 h_0 + p_0 h_1))]$$

$$= \lambda^{t+1} f_1 p_0 h_0 + \lambda^{2t+2} f_0 p_1 h_0 + \lambda^{2t+2} f_0 p_0 h_1.$$

Hence,

$$\lambda^{2t+2} = \lambda^{t+1}.$$

The third components yield:

$$((\mathbf{f} \circ_t \mathbf{p}) \circ_t \mathbf{h})_3 = \mu^{t+1}[(\mu^{t+1}(f_2p_0 + f_0p_2))h_0 + f_0p_0h_2]$$

$$= \mu^{2t+2}f_2p_0h_0 + \mu^{2t+2}f_0p_2h_0 + \mu^{t+1}f_0p_0h_2$$

$$(\mathbf{f} \circ_t (\mathbf{p} \circ_t \mathbf{h}))_3 = \mu^{t+1}[(f_2p_0h_0) + f_0(\mu^{t+1}(p_2h_0 + p_0h_2))]$$

$$= \mu^{t+1}f_2p_0h_0 + \mu^{2t+2}f_0p_2h_0 + \mu^{2t+2}f_0p_0h_2.$$

So,

$$\mu^{2t+2} = \mu^{t+1}.$$

The fourth components imply:

$$((\mathbf{f} \circ_t \mathbf{p}) \circ_t \mathbf{h})_4 = f_0p_0h_0(a - \frac{a}{2^t} + \frac{a}{2^{t+1}})$$

$$+\frac{1}{2^{t+1}}[(f_0p_0(a - \frac{a}{2^t} + \frac{a}{2^{t+1}}) + \frac{1}{2^{t+1}}(f_3p_0 + f_0p_3)$$

$$-\frac{a}{2^{t+1}}f_3p_3)h_0] + \frac{1}{2^{t+1}}(f_0p_0h_3)$$

$$-\frac{a}{2^{t+1}}h_3[f_0p_0(a - \frac{a}{2^t} + \frac{a}{2^{t+1}}) + \frac{1}{2^{t+1}}(f_3p_0 + f_0p_3) - \frac{a}{2^t+1}f_3p_3]$$

$$= f_0p_0h_0(a - \frac{a}{2^t} + \frac{a}{2^{t+1}}) + f_0p_0h_0(\frac{a}{2^{t+1}} - \frac{a}{2^{2t+1}} + \frac{a}{2^{2t+2}})$$

$$+\frac{1}{2^{2t+2}}f_3p_0h_0 + \frac{1}{2^{2t+2}}(f_0p_3h_0) - \frac{a}{2^{2t+2}}(f_3p_3h_0)$$

$$+\frac{1}{2^{t+1}}h_3f_0p_0 - \frac{a^2}{2^{t+1}}f_0p_0h_3 + \frac{a^2}{2^{2t+1}}f_0p_0h_3 - \frac{a^2}{2^{2t+2}}f_0p_0h_3$$

$$-f_3p_0h_3\frac{a}{2^{2t+2}} - \frac{a}{2^{2t+2}}f_0p_3h_3 + \frac{a^2}{2^{2t+2}}f_3p_3h_3,$$

and

$$
\begin{aligned}
&(\mathbf{f}\circ_t(\mathbf{p}\circ_t\mathbf{h}))_4 = f_0p_0h_0(a-\frac{a}{2^t}+\frac{a}{2^{t+1}})\\
&+\frac{1}{2^{t+1}}[f_3p_0h_0+f_0(p_0h_0(a-\frac{a}{2^t}+\frac{a}{2^{t+1}}))+\frac{1}{2^{t+1}}(p_3h_0+p_0h_3)\\
&-\frac{a}{2^{t+1}}(p_3h_3)]-\frac{a}{2^{t+1}}[f_3(p_0h_0(a-\frac{a}{2^t}+\frac{a}{2^{t+1}})\\
&+\frac{1}{2^{t+1}}(p_3h_0+p_0h_3)-\frac{a}{2^{t+1}}(p_3h_3)]\\
&=(a-\frac{a}{2^t}+\frac{a}{2^{t+1}})f_0p_0h_0+\frac{1}{2^{t+1}}f_3p_0h_0\\
&+f_0p_0h_0(\frac{a}{2^{t+1}}-\frac{a}{2^{2t+1}}+\frac{a}{2^{2t+2}})+f_0p_3h_0\frac{1}{2^{2t+2}}\\
&+\frac{1}{2^{2t+2}}p_0h_3f_0-\frac{a}{2^{2t+2}}p_3h_3f_0-\frac{a^2}{2^{t+1}}f_3p_0h_0+\frac{a^2}{2^{2t+1}}f_3p_0h_0\\
&-\frac{a^2}{2^{2t+2}}f_3p_0h_0-\frac{a}{2^{2t+2}}h_0f_3p_3-\frac{a}{2^{2t+2}}f_3p_0h_3+\frac{a^2}{2^{2t+2}}f_3p_3h_3
\end{aligned}
$$

which yields:

$$\frac{1}{2^{2t+2}}=\frac{1}{2^{t+1}}+\frac{a^2}{2^{2t+1}}-\frac{a^2}{2^{2t+2}}-\frac{a^2}{2^{t+1}}.$$

Finally, one gets the following system of equations:

$$
\begin{aligned}
&\lambda^{2t+2}=\lambda^{t+1},\\
&\mu^{2t+2}=\mu^{t+1},\\
&1=2^{t+1}(1-a^2)+a^2.
\end{aligned}
$$

Hence,

$$1=(1+a^2)(1-a^2)+a^2,$$

which yield $|a| = 1$ and $t \geq 0$. The first two conditions imply that $\mu^2 = \mu$ or $\mu = 0, 1$ and $\lambda^2 = \lambda$ or $\lambda = 0, 1$. By using the fact that:

$$\max\{|\lambda|, |\mu|\} \leq \frac{\sqrt{1-|a|}}{2}.$$

This completes the proof. □

A commutative algebra $(A, \circ)$ is called alternative if $(\mathbf{x} \circ \mathbf{x}) \circ \mathbf{y} = \mathbf{x} \circ (\mathbf{x} \circ \mathbf{y})$, for all $\mathbf{x}, \mathbf{y} \in A$.

Theorem 4.1.4 *A FQGLV-A $(A_t, \circ_t)$ is alternative if and only if $t \geq 0, |a| = 1$, and $\lambda = \mu = 0$.*

Proof. One can find

$$\mathbf{f} \circ_t \mathbf{f} = (f_0^2, 2f_0 f_1 \lambda^{t+1}, 2f_0 f_2 \mu^{t+1}, (a + \frac{a}{2^{t+1}} - \frac{a}{2^t}) f_0^2 + \frac{1}{2^t} f_0 f_3 - \frac{1}{2^{t+1}} a f_3^2). \quad (4.6)$$

Now, it is need it to find all coordinates of $(\mathbf{f} \circ_t \mathbf{f}) \circ_t \mathbf{p}$ and $\mathbf{f} \circ_t (\mathbf{f} \circ_t \mathbf{p})$ The first components are:

$$((\mathbf{f} \circ_t \mathbf{f}) \circ_t \mathbf{p})_1 = f_0^2 p_0$$

$$(\mathbf{f} \circ_t (\mathbf{f} \circ_t \mathbf{p}))_1 = f_0^2 p_0$$

which do not produce any condition. The second components are:

$$((\mathbf{f} \circ_t \mathbf{f}) \circ_t \mathbf{p})_2 = \lambda^{t+1}[(2f_0f_1\lambda^{t+1}p_0) + f_0^2p_1] = 2\lambda^{2t+2}f_0f_1p_0 + \lambda^{t+1}f_0^2p_1$$

$$(\mathbf{f} \circ_t (\mathbf{f} \circ_t \mathbf{p}))_2 = \lambda^{t+1}[f_0f_1p_0 + f_0\lambda^{t+1}(f_1p_0 + f_0p_1)]$$

$$= \lambda^{t+1}f_1f_0p_0 + \lambda^{2t+2}f_0f_1p_0 + \lambda^{2t+2}f_0^2p_1$$

which yield that:

$$\lambda^{t+1} = \lambda^{2t+2}.$$

Similarly, the third components are given by

$$((\mathbf{f} \circ_t \mathbf{f}) \circ_t \mathbf{p})_3 = \mu^{t+1}[(2f_0f_2\mu^{t+1}p_0) + f_0^2p_2] = 2\mu^{2t+2}f_0f_2p_0 + \mu^{t+1}f_0^2p_2$$

$$(\mathbf{f} \circ_t (\mathbf{f} \circ_t \mathbf{p}))_3 = \mu^{t+1}[f_0f_2p_0 + f_0\mu^{t+1}(f_2p_0 + f_0p_2)]$$

$$= \mu^{t+1}f_2f_0p_0 + \mu^{2t+2}f_0f_2p_0 + \mu^{2t+2}f_0^2p_2$$

which give the following ones:

$$\mu^{t+1} = \mu^{2t+2}.$$

The fourth components are calculated as follows:

$$((\mathbf{f} \circ_t \mathbf{f}) \circ_t \mathbf{p})_4 = f_0^2p_0(a - \frac{a}{2^t} + \frac{a}{2^{t+1}}) + \frac{1}{2^{t+1}}((a + \frac{a}{2^{t+1}} - \frac{a}{2^t})f_0^2p_0$$
$$+ \frac{1}{2^t}f_0f_3p_0 - \frac{1}{2^{t+1}}af_3^2p_0 + f_0^2p_3) - \frac{a}{2^{t+1}}((a + \frac{a}{2^{t+1}} - \frac{a}{2^t})f_0^2p_3$$
$$+ \frac{1}{2^t}f_0f_3p_3 - \frac{1}{2^{t+1}}af_3^2p_3)$$

and

$$(\mathbf{f} \circ_t (\mathbf{f} \circ_t \mathbf{p}))_4 = f_0^2 p_0 (a - \frac{a}{2^t} + \frac{a}{2^{t+1}}) + \frac{1}{2^{t+1}}[f_3 f_0 p_0 + f_0 (f_0 p_0 (a - \frac{a}{2^t} + \frac{a}{2^{t+1}})$$
$$+ \frac{1}{2^{t+1}}(f_3 p_0 + f_0 p_3) - \frac{a}{2^{t+1}} f_3 p_3)] - \frac{a}{2^{t+1}} f_3 [f_0 p_0 (a - \frac{a}{2^t} + \frac{a}{2^{t+1}})$$
$$+ \frac{1}{2^{t+1}}(f_3 p_0 + f_0 p_3) - \frac{a}{2^{t+1}}(f_3 p_3)]$$

which imply that:

$$\frac{-1}{2^{2t+2}}(-1 + 2^{t+1})(-1 + a^2) = 0.$$

Finally, under the assumption $t \geq 0$, one finds:

- The condition on second components implies that $\lambda = 0$ or $\lambda = 1$.
- The condition on third components implies that $\mu = 0$ or $\mu = 1$.
- The condition on fourth components implies that $a = -1$ or $a = 1$.

The condition $\max\{|\lambda|, |\mu|\} \leq \frac{\sqrt{1-|a|}}{2}$, yields $|a| = 1$ and $\lambda = \mu = 0$. □

An algebra $(A, \circ)$ is called Jordan algebra if $(\mathbf{x} \circ \mathbf{y}) \circ \mathbf{x}^2 = \mathbf{x} \circ (\mathbf{y} \circ \mathbf{x}^2)$, for all $\mathbf{x}, \mathbf{y} \in A$.

Theorem 4.1.5 *A FQGLV-A $(A_t, \circ_t)$ is Jordan algebra if and only if either one of the followings hold:*

- $|a| = 1, \lambda = \mu = 0, t \geq 0,$
- $a = 0, \lambda, \mu \in \{0, \frac{1}{2}\}, t = 0,$

Proof. One can find:

$$\mathbf{p}\circ_t\mathbf{f}^2=(f_0^2p_0,\lambda^{t+1}(f_0^2p_1+2f_0f_1p_0\lambda^{t+1}),$$
$$\mu^{t+1}(f_0^2p_2+2f_0f_2p_0\mu^{t+1}),(a-\frac{a}{2^t}+\frac{a}{2^{t+1}})f_0^2p_0$$
$$-\frac{a}{2^{t+1}}((a-\frac{a}{2^t}+\frac{a}{2^{t+1}})f_0^2+\frac{1}{2^t}f_0f_3-\frac{a}{2^{t+1}}f_3^2)p_3$$
$$+\frac{1}{2^{t+1}}(((a-\frac{a}{2^t}+\frac{a}{2^{t+1}})f_0^2+\frac{1}{2^t}f_0f_3-\frac{a}{2^{t+1}}f_3^2)p_0+f_0^2p_3)).$$

Now, it is need it to find all coordinates of $(\mathbf{f}\circ_t\mathbf{p})\circ_t\mathbf{f}^2$ and $\mathbf{f}\circ_t(\mathbf{p}\circ_t\mathbf{f}^2)$.

The first components are:

$$((\mathbf{f}\circ_t\mathbf{p})\circ_t\mathbf{f}^2)_1=f_0^3p_0,$$
$$(\mathbf{f}\circ_t(\mathbf{p}\circ_t\mathbf{f}^2))_1=f_0^3p_0,$$

which do not produce any condition. The second components imply that:

$$((\mathbf{f}\circ_t\mathbf{p})\circ_t\mathbf{f}^2)_2=\lambda^{t+1}[\lambda^{t+1}(f_1p_0+f_0p_1)f_0^2+f_0p_0(2f_0f_1\lambda^{t+1})]$$
$$=\lambda^{2t+2}f_1p_0f_0^2+\lambda^{2t+2}f_0^3p_1+2\lambda^{2t+2}f_0^2p_0f_1$$
$$=3\lambda^{2t+2}f_1p_0f_0^2+\lambda^{2t+2}f_0^3p_1,$$
$$(\mathbf{f}\circ_t(\mathbf{p}\circ_t\mathbf{f}^2))_2=\lambda^{t+1}[f_1f_0^2p_0+f_0(\lambda^{t+1}(f_0^2p_1+2f_0f_1p_0\lambda^{t+1}))]$$
$$=\lambda^{t+1}f_1p_0f_0^2+\lambda^{2t+2}f_0^3p_1+2\lambda^{3t+3}f_0^2p_0f_1,$$

which yield that:

$$3\lambda^{2t+2}=\lambda^{t+1}+2\lambda^{3t+3}.$$

Similarly, the third components are give by

$$((\mathbf{f} \circ_t \mathbf{p}) \circ_t \mathbf{f}^2)_3 = \mu^{t+1}[\mu^{t+1}(f_2 p_0 + f_0 p_2) f_0^2 + f_0 p_0 (2 f_0 f_2 \mu^{t+1})]$$

$$= \mu^{2t+2} f_2 p_0 f_0^2 + \mu^{2t+2} f_0^3 p_2 + 2\mu^{2t+2} f_0^2 p_0 f_2$$

$$= 3\mu^{2t+2} f_2 p_0 f_0^2 + \mu^{2t+2} f_0^3 p_2,$$

$$(\mathbf{f} \circ_t (\mathbf{p} \circ_t \mathbf{f}^2))_3 = \mu^{t+1}[f_2 f_0^2 p_0 + f_0(\mu^{t+1}(f_0^2 p_2 + 2 f_0 f_2 p_0 \mu^{t+1}))]$$

$$= \mu^{t+1} f_2 p_0 f_0^2 + \mu^{2t+2} f_0^3 p_2 + 2\mu^{3t+3} f_0^2 p_0 f_2,$$

which give the following one:

$$3\mu^{2t+2} = \mu^{t+1} + 2\mu^{3t+3}.$$

The fourth components are calculated as follows:

$$((\mathbf{f} \circ_t \mathbf{p}) \circ_t \mathbf{f}^2)_4 = f_0^3 p_0 (a - \frac{a}{2^t} + \frac{a}{2^{t+1}})$$
$$+ \frac{1}{2^{t+1}} f_0^2 [f_0 p_0 (a - \frac{a}{2^t} + \frac{a}{2^{t+1}}) + \frac{1}{2^{t+1}} (f_3 p_0 + f_0 p_3) - \frac{a f_3 p_3}{2^{t+1}}]$$
$$+ \frac{1}{2^{t+1}} f_0 p_0 (f_0^2 (a - \frac{a}{2^t} + \frac{a}{2^{t+1}}) + \frac{1}{2^t} - \frac{a f_3^2}{2^{t+1}})$$
$$- \frac{a}{2^{t+1}} [(f_0 p_0 (a - \frac{a}{2^t} + \frac{a}{2^{t+1}}) + \frac{1}{2^{t+1}} (f_3 p_0 + f_0 p_3) - \frac{a f_3 p_3}{2^{t+1}}]$$
$$.[f_0^2 (a - \frac{a}{2^t} + \frac{a}{2^{t+1}}) + \frac{1}{2^t} - \frac{a f_3^2}{2^{t+1}}]$$

and

$$(\mathbf{f}\circ_t(\mathbf{p}\circ_t\mathbf{f}^2))_4 = f_0^3p_0(a-\frac{a}{2^t}+\frac{a}{2^{t+1}})+\frac{1}{2^{t+1}}[f_3f_0^2p_0+f_0(a-\frac{a}{2^t}+\frac{a}{2^{t+1}})f_0^2p_0$$
$$-\frac{a}{2^{t+1}}((a-\frac{a}{2^t}+\frac{a}{2^{t+1}})f_0^2+\frac{f_0f_3}{2^t}-\frac{af_3^2}{2^{t+1}})p_3+\frac{1}{2^{t+1}}(((a-\frac{a}{2^t}+\frac{a}{2^{t+1}})f_0^2$$
$$+\frac{f_0f_3}{2^t}-\frac{af_3^2}{2^{t+1}})p_0+f_0^2p_3))]-\frac{a}{2^{t+1}}f_3[(a-\frac{a}{2^t}+\frac{a}{2^{t+1}})f_0^2p_0$$
$$-\frac{a}{2^{t+1}}((a-\frac{a}{2^t}+\frac{a}{2^{t+1}})f_0^2+\frac{f_0f_3}{2^t}-\frac{af_3^2}{2^{t+1}})p_3+\frac{1}{2^{t+1}}(((a-\frac{a}{2^t}+\frac{a}{2^{t+1}})f_0^2$$
$$+\frac{f_0f_3}{2^t}-\frac{af_3^2}{2^{t+1}})p_0+f_0^2p_3)),$$

which imply that:

$$\frac{-a}{2^{2t+1}}+\frac{a}{2^{t+1}}+\frac{a}{2^{3t+3}}+\frac{a^3}{2^{2t+1}}-\frac{a^3}{2^{t+1}}-\frac{a^3}{2^{3t+3}}=0,$$
$$\frac{3}{2^{2t+2}}-\frac{1}{2^{3t+2}}-\frac{1}{2^{t+1}}+\frac{a^2}{2^{3t+2}}-\frac{3a^2}{2^{2t+2}}+\frac{a^2}{2^{t+1}}=0$$
$$\frac{-a}{2^{2t+2}}+\frac{a}{2^{3t+3}}+\frac{a^3}{2^{2t+2}}-\frac{a^3}{2^{3t+3}}=0.$$

Now, one has the following conditions:

- The first condition $-\lambda^{t+1}+3\lambda^{2t+2}-2\lambda^{3t+3}=0$ yields:

$$\lambda^{t+1}(1-2\lambda^{t+1})(1-\lambda^{t+1})=0.$$

- The second condition can simplify as:

$$\mu^{t+1}(1-2\mu^{t+1})(1-\mu^{t+1})=0.$$

- The third condition yields:

$$\frac{-a}{2^{2t+1}}+\frac{a}{2^{t+1}}+\frac{a}{2^{3t+3}}+\frac{a^3}{2^{2t+1}}-\frac{a^3}{2^{t+1}}-\frac{a^3}{2^{3t+3}}=0$$

which implies:

$$2^{2+t}(a-a^3)-2^{2t+2}(a-a^3)-(a-a^3)=0.$$

So,

$$(2^{2+t}-1)^2a(a-1)(a+1)=0.$$

- From the fourth condition, one gets:

$$\frac{3}{2^{2t+2}}-\frac{1}{2^{3t+2}}-\frac{1}{2^{t+1}}+\frac{a^2}{2^{3t+2}}-\frac{3a^2}{2^{2t+2}}+\frac{a^2}{2^{t+1}}=0$$

which gives that:

$$(2^{1+t}-1)(2^t-1)(1-a)(a+1)=0.$$

- The fifth condition yields:

$$\frac{-a}{2^{2t+2}}+\frac{a}{2^{3t+3}}+\frac{a^3}{2^{2t+2}}-\frac{a^3}{2^{3t+3}}=0.$$

So,

$$(2^{t+1}-1)a(a-1)(a+1)=0.$$

One can summarize the obtained conditions as follows:

$$\lambda^{t+1}(1-2\lambda^{t+1})(1-\lambda^{t+1})=0,$$

$$\mu^{t+1}(1-2\mu^{t+1})(1-\mu^{t+1})=0,$$

$$(2^{1+t}-1)(2^{t}-1)(1-a)(a+1)=0,$$

$$(2^{t+1}-1)a(a-1)(a+1)=0.$$

These conditions can simplified by considering the following cases:

Case 1 Let $a=0$. Then, $(2^{t+1}-1)(2^{t}-1)=0$ which gives $t=0$. Then, $\lambda=0$, or $1-2\lambda=0$, or $1-\lambda=0$. Since $\max\{|\lambda|,|\mu|\}\leq\frac{\sqrt{1-|a|}}{2}$, then $\lambda\in\{0,\frac{1}{2}\}$. Similarly, $\mu\in\{0,\frac{1}{2}\}$.

Case 2 Let $a=\pm1$ and $t\geq0$. Then, $\lambda^{t+1}=0$, or $1-2\lambda^{t+1}=0$, or $1-\lambda^{t+1}=0$. Since $\max\{|\lambda|,|\mu|\}\leq\frac{\sqrt{1-|a|}}{2}$, then $\lambda=0$. Similarly, $\mu=0$.

This completes the proof. □

Remark If $a=0$ and $\lambda=\mu=0,t=0$, the algebra is Jordan, but not alternative. More information about Jordan and alternative algebra can be found in [42].

4.2 Idempotents

This section is devoted to the description of idempotents of the flow quantum genetic Lotka-Volterra algebras. Recall that an element $\mathbf{q}\in A$ is called idempotent if $\mathbf{q}\circ\mathbf{q}=\mathbf{q}$.

Theorem 4.2.1 *Let $(A_t,\circ_t)$ be a FQGLV-A. Then, $\mathbf{q}\in A_t$ is an idempotent if and only if one of the followings hold:*

1. $\mathbf{q}=(0,0,0,0)$,

2. $\mathbf{q} = (0,0,0,\frac{-2^{t+1}}{a})$, *if* $a \neq 0$,

3. $\mathbf{q} = (1,0,0,\frac{2-2^{t+1}\pm\sqrt{2^{t+2}-2^{t+3}+4-4a^2+4a^2 2^{t+1}}}{2a})$, *if* $a \neq 0, |\lambda|, |\mu| < \frac{1}{2}$.

4. $\mathbf{q} = (1,0,0,0)$ *if* $a = 0, |\lambda|, |\mu| < \frac{1}{2}$.

5. $\mathbf{q} = (1,q_1,0,q_3)$ *if* $a = 0, t = 0, |\lambda| = \frac{1}{2}, |\mu| < \frac{1}{2}$.

6. $\mathbf{q} = (1,0,q_2,q_3)$ *if* $a = 0, t = 0, |\lambda| < \frac{1}{2}, |\mu| = \frac{1}{2}$.

7. $\mathbf{q} = (1,q_1,q_2,q_3)$ *if* $a = 0, t = 0, |\lambda| = |\mu| = \frac{1}{2}$.

Proof. Let $\mathbf{q} = (q_0,q_1,q_2,q_3)$ be an idempotent vector in A_t. Then, $\mathbf{q} \circ_t \mathbf{q} = \mathbf{q}$ which implies that:

$$\left(q_0^2, 2q_0q_1\lambda^{t+1}, 2q_0q_2\mu^{t+1}, (a - \frac{a}{2^t} + \frac{a}{2^{t+1}})q_0^2 + \frac{q_0q_3}{2^t} - \frac{aq_3^2}{2^{t+1}}\right) = (q_0,q_1,q_2,q_3).$$

By comparing the components, one finds the following system of equations:

$$q_0^2 = q_0, \tag{4.7}$$

$$q_1(2q_0\lambda^{t+1} - 1) = 0, \tag{4.8}$$

$$q_2(2q_0\mu^{t+1} - 1) = 0, \tag{4.9}$$

$$aq_0^2 + \frac{aq_0^2}{2^{t+1}} - \frac{aq_0^2}{2^t} + \frac{q_0q_3}{2^t} - \frac{aq_3^2}{2^{t+1}} = q_3. \tag{4.10}$$

Equation (4.7) yields that $q_0 = 0$ or $q_0 = 1$. Thus, two cases should be considered, separately.

Case 1. Let $q_0 = 0$. Then, Equation (4.8) implies that $q_1 = 0$. Also, Equation (4.9) yields $q_2 = 0$. Finally, Equation (4.10) gives that $\frac{-aq_3^2}{2^{t+1}} = q_3$. Hence, $q_3 = 0$ or $q_3 = \frac{-2^{t+1}}{a}$ provided that $a \neq 0$. Thus, $\mathbf{q}$ can be either $\mathbf{q} = (0,0,0,0)$ or $\mathbf{q} =$

$(0,0,0,\frac{-2^{t+1}}{a})$. Note that if $a=0$, then $\mathbf{q}=(0,0,0,0)$.

Case 2. Let $q_0=1$. Assume that $a\neq 0$, and $t>0$. Thus, $q_1=0$ provided that $2\lambda^{t+1}\neq 1$ by Equation (4.8) (since $|\lambda|<\frac{1}{2}$, see (3.23)). Also, Equation (4.9) gives that $q_2=0$ provided that $2\mu^{t+1}\neq 1$ (since $|\mu|<\frac{1}{2}$, see (3.23)). Then, Equation (4.10) yields $\frac{-aq_3^2}{2^{t+1}}+(\frac{1}{2^t}-1)q_3+a(1-\frac{1}{2^{t+1}})=0$. Hence, $q_3^2+(\frac{2^{t+1}-2}{a})q_3+(1-2^{t+1})=0$. Thus,

$$q_3=\frac{2-2^{t+1}\pm\sqrt{2^{2t+2}-2^{t+3}+4-4a^2+4a^2 2^{t+1}}}{2a}.$$

Hence,

$$\mathbf{q}=(1,0,0,\frac{2-2^{t+1}\pm\sqrt{2^{2t+2}-2^{t+3}+4-4a^2+4a^2 2^{t+1}}}{2a}).$$

Assume that $t=0$. If $|\lambda|,|\mu|<\frac{1}{2}$. This case is similar to the one considered above. Now, suppose that $a=0$ and $t>0$. Then, due to $|\lambda|\neq\frac{1}{2},|\mu|\neq\frac{1}{2}$, one has if $q_1=0,q_2=0$, then by the same argument as above, one finds $\mathbf{q}=(1,0,0,0)$. Let $t=0$. If $|\lambda|<\frac{1}{2},|\mu|<\frac{1}{2}$, then $\mathbf{q}=(1,0,0,0)$. Let $|\lambda|=\frac{1}{2},|\mu|<\frac{1}{2}$, then $q_1\neq 0,q_2=0$, and $q_3=\frac{q_0q_3}{2^0}$ which imply q_3 is arbitrary. So, $\mathbf{q}=(1,q_1,0,q_3)$. Let $|\lambda|<\frac{1}{2},|\mu|=\frac{1}{2}$. Then, one similarly finds $\mathbf{q}=(1,0,q_2,q_3)$. Let $|\lambda|=|\mu|=\frac{1}{2}$, then $\mathbf{q}=(1,q_1,q_2,q_3)$.

□

Remark If Δ_t is not positive, then one can find, in addition to the seven idempotents in Theorem 4.2.1, three more idempotents which are given by

1. $\mathbf{q}=(1,q_1,0,\pm 1)$ if $a\neq 0,t=0,|\lambda|=\frac{1}{2},|\mu|<\frac{1}{2}$.

2. $\mathbf{q} = (1, 0, q_2, \pm 1)$ if $a \neq 0, t = 0, |\lambda| < \frac{1}{2}, |\mu| = \frac{1}{2}$.

3. $\mathbf{q} = (1, q_1, q_2, \pm 1)$ if $a \neq 0, t = 0, |\lambda| = |\mu| = \frac{1}{2}$.

Let $\mathbf{x} \in A_t$. Denote $P_{\mathbf{x}} = \{\lambda \mathbf{x} | \lambda \in \mathbb{C}\}$. Now, it is natural to know when $P_{\mathbf{x}}$ is a subalgebra, which means there exists $\tau \in \mathbb{C}$ such that $\mathbf{x} \circ_t \mathbf{x} = \tau \mathbf{x}$.

Theorem 4.2.2 *Let $(A_t, \circ_t)$ be a FQGLV-A and let $\mathbf{x} \in A_t, \mathbf{x} \neq 0$. Assume that $t > 0$ and $a \neq 0$. Then, $P_{\mathbf{x}}$ is a 1-dimensional subalgebra of A_t if and only if one of the followings hold:*

(i) $\mathbf{x} = (0, 0, 0, \frac{-2^{t+1}\tau}{a}), \tau \in \mathbb{C}$

(ii) $\mathbf{x} = (\tau, 0, 0, \frac{2\tau - 2^{t+1}\tau \pm \sqrt{2^{2t+2}\tau^2 - 2^{t+3}\tau^2 + 4\tau^2 - 4a^2\tau^2 + 4a^2 2^{t+1}\tau^2}}{2a}), \tau \in \mathbb{C}$.

Proof. To show that the set $P_{\mathbf{x}} = \{\alpha \mathbf{x} | \alpha \in \mathbb{C}\} \subset A_t$ is subalgebra, it is enough to prove $\mathbf{x} \circ_t \mathbf{x} \in P_{\mathbf{x}}$. If $\mathbf{x} \circ_t \mathbf{x} = \tau \mathbf{x}$ where $\tau \in \mathbb{C}, \tau \neq 1$, then

$$(x_0^2, 2x_0x_1\lambda^{t+1}, 2x_0x_2\mu^{t+1}, (a - \frac{a}{2^t} + \frac{a}{2^{t+1}})x_0^2 + \frac{x_0x_3}{2^t} - \frac{ax_3^2}{2^{t+1}}) = \tau(x_0, x_1, x_2, x_3).$$

Now, comparing the coordinates, one gets:

$$x_0^2 = \tau x_0, \tag{4.11}$$

$$2x_0x_1\lambda^{t+1} = \tau x_1, \tag{4.12}$$

$$2x_0x_2\mu^{t+1} = \tau x_2, \tag{4.13}$$

$$ax_0^2 - \frac{ax_0^2}{2^{t+1}} - \frac{ax_0^2}{2^t} + \frac{x_0x_3}{2^t} - \frac{ax_3^2}{2^{t+1}} = \tau x_3. \tag{4.14}$$

Equation (4.11) yields that $x_0 = 0$ or $x_0 = \tau$. Thus, two cases should be considered.

1. Let $x_0 = 0$. Then, $x_1 = 0$ provided that $\tau \neq 0$ by Equation (4.12). Also, Equation (4.13) gives that $x_2 = 0$ provided that $\tau \neq 0$. Finally, Equation (4.14) implies that $\frac{-ax_3^2}{2^{t+1}} = \tau x_3$. Thus, $x_3 = 0$ or $x_3 = \frac{-2^{t+1}\tau}{a}$ where $a \neq 0$. Hence, $x = (0,0,0,0)$ which will be rejected since $x \neq 0$ or $x = (0,0,0,\frac{-2^{t+1}\tau}{a})$.

2. Let $x_0 = \tau$. Then, $x_1 = 0$ provided that $2\lambda^{t+1} \neq 1$ by Equation (4.12). Equations (4.13) gives that $x_2 = 0$ provided that $2\mu^{t+1} \neq 1$. Finally, Equation (4.14) yields that:

$$a\tau^2 + \frac{a\tau^2}{2^{t+1}} - \frac{a\tau^2}{2^t} + \frac{\tau x_3}{2^t} - \frac{ax_3^2}{2^{t+1}} = \tau x_3$$

or

$$\begin{aligned} x_3^2 - \frac{2^{t+1}}{a}(\frac{\tau}{2^t} - \tau)x_3 - \frac{2^{t+1}}{a}a(\tau^2 - \frac{\tau^2}{2^{t+1}}) = 0, \\ x_3^2 + (\frac{2^{t+1}\tau - 2\tau}{a})x_3 + (\tau^2 - 2^{t+1}\tau^2) = 0. \end{aligned}$$

Hence,

$$x_3 = \frac{2\tau - 2^{t+1}\tau \pm \sqrt{2^{2t+2}\tau^2 - 2^{t+3}\tau^2 + 4\tau^2 - 4a^2\tau^2 + 4a^2 2^{t+1}\tau^2}}{2a}.$$

Thus, the vector x is given by

$$x = (\tau, 0, 0, \frac{2\tau - 2^{t+1}\tau \pm \sqrt{2^{2t+2}\tau^2 - 2^{t+3}\tau^2 + 4\tau^2 - 4a^2\tau^2 + 4a^2 2^{t+1}\tau^2}}{2a}).$$

□

Remark From the proved theorem with Theorem 4.2.1, one infers that 1-dimensional subalgebras are generated only by idempotents.

4.3 An algebra generated by the idempotents

In this section, it is investigated an algebra generated by the idempotents given by Theorem (4.2.1). Throughout this section, it is assumed that $t > 0$ and $a \neq 0$. Then, by Theorem (4.2.1), one concludes that the idempotents of FQGLV-A are given by $\mathbf{q}_1 = (0,0,0,\frac{-2^{t+1}}{a})$, $\mathbf{q}_2 = (1,0,0,\frac{2-2^{t+1}+\alpha_t}{2a})$, $\mathbf{q}_3 = (1,0,0,\frac{2-2^{t+1}-\alpha_t}{2a})$ where

$$\alpha_t = \sqrt{2^{2t+2} - 2^{t+3} + 4 - 4a^2 + 4a^2 2^{t+1}}, a \neq 0.$$

Proposition 4.3.1 *Let $t > 0$ and $a \neq 0$, and $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ be idempotents of A_t, then the following statements hold:*

(i) *The vectors $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ are linearly dependent,*

(ii) *Each two idempotents are linearly independent.*

Proof. (i) Let $\lambda_1\mathbf{q}_1 + \lambda_2\mathbf{q}_2 + \lambda_3\mathbf{q}_3 = 0$. Then,

$$(\lambda_2 + \lambda_3, 0, 0, \frac{-\lambda_1 2^{t+1}}{a} + \lambda_2\frac{2-2^{t+1}+\alpha_t}{2a} + \lambda_3\frac{2-2^{t+1}-\alpha_t}{2a}) = 0.$$

Thus, $\lambda_2 + \lambda_3 = 0$ which yields that $\lambda_3 = -\lambda_2$. Also,

$$\begin{aligned} 0 &= \frac{-\lambda_1 2^{t+1}}{a} + \lambda_2\frac{2-2^{t+1}+\alpha_t}{2a} + \lambda_3\frac{2-2^{t+1}-\alpha_t}{2a} \\ &= \frac{-\lambda_1 2^{t+1}}{a} + \lambda_2(\frac{2-2^{t+1}+\alpha_t}{2a} - \frac{2-2^{t+1}-\alpha_t}{2a}) \\ &= \frac{-\lambda_1 2^{t+1} + \alpha_t\lambda_2}{a}. \end{aligned}$$

Then, $-\lambda_1 2^{t+1} + \alpha_t\lambda_2 = 0$ which gives that $\lambda_1 = \frac{\alpha_t\lambda_2}{2^{t+1}}$, where $\lambda_2 \in \mathbb{C}$. Thus, $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ are linearly dependant.

(ii) Now, the remaining of the proof is to check that each two idempotents are linearly independent. Let $\lambda_1\mathbf{q}_1+\lambda_2\mathbf{q}_2=0$. Then,

$$(\lambda_2,0,0,\frac{-\lambda_1 2^{t+1}}{a}+\lambda_2\frac{2-2^{t+1}+\alpha_t}{2a})=0.$$

Thus, $\lambda_2=0$. Also, $-\frac{\lambda_1 2^{t+1}}{a}=0$ which implies that $\lambda_1=\lambda_2=0$. Thus, $\{\mathbf{q}_1,\mathbf{q}_2\}$ are linearly independent. Using similar argument, one can show that $\{\mathbf{q}_1,\mathbf{q}_3\}$ are linearly independent. Let $\lambda_2\mathbf{q}_2+\lambda_3\mathbf{q}_3=0$. Then,

$$(\lambda_2+\lambda_3,0,0,\lambda_2\frac{2-2^{t+1}+\alpha_t}{2a}+\lambda_3\frac{2-2^{t+1}-\alpha_t}{2a})=0.$$

Thus, $\lambda_2+\lambda_3=0$ which gives that $\lambda_3=-\lambda_2$. Also,

$$0=\lambda_2\frac{2-2^{t+1}+\alpha_t}{2a}+\lambda_3\frac{2-2^{t+1}-\alpha_t}{2a}=\lambda_2(\frac{2-2^{t+1}+\alpha_t}{2a}-\frac{2-2^{t+1}-\alpha_t}{2a})=\lambda_2\frac{\alpha_t}{a}.$$

Then, $\lambda_2=0$ which implies that $\lambda_2=\lambda_3=0$. Thus, $\{\mathbf{q}_2,\mathbf{q}_3\}$ are linearly independent.

□

Let $\mathbf{F}=\{\lambda\mathbf{q}_1+\mu\mathbf{q}_2|\lambda,\mu\in\mathbb{C}\}$ be an algebra generated by the following two idempotents, where

$$\mathbf{q}_1=\left(0,0,0,\frac{-2^{t+1}}{a}\right),\mathbf{q}_2=\left(1,0,0,\frac{2-2^{t+1}+\alpha_t}{2a}\right).$$

Now, let us calculate $\mathbf{q}_1 \circ_t \mathbf{q}_2$. Using (4.5), one gets:

$$\mathbf{q}_1 \circ_t \mathbf{q}_2 = \left(0,0,0,\frac{1}{2^{t+1}}\left(\frac{-2^{t+1}}{a}\right) - \frac{a}{2^{t+1}}\left(\frac{-2^{t+1}}{a}.\frac{2-2^{t+1}+\alpha_t}{2a}\right)\right) \quad (4.15)$$
$$= \left(0,0,0,\frac{-2}{2a}+\frac{2-2^{t+1}+\alpha_t}{2a}\right)$$
$$= \left(0,0,0,\frac{\alpha_t - 2^{t+1}}{2a}\right).$$

Now, the task is to study some properties of the subalgebra $\mathbf{F}$. Let $\mathbf{x} = \lambda_1 \mathbf{q}_1 + \mu_1 \mathbf{q}_2$ and $\mathbf{y} = \lambda_2 \mathbf{q}_1 + \mu_2 \mathbf{q}_2$, where $\mathbf{x}, \mathbf{y} \in \mathbf{F}$. Then, using (4.15), one finds:

$$\mathbf{x} \circ_t \mathbf{y} = \lambda_1 \lambda_2 \ \mathbf{q}_1 \circ_t \ \mathbf{q}_1 + \lambda_1 \mu_2 \ \mathbf{q}_1 \circ_t \ \mathbf{q}_2 + \mu_1 \ \lambda_2 \ \mathbf{q}_2 \circ_t \ \mathbf{q}_1 + \mu_1 \mu_2 \ \mathbf{q}_2 \circ_t \ \mathbf{q}_2$$
$$= \lambda_1 \ \lambda_2 \ \mathbf{q}_1 + (\ \lambda_1 \mu_2 + \mu_1 \ \lambda_2) \ \mathbf{q}_1 \circ_t \ \mathbf{q}_2 \ + \mu_1 \mu_2 \ \mathbf{q}_2$$
$$= \lambda_1 \ \lambda_2 \ \mathbf{q}_1 - (\ \lambda_1 \mu_2 + \mu_1 \ \lambda_2) \ (\gamma_t \ \beta_t \ \ \mathbf{q}_1) + \mu_1 \mu_2 \ \mathbf{q}_2$$

where $\beta_t = \frac{2^{2t+2} - 2^{t+1}\alpha_t}{2a^2}$ and $\gamma_t = -\frac{a}{2^{t+1}}$ Then , $\mathbf{F}$ is subalgebra of A.

Remark It is noted that the subalgebra $\mathbf{F}$ is not ideal. Indeed, let $\mathbf{x} = (x_0, x_1, x_2, x_3) \in A$. Then,

$$\mathbf{x} \circ_t \mathbf{q}_1 = (0,0,0,ax_0 + x_3),$$
$$\mathbf{x} \circ_t \mathbf{q}_2 = \left(x_0, \lambda^{t+1} x_1, \mu^{t+1} x_2, x_3\left(\frac{2^{t+1} \pm \alpha_t}{2^{t+1}}\right) + x_0\left(\frac{a2^{t+1}-1}{2^{t+1}} + \frac{2-2^{t+1} \pm \alpha_t}{a2^{t+1}}\right)\right).$$

As a special case, let $\widetilde{\mathbf{x}} = (0,1,1,0)$. Then,

$$\widetilde{\mathbf{x}} \circ_t \mathbf{q}_2 = (0, \lambda^{t+1}, \mu^{t+1}, 0)$$

which does not belong to $\mathbf{F}$ since the second and third components of all elements in $\mathbf{F}$ are zeros. Thus, $\mathbf{F}$ is not ideal.

Theorem 4.3.2 *An algebra* $\mathbf{F} = \{\lambda \mathbf{q}_1 + \mu \mathbf{q}_2 | \lambda, \mu \in \mathbb{C}\}$ *is associative if and only if* $|a| = 1$ *or* $a \in \left(\frac{1}{2}, \frac{-1+\sqrt{5}}{2}\right)$.

Proof. Let $\mathbf{x} = \lambda_1 \mathbf{q}_1 + \mu_1 \mathbf{q}_2, \mathbf{y} = \lambda_2 \mathbf{q}_1 + \mu_2 \mathbf{q}_2, \mathbf{z} = \lambda_3 \mathbf{q}_1 + \mu_3 \mathbf{q}_2$. Then, by equation (4.5)

$$\mathbf{x} \circ_t \mathbf{y} = (\lambda_1\lambda_2 - (\lambda_1\mu_2 + \mu_1\lambda_2)\beta_t\gamma_t)\mathbf{q}_1 + \mu_1\mu_2\mathbf{q}_2,$$

and

$$\begin{aligned}(\mathbf{x} \circ_t \mathbf{y}) \circ_t \mathbf{z} = {}& [(\lambda_1\lambda_2 - (\lambda_1\mu_2 + \mu_1\lambda_2)\beta_t\gamma_t)\lambda_3 \\ & -((\lambda_1\lambda_2 - (\lambda_1\mu_2 + \mu_1\lambda_2)\beta_t\gamma_t)\mu_3 + \lambda_3\mu_1\mu_2)\beta_t\gamma_t]\mathbf{q}_1 + \mu_1\mu_2\mu_3\mathbf{q}_2.\end{aligned} \tag{4.16}$$

Now,

$$\mathbf{y} \circ_t \mathbf{z} = (\lambda_2\lambda_3 - (\lambda_2\mu_3 + \mu_2\lambda_3)\beta_t\gamma_t)\mathbf{q}_1 + \mu_2\mu_3\mathbf{q}_2,$$

and

$$\begin{aligned}\mathbf{x} \circ_t (\mathbf{y} \circ_t \mathbf{z}) = {}& [(\lambda_1(\lambda_2\lambda_3 - (\lambda_2\mu_3 + \mu_2\lambda_3)\beta_t\gamma_t) \\ & -(\lambda_1\mu_2\mu_3 + \mu_1(\lambda_2\lambda_3 - (\lambda_2\mu_3 + \mu_2\lambda_3)\beta_t\gamma_t))\beta_t\gamma_t]\mathbf{q}_1 + \mu_1\mu_2\mu_3\mathbf{q}_2.\end{aligned} \tag{4.17}$$

Comparing the coefficients in equations (4.16) and (4.17), one gets:

$$\begin{aligned}&\lambda_1\lambda_2\lambda_3 - \lambda_1\mu_2\lambda_3\beta_t\gamma_t - \lambda_2\lambda_3\mu_1\beta_t\gamma_t - \lambda_1\lambda_2\mu_3\beta_t\gamma_t \\ &+\lambda_1\mu_2\mu_3\beta_t^2\gamma_t^2 + \mu_1\lambda_2\mu_3\beta_t^2\gamma_t^2 - \lambda_3\mu_1\mu_2\beta_t\gamma_t = \lambda_1\lambda_2\lambda_3 - \lambda_1\lambda_2\mu_3\beta_t\gamma_t \\ &-\lambda_1\mu_2\lambda_3\beta_t\gamma_t - \lambda_1\mu_2\mu_3\beta_t\gamma_t - \mu_1\lambda_2\lambda_3\beta_t\gamma_t + \mu_1\lambda_2\mu_3\beta_t^2\gamma_t^2 + \mu_1\mu_2\lambda_3\beta_t^2\gamma_t^2.\end{aligned}$$

Simplify the last equation, one find:

$$\lambda_1\mu_2\mu_3(\beta_t^2\gamma_t^2+\beta_t\gamma_t)-\lambda_3\mu_1\mu_2(\beta_t^2\gamma_t^2+\beta_t\gamma_t)=0$$

or

$$(\beta_t^2\gamma_t^2+\beta_t\gamma_t)(\lambda_1\mu_2\mu_3-\lambda_3\mu_1\mu_2)=0.$$

Hence, $\beta_t\gamma_t=0$ or $\beta_t\gamma_t=-1$. Let $\beta_t\gamma_t=0$, then

$$\beta_t\gamma_t=\left(\frac{2^{2t+2}-2^{t+1}\alpha_t}{2a^2}\right)\left(\frac{-a}{2^{t+1}}\right)=\frac{\alpha_t-2^{t+1}}{2a}=0.$$

Thus,

$$\begin{aligned}
&2^{t+2}-2^{t+3}+4-4a^2+4a^2 2^{t+1}=2^{2t+2}\\
&4a^2(2^{t+1}-1)+4(1-2^{t+1})=0\\
&4(2^{t+1}-1)(a^2-1)=0.
\end{aligned}$$

Hence, $a^2=1$ since $2^{t+1}-1\neq 0$.

Let $\beta_t\gamma_t+1=0$, by substituting the values of β_t and γ_t and simplifying the last equation one has

$$\frac{2^{t+1}-\alpha_t}{2a}=1,$$

or

$$\alpha_t=2^{t+1}-2a.$$

Square both sides and simplify the result to get

$$2^t(2a^2+2a-2)+(1-2a^2)=0,$$

or

$$2^t=\frac{2a^2-1}{2a^2+2a-2}.$$

Taking logarithm base two for both sides yields:

$$t=log_2\left(\frac{2a^2-1}{2a^2+2a-2}\right)>0$$

which implies that

$$\frac{2a^2-1}{2a^2+2a-2}>1.$$

Solving the last inequality gives $a\in\left(\frac{1}{2},\frac{-1+\sqrt{5}}{2}\right)$. One can see that the range of the function:

$$log_2\left(\frac{2a^2-1}{2a^2+2a-2}\right)>0$$

on $a\in\left(\frac{1}{2},\frac{-1+\sqrt{5}}{2}\right)$ is $(0,\infty)$. This completes the proof. □

Remark Comparing Theorems (4.1.3) and (4.3.2), one can see that if the algebra which is generated by the idempotents of the FQGLV-A is associative if $|a|=1$ or $a\in\left(\frac{1}{2},\frac{-1+\sqrt{5}}{2}\right)$ while the FQGLV-A itself is associative if $|a|=1$. This shows the influence of the idempotents on the associativity property.

Chapter 5: Derivations of Flow Quantum Genetic Lotka-Volterra Algebras

5.1 Derivations in $M_4(\mathbb{C})$

Let $(A_t, \circ_t)$ be a flow quantum genetic Lotka-Volterra algebra. Then, recall that for $\mathbf{f} = (f_0, f_1, f_2, f_3)$ and $\mathbf{p} = (p_0, p_1, p_2, p_3)$, one has:

$$\begin{aligned} \mathbf{f} \circ_t \mathbf{p} = (&f_0 p_0, \lambda^{t+1}(f_1 p_0 + f_0 p_1), \mu^{t+1}(f_2 p_0 + f_0 p_2), \\ &f_0 p_0 \xi + \frac{1}{2^{t+1}}(f_3 p_0 + f_0 p_3) - \frac{a}{2^{t+1}}(f_3 p_3)) \end{aligned} \tag{5.1}$$

where $\xi = a - \frac{a}{2^t} + \frac{a}{2^{t+1}} = a\left(\frac{2^{t+1}-1}{2^{t+1}}\right)$.

Definition 5.1.1 A linear mapping $d : A \to A$ is called derivation if

$$d(x \circ_t y) = d(x) \circ_t y + x \circ_t d(y), \forall x, y \in A \tag{5.2}$$

Proposition 5.1.1 *Let* $\mathbf{e_0} = (1,0,0,0), \mathbf{e_1} = (0,1,0,0), \mathbf{e_2} = (0,0,1,0)$*, and* $\mathbf{e_3} = (0,0,0,1)$*. Then, d is derivation if and only if*

$$d(\mathbf{e_i} \circ_t \mathbf{e_j}) = d(\mathbf{e_i}) \circ \mathbf{e_j} + \mathbf{e_i} \circ_t d(\mathbf{e_j}) \tag{5.3}$$

Proof. ($\Leftarrow$) Let $x = \sum_{i=0}^{3} x_i \mathbf{e_i}$ and $y = \sum_{j=0}^{3} y_j \mathbf{e_j}$. Then,

$$
\begin{aligned}
d(x \circ_t y) &= d\left(\left(\sum_{i=0}^{3} x_i \mathbf{e_i}\right) \circ_t \left(\sum_{j=0}^{3} y_j \mathbf{e_j}\right)\right)\\
&= d\left(\sum_{i=0}^{3}\sum_{j=0}^{3} x_i y_j (\mathbf{e_i} \circ_t \mathbf{e_j})\right)\\
&= \sum_{i=0}^{3}\sum_{j=0}^{3} x_i y_j d(\mathbf{e_i} \circ_t \mathbf{e_j}).
\end{aligned}
$$

From (5.3), one gets:

$$
\begin{aligned}
d(x \circ_t y) &= \sum_{i=0}^{3}\sum_{j=0}^{3} x_i y_j (d(\mathbf{e_i}) \circ \mathbf{e_j} + \mathbf{e_i} \circ_t d(\mathbf{e_j}))\\
&= \sum_{i=0}^{3}\sum_{j=0}^{3} x_i y_j (d(\mathbf{e_i}) \circ \mathbf{e_j}) + \sum_{i=0}^{3}\sum_{j=0}^{3} x_i y_j (\mathbf{e_i} \circ_t d(\mathbf{e_j}))\\
&= d(\sum_{i=0}^{3} x_i \mathbf{e_i}) \circ_t (\sum_{j=0}^{3} y_j \mathbf{e_j}) + (\sum_{i=0}^{3} x_i \mathbf{e_i}) \circ_t d(\sum_{j=0}^{3} y_j \mathbf{e_j})\\
&= d(x) \circ_t y + x \circ_t d(y).
\end{aligned}
$$

Thus, d is derivation.

($\Rightarrow$) Follows directly from the definition. □

From (5.1), one gets:

$$
\begin{aligned}
\mathbf{e_0} \circ_t \mathbf{e_0} &= (1,0,0,\xi) = \mathbf{e_0} + \xi \mathbf{e_3},\\
\mathbf{e_0} \circ_t \mathbf{e_1} &= (0,\lambda^{t+1},0,0) = \lambda^{t+1}\mathbf{e_1},\\
\mathbf{e_0} \circ_t \mathbf{e_2} &= (0,0,\mu^{t+1},0) = \mu^{t+1}\mathbf{e_2},\\
\mathbf{e_0} \circ_t \mathbf{e_3} &= (0,0,0,\frac{1}{2^{t+1}}) = \frac{1}{2^{t+1}}\mathbf{e_3}.
\end{aligned}
$$

Similarly, other terms can be computed and they summarized in the following Table.

Table 5.1: $\mathbf{e_i} \circ_t \mathbf{e_j}$ for $i, j \in \{0,1,2,3\}$

$\circ_t$	e_0	e_1	e_2	e_3
e_0	$e_0 + \xi e_3$	$\lambda^{t+1} e_1$	$\mu^{t+1} e_2$	$\frac{1}{2^{t+1}} e_3$
e_1	$\lambda^{t+1} e_1$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
e_2	$\mu^{t+1} e_2$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
e_3	$\frac{1}{2^{t+1}} e_3$	$\mathbf{0}$	$\mathbf{0}$	$\frac{-a}{2^{t+1}} e_3$

Let $d : A \to A$ be a derivation. Then, one has

$$d(\mathbf{e_i}) = \sum_{j=0}^{3} d_{i,j} \mathbf{e_j}.$$

The goal now is to find the conditions on $d_{i,j}$ such that

$$d(\mathbf{e_i} \circ_t \mathbf{e_j}) = d(\mathbf{e_i}) \circ_t \mathbf{e_j} + \mathbf{e_i} \circ_t d(\mathbf{e_j})$$

for $i, j \in \{0,1,2,3\}$ and $i \leq j$. To explain the procedure, let us assume that

$$d(\mathbf{e_0} \circ_t \mathbf{e_0}) = d(\mathbf{e_0}) \circ_t \mathbf{e_0} + \mathbf{e_0} \circ_t d(\mathbf{e_0}).$$

Then, from Table (5.1) and since $\mathbf{e_0} \circ_t \mathbf{e_0} = \mathbf{e_0} + \xi \mathbf{e_3}$, one gets

$$d(\mathbf{e_0}) + \xi d(\mathbf{e_3}) = d(\mathbf{e_0}) \circ_t \mathbf{e_0} + \mathbf{e_0} \circ_t d(\mathbf{e_0}).$$

By comparing the components, one finds:

$$d_{00} + \xi d_{30} = 2d_{00}, d_{01} + \xi d_{31} = 2\lambda^{t+1} d_{01}$$
$$d_{02} + \xi d_{32} = 2\mu^{t+1} d_{02}, d_{03} + \xi d_{33} = 2\left(\frac{1}{2^{t+1}} d_{03} + \xi d_{00}\right).$$

These equations can be simplified as follows:

$$-d_{00}+\xi d_{30}=0, \tag{5.4}$$

$$d_{01}-2\lambda^{t+1}d_{01}+\xi d_{31}=0, \tag{5.5}$$

$$d_{02}-2\mu^{t+1}d_{02}+\xi d_{32}=0, \tag{5.6}$$

$$d_{03}-\frac{1}{2^t}d_{03}-2\xi d_{00}+\xi d_{33}=0. \tag{5.7}$$

Using similar argument, one can generate the following system of equations for the cases $d(\mathbf{e_i}\circ_t\mathbf{e_j})=d(\mathbf{e_i})\circ_t\mathbf{e_j}+\mathbf{e_i}\circ_t d(\mathbf{e_j})$ for $i,j\in\{1,2,3\}$ with $i\leq j$.

$$-d_{01}+\lambda^{t+1}d_{10}=0, \tag{5.8}$$

$$d_{00}\lambda^{t+1}=0, \tag{5.9}$$

$$d_{12}\lambda^{t+1}-d_{12}\mu^{t+1}=0, \tag{5.10}$$

$$-\frac{d_{13}}{2^{t+1}}+d_{13}\lambda^{t+1}-d_{10}\xi=0, \tag{5.11}$$

$$2d_{10}\lambda^{t+1}=0, \tag{5.12}$$

$$-d_{20}+d_{20}\mu^{t+1}=0, \tag{5.13}$$

$$-d_{20}+d_{20}\mu^{t+1}=0, \tag{5.14}$$

$$-d_{21}\lambda^{t+1}+d_{21}\mu^{t+1}=0, \tag{5.15}$$

$$d_{00}\mu^{t+1}=0, \tag{5.16}$$

$$-\frac{d_{23}}{2^{t+1}}+d_{23}\mu^{t+1}-d_{20}\xi=0, \tag{5.17}$$

$$d_{20}\lambda^{t+1} = 0, \tag{5.18}$$

$$d_{10}\mu^{t+1} = 0, \tag{5.19}$$

$$d_{20}\mu^{t+1} = 0, \tag{5.20}$$

$$\frac{d_{30}}{2^{t+1}} - d_{30} = 0, \tag{5.21}$$

$$\frac{d_{31}}{2^{t+1}} - d_{31}\lambda^{t+1} = 0, \tag{5.22}$$

$$\frac{1}{2^{t+1}}d_{32} - d_{32}\mu^{t+1} = 0, \tag{5.23}$$

$$\frac{-d_{00} + ad_{03}}{2^{t+1}} - d_{30}\xi = 0, \tag{5.24}$$

$$d_{30}\lambda^{t+1} = 0, \tag{5.25}$$

$$-d_{10} + ad_{13} = 0, \tag{5.26}$$

$$d_{30}\mu^{t+1} = 0, \tag{5.27}$$

$$-d_{20} + ad_{23} = 0, \tag{5.28}$$

$$ad_{30} = 0, \tag{5.29}$$

$$ad_{32} = 0, \tag{5.30}$$

$$-2d_{30} + ad_{33} = 0. \tag{5.31}$$

Equations (5.13) and (5.20) give:

$$-d_{20} + d_{20}\mu^{t+1} = 0, d_{20}\mu^{t+1} = 0$$

which imply that $d_{20} = 0$. Hence, Equation (5.21) yields that:

$$-d_{30} + \frac{d_{30}}{2^{t+1}} = d_{30}\left(\frac{1}{2^{t+1}} - 1\right) = 0.$$

Since $t \geq 0$ and $\frac{1}{2^{t+1}} - 1 \neq 0$, one finds $d_{30} = 0$. Equation (5.4) gives $d_{00} = 0$, since $d_{30} = 0$. Moreover, Equations (5.8) and (5.12) yield $-d_{01} + \lambda^{t+1}d_{10} = 0$ and $2d_{10}\lambda^{t+1} =$

0. Hence, $d_{01} = 0$. Thus, Equations (5.4-5.31) are reduced to the following ones:

$$\xi d_{31} = 0, \tag{5.32}$$

$$d_{02}(1 - 2\mu^{t+1}) + \xi d_{32} = 0, \tag{5.33}$$

$$d_{03}(1 - \frac{1}{2^t}) + \xi d_{33} = 0, \tag{5.34}$$

$$d_{12}(\lambda^{t+1} - \mu^{t+1}) = 0, \tag{5.35}$$

$$d_{13}(\lambda^{t+1} - \frac{1}{2^{t+1}}) - d_{10}\xi = 0, \tag{5.36}$$

$$d_{10}\lambda^{t+1} = 0, \tag{5.37}$$

$$-d_{21}(\lambda^{t+1} - \mu^{t+1}) = 0, \tag{5.38}$$

$$d_{23}(\mu^{t+1} - \frac{1}{2^{t+1}}) = 0, \tag{5.39}$$

$$d_{10}\mu^{t+1} = 0, \tag{5.40}$$

$$d_{31}(\frac{1}{2^{t+1}} - \lambda^{t+1}) = 0, \tag{5.41}$$

$$d_{32}(\frac{1}{2^{t+1}} - \mu^{t+1}) = 0, \tag{5.42}$$

$$ad_{03} = 0, \tag{5.43}$$

$$-d_{10} + ad_{13} = 0, \tag{5.44}$$

$$ad_{23} = 0, \tag{5.45}$$

$$ad_{32} = 0, \tag{5.46}$$

$$ad_{33} = 0. \tag{5.47}$$

(I) Let $a \neq 0$. Then, $\xi = a\left(\frac{2^{t+1}-1}{2^{t+1}}\right) \neq 0$. Thus, Equations (5.43) and (5.45-5.47) give that $d_{03} = d_{23} = d_{32} = d_{33} = 0$. Moreover, due to $d_{32} = 0$ and $1 - 2\mu^{t+1} \neq 0$, Equation (5.33) implies that $d_{02} = 0$. Since $\xi \neq 0$ Equation (5.32) yields that

$d_{31} = 0$. Hence, Equations (5.32-5.47) can be reduced to the following system.

$$d_{12}(\lambda^{t+1} - \mu^{t+1}) = 0, \tag{5.48}$$

$$d_{13}(\lambda^{t+1} - \frac{1}{2^{t+1}}) - d_{10}\xi = 0, \tag{5.49}$$

$$d_{10}\lambda^{t+1} = 0, \tag{5.50}$$

$$-d_{21}(\lambda^{t+1} - \mu^{t+1}) = 0, \tag{5.51}$$

$$d_{10}\mu^{t+1} = 0, \tag{5.52}$$

$$-d_{10} + ad_{13} = 0. \tag{5.53}$$

Thus, four cases should be considered separately.

Case 1.1 Let $\lambda \neq 0$ and $\lambda \neq \mu$. Then, Equations (5.48) and (5.51) imply that $d_{12} = d_{21} = 0$. Also, Equation (5.50) gives $d_{10} = 0$. Since $d_{10} = 0$ and $a \neq 0$, Equation (5.53) yields $d_{13} = 0$. Therefore,

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & 0 & 0 \\ 0 & 0 & d_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Case 1.2 Let $\lambda \neq 0$ and $\lambda = \mu$. Then, Equations (5.50) implies that $d_{10} = 0$. Since $d_{10} = 0$ and $a \neq 0$, Equation (5.53) yields $d_{13} = 0$. Therefore,

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & d_{12} & 0 \\ 0 & d_{21} & d_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Case 1.3 Let $\lambda = 0$ and $\mu \neq 0$. Then, Equations (5.48) and (5.51-5.52) imply that $d_{12} = d_{21} = d_{10} = 0$. Since $d_{10} = 0$ and $a \neq 0$, Equation (5.53) yields

$d_{13} = 0$. Therefore,

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & 0 & 0 \\ 0 & 0 & d_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Case 1.4 Let $\lambda = \mu = 0$. Then, Equations (5.53) gives that $-d_{10} + ad_{13} = 0$ which implies that $d_{10} = ad_{13}$. Equation (5.49) yields $\frac{-d_{13}}{2^{t+1}} - d_{10}\xi = d_{13}\left(\frac{1}{2^{t+1}} + a\xi\right) = 0$ which implies that $d_{13}\left(\frac{1}{2^{t+1}} + a^2\left(\frac{2^{t+1}-1}{2^{t+1}}\right)\right)$. Thus, $d_{13}(a^2(2^{t+1}-1)+1) = 0$. Since $t \geq 0$ and $a \neq 0$, $a^2(2^{t+1}-1)+1 > 0$. Thus, $d_{13} = 0$ which gives $d_{10} = 0$ since $d_{10} = ad_{13}$. Therefore,

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & d_{12} & 0 \\ 0 & d_{21} & d_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(II) Now, let $a = 0$. Then, $\xi = a\left(\frac{2^{t+1}-1}{2^{t+1}}\right) = 0$. Thus, Equations (5.4) and (5.26) yield that: $d_{00} = d_{10} = 0$. Equations (5.13) and (5.20) give that $d_{20} = 0$. Since $\frac{1}{2^{t+1}} - 1 \neq 0$, Equation (5.21) implies that $d_{30} = 0$. Equations (5.8) and (5.12) yield that $d_{01} = 0$. Thus, Equations (5.4)-(5.31) can be reduced to the following system:

$$d_{03}\left(1-\frac{1}{2^t}\right)=0, \tag{5.54}$$

$$d_{12}\left(\lambda^{t+1}-\mu^{t+1}\right)=0, \tag{5.55}$$

$$d_{13}\left(\lambda^{t+1}-\frac{1}{2^{t+1}}\right)=0, \tag{5.56}$$

$$d_{21}\left(\mu^{t+1}-\lambda^{t+1}\right)=0, \tag{5.57}$$

$$d_{23}\left(\mu^{t+1}-\frac{1}{2^{t+1}}\right)=0, \tag{5.58}$$

$$d_{31}\left(\frac{1}{2^{t+1}}-\lambda^{t+1}\right)=0, \tag{5.59}$$

$$d_{32}\left(\frac{1}{2^{t+1}}-\mu^{t+1}\right)=0. \tag{5.60}$$

Hence, ten cases should be considered, separately.

Case 2.1 Let $\lambda=\mu=\frac{1}{2}, t=0$. Then,

$$d=\begin{pmatrix} 0 & 0 & 0 & d_{03} \\ 0 & d_{11} & d_{12} & d_{13} \\ 0 & d_{21} & d_{22} & d_{23} \\ 0 & d_{31} & d_{32} & d_{33} \end{pmatrix}.$$

Case 2.2 Let $\lambda=\mu\neq\frac{1}{2}, t=0$. Then, Equations (5.56) and (5.58-5.60) give that $d_{13}=d_{23}=d_{31}=d_{32}=0$. Hence,

$$d=\begin{pmatrix} 0 & 0 & 0 & d_{03} \\ 0 & d_{11} & d_{12} & 0 \\ 0 & d_{21} & d_{22} & 0 \\ 0 & 0 & 0 & d_{33} \end{pmatrix}.$$

Case 2.3 Let $\lambda=\frac{1}{2}, \mu\neq\frac{1}{2}, t=0$. Then, Equations (5.55), (5.57)-(5.58), and

(5.60) imply that: $d_{12} = d_{21} = d_{32} = d_{23} = 0$. Hence,

$$d = \begin{pmatrix} 0 & 0 & 0 & d_{03} \\ 0 & d_{11} & 0 & d_{13} \\ 0 & 0 & d_{22} & 0 \\ 0 & d_{31} & 0 & d_{33} \end{pmatrix}.$$

Case 2.4 Let $\lambda \neq \frac{1}{2}, \mu = \frac{1}{2}, t = 0$. Then, Equations (5.55-5.57) and (5.59) yield that $d_{12} = d_{21} = d_{31} = 0$. Hence,

$$d = \begin{pmatrix} 0 & 0 & 0 & d_{03} \\ 0 & d_{11} & 0 & 0 \\ 0 & 0 & d_{22} & d_{23} \\ 0 & 0 & d_{32} & d_{33} \end{pmatrix}.$$

Case 2.5 Let $\lambda \neq \frac{1}{2}, \mu \neq \frac{1}{2}, \lambda \neq \mu, t = 0$. Then, Equations (5.55-5.60) give that $d_{12} = d_{13} = d_{21} = d_{23} = d_{31} = d_{32} = 0$. Hence,

$$d = \begin{pmatrix} 0 & 0 & 0 & d_{03} \\ 0 & d_{11} & 0 & 0 \\ 0 & 0 & d_{22} & 0 \\ 0 & 0 & 0 & d_{33} \end{pmatrix}.$$

Case 2.6 Let $\lambda = \mu = \frac{1}{2}, t \neq 0$. Then, Equations (5.54) implies that: $d_{03} = 0$. Hence,

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & d_{12} & d_{13} \\ 0 & d_{21} & d_{22} & d_{23} \\ 0 & d_{31} & d_{32} & d_{33} \end{pmatrix}.$$

Case 2.7 Let $\lambda = \mu \neq \frac{1}{2}, t \neq 0$. Then, Equations (5.56) and (5.58-5.60) yield that $d_{13} = d_{23} = d_{31}d_{32} = 0$. Hence,

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & d_{12} & 0 \\ 0 & d_{21} & d_{22} & 0 \\ 0 & 0 & 0 & d_{33} \end{pmatrix}.$$

Case 2.8 Let $\lambda = \frac{1}{2}, \mu \neq \frac{1}{2}, t \neq 0$. Then, Equations (5.54-5.55), (5.57-5.59) give that $d_{03} = d_{12} = d_{21} = d_{32} = d_{23} = 0$. So,

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & 0 & d_{13} \\ 0 & 0 & d_{22} & 0 \\ 0 & d_{31} & 0 & d_{33} \end{pmatrix}.$$

Case 2.9 Let $\lambda \neq \frac{1}{2}, \mu = \frac{1}{2}, t \neq 0$. Then, Equations (5.54-5.57) and (5.59) imply that $d_{03} = d_{12} = d_{13} = d_{21} = d_{31} = 0$. Hence,

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & 0 & 0 \\ 0 & 0 & d_{22} & d_{23} \\ 0 & 0 & d_{32} & d_{33} \end{pmatrix}.$$

Case 2.10 Let $\lambda \neq \frac{1}{2}, \mu \neq \frac{1}{2}, \lambda \neq \mu, t \neq 0$. Then, Equations (5.54-5.60) yield that $d_{03} = d_{12} = d_{13} = d_{21} = d_{23} = d_{31} = d_{32} = 0$. Hence,

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & 0 & 0 \\ 0 & 0 & d_{22} & 0 \\ 0 & 0 & 0 & d_{33} \end{pmatrix}.$$

The obtained results are summarized in the following theorem.

Theorem 5.1.2 *Let $(A, \circ_t)$ be a FQGLV-A. Then, its derivations are given in the following Tables (5.2-5.4).*

Table 5.2: The first five derivations of the given flow

Conditions	derivation
$a \neq 0, \lambda \neq 0, \lambda \neq \mu$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & 0 & 0 \\ 0 & 0 & d_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$a \neq 0, \lambda \neq 0, \lambda = \mu$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & d_{12} & 0 \\ 0 & d_{21} & d_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$a \neq 0, \lambda = 0, \mu \neq 0$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & 0 & 0 \\ 0 & 0 & d_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$a \neq 0, \lambda = \mu = 0$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & d_{12} & 0 \\ 0 & d_{21} & d_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$a = 0, \lambda = \mu = \frac{1}{2}, t = 0$	$\begin{pmatrix} 0 & 0 & 0 & d_{03} \\ 0 & d_{11} & d_{12} & d_{13} \\ 0 & d_{21} & d_{22} & d_{23} \\ 0 & d_{31} & d_{32} & d_{33} \end{pmatrix}$

Table 5.3: The second five derivations of the given flow

Conditions	derivation
$a=0,\lambda=\mu\neq\frac{1}{2},t=0$	$\begin{pmatrix} 0 & 0 & 0 & d_{03} \\ 0 & d_{11} & d_{12} & 0 \\ 0 & d_{21} & d_{22} & 0 \\ 0 & 0 & 0 & d_{33} \end{pmatrix}$
$a=0,\lambda=\frac{1}{2},\mu\neq\frac{1}{2},t=0$	$\begin{pmatrix} 0 & 0 & 0 & d_{03} \\ 0 & d_{11} & 0 & d_{13} \\ 0 & 0 & d_{22} & 0 \\ 0 & d_{31} & 0 & d_{33} \end{pmatrix}$
$a=0,\lambda\neq\frac{1}{2},\mu=\frac{1}{2},t=0$	$\begin{pmatrix} 0 & 0 & 0 & d_{03} \\ 0 & d_{11} & 0 & 0 \\ 0 & 0 & d_{22} & d_{23} \\ 0 & 0 & d_{32} & d_{33} \end{pmatrix}$
$a=0,\lambda\neq\mu,\lambda\neq\frac{1}{2},\mu\neq\frac{1}{2},t=0$	$\begin{pmatrix} 0 & 0 & 0 & d_{03} \\ 0 & d_{11} & 0 & 0 \\ 0 & 0 & d_{22} & 0 \\ 0 & 0 & 0 & d_{33} \end{pmatrix}$
$a=0,\lambda=\mu=\frac{1}{2},t\neq 0$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & d_{12} & d_{13} \\ 0 & d_{21} & d_{22} & d_{23} \\ 0 & d_{31} & d_{32} & d_{33} \end{pmatrix}$

Remark It is notice that subalgebra $\mathbb{F}$ (see Section 4.3) has the following form ($a\neq 0, t>0$):

$$\mathbb{F}=\{(\lambda,0,0,\mu):\lambda,\mu\in\mathbb{C}\}.$$

It is interesting to know about $d(\mathbb{F})$ for derivation d. Taking into account the formula:

$$d(\mathbf{x})=\mathbf{x}^T(d_{ij}).$$

Using Table (5.2), one immediately finds that $d(\mathbb{F})=\{\mathbf{0}\}$. Indeed, let $\mathbf{g}\in\mathbb{F}$, i.e.,

$\mathbf{g} = (\lambda, 0, 0, \mu)$. Then, by Table (5.2) (case $a \neq 0, \lambda \neq 0, \lambda \neq \mu$), one gets:

$$d(\mathbf{g}) = (\lambda, 0, 0, \mu) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & 0 & 0 \\ 0 & 0 & d_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = (0, 0, 0, 0).$$

Using the same argument, one arrives at the required equality.

Table 5.4: The last four derivations of the given flow

Conditions	derivation
$a = 0, \lambda = \mu \neq \frac{1}{2}, t \neq 0$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & d_{12} & 0 \\ 0 & d_{21} & d_{22} & 0 \\ 0 & 0 & 0 & d_{33} \end{pmatrix}$
$a = 0, \lambda = \frac{1}{2}, \mu \neq \frac{1}{2}, t \neq 0$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & 0 & d_{13} \\ 0 & 0 & d_{22} & 0 \\ 0 & d_{31} & 0 & d_{33} \end{pmatrix}$
$a = 0, \lambda \neq \frac{1}{2}, \mu = \frac{1}{2}, t \neq 0$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & 0 & 0 \\ 0 & 0 & d_{22} & d_{23} \\ 0 & 0 & d_{32} & d_{33} \end{pmatrix}$
$a = 0, \lambda \neq \mu, \lambda \neq \frac{1}{2}, \mu \neq \frac{1}{2}, t \neq 0$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{11} & 0 & 0 \\ 0 & 0 & d_{22} & 0 \\ 0 & 0 & 0 & d_{33} \end{pmatrix}$

Chapter 6: Automorphisms of FQGLVA

6.1 Preliminaries facts on unital maps

Recall that $\{\mathbb{1}, \sigma_1, \sigma_2, \sigma_3\}$ is a basis for $M_2(\mathbb{C})$. For any $x \in M_2(\mathbb{C})$ one has,

$$x = \omega_0 \mathbb{1} + \omega \cdot \sigma \tag{6.1}$$

where $\omega \cdot \sigma = \omega_1 \sigma_1 + \omega_2 \sigma_2 + \omega_3 \sigma_3$. Correspondingly, for any linear functional $\varphi : M_2(\mathbb{C}) \to \mathbb{C}$, one has $\varphi(x) = \sum_{i=0}^{3} f_i \cdot \omega_i$ where $f_i = \varphi(\sigma_i), f_0 = \varphi(\mathbb{1})$. Then, φ can be written in terms of its coordinate vector $(f_0, f_1, f_2, f_3) \in \mathbb{C}^4$. Recall that

1. φ is a state if $\varphi(\mathbb{1}) = 1$.
2. $\varphi \geq 0$ (positive) if $\|\varphi\|^2 = |\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2 \leq 1$ and $\varphi_1, \varphi_2, \varphi_3$ are real.

Thus, there is one-to-one correspondence between all functionals on $M_2^*(\mathbb{C})$ and $\mathbb{C}^4$. Let $A = M_2^*(\mathbb{C})$. By S, it was denoted the set of all states on $M_2(\mathbb{C})$. Now, it is interested to describe a mapping $\alpha : A \to A$ such that $\alpha(S) \subset S$. Since A has a dimension four, then the mapping α in the standard basis is represented as follows:

$$\alpha = \begin{pmatrix} a_{00} & a_{10} & a_{20} & a_{30} \\ a_{01} & a_{11} & a_{21} & a_{31} \\ a_{01} & a_{12} & a_{22} & a_{32} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

Therefore, α acts on φ as follows:

$$\alpha(\varphi) = \left(\sum_{j=0}^{3} a_{0j}\varphi_j, \sum_{j=0}^{3} a_{1j}\varphi_j, \sum_{j=0}^{3} a_{2j}\varphi_j, \sum_{j=0}^{3} a_{3j}\varphi_j \right)$$
$$= (\alpha(\varphi)_0, \alpha(\varphi)_1, \alpha(\varphi)_2, \alpha(\varphi)_3).$$

Now, using duality representation, one gets:

$$\alpha(\varphi)(x) = \langle \alpha(\varphi), x \rangle = \langle \varphi, \alpha^*(x) \rangle.$$

Using (6.1), one finds

$$\alpha^*(x) = (\alpha^*(x)_0, \alpha^*(x)_1, \alpha^*(x)_2, \alpha^*(x)_3) = \alpha^T(\omega)$$

where $\omega = (\omega_0, \omega_1, \omega_2, \omega_3)$ and

$$\alpha^T = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Remark If α preserve all states, i.e; $\alpha(S) \subset S$, then

$$\begin{pmatrix} a_{00} & a_{10} & a_{20} & a_{30} \\ a_{01} & a_{11} & a_{21} & a_{31} \\ a_{01} & a_{12} & a_{22} & a_{32} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which implies that:

$$\begin{pmatrix} a_{00} \\ a_{01} \\ a_{02} \\ a_{03} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, $a_{00} = 1, a_{01} = a_{02} = a_{03} = 0$. Therefore,

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Now, automorphisms of flow of quantum genetic Lotka-Volterra algebras are going to be considered.

Definition 6.1.1 An automorphism of $(A_t, \circ_t)$ is a map $\alpha : A_t \to A_t$ such that

- α is homomorphism, $\alpha(a \circ_t b) = \alpha(a) \circ_t \alpha(b)$ for all $a, b \in A_t$.
- α is one-to-one map.

Using the argument of Proposition (5.1.1), one can prove the following fact.

Theorem 6.1.1 *α is an automorphism of $(A_t, \circ_t)$ if and only if $\alpha(\mathbf{e}_i \circ_t \mathbf{e}_j) = \alpha(\mathbf{e}_i) \circ_t \alpha(\mathbf{e}_j)$ for all $i, j \in \{0, 1, 2, 3\}$ where $\{\mathbf{e}_j\}_{j=0}^{3}$ is the standard basis.*

6.2 Automorphisms of FQGLVA

From the previous section, α can be written as follows:

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

In this section, the necessary conditions on the matrix α and on λ, μ, a are investigated, when

$$\alpha(\mathbf{e}_i \circ \mathbf{e}_j) = \alpha(\mathbf{e}_i) \circ \alpha(\mathbf{e}_j)$$

for $i,j \in \{1,2,3\}$. Now, $\alpha(\mathbf{e}_i)$ for $i=0,1,2,3$ are going to be computed. Then,

$$\alpha(\mathbf{e}_0)=\begin{pmatrix}1&0&0&0\\a_{10}&a_{11}&a_{12}&a_{13}\\a_{20}&a_{21}&a_{22}&a_{23}\\a_{30}&a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}1\\a_{10}\\a_{20}\\a_{30}\end{pmatrix}=\mathbf{e}_0+a_{10}\mathbf{e}_1+a_{20}\mathbf{e}_2+a_{30}\mathbf{e}_3,$$

$$\alpha(\mathbf{e}_1)=\begin{pmatrix}1&0&0&0\\a_{10}&a_{11}&a_{12}&a_{13}\\a_{20}&a_{21}&a_{22}&a_{23}\\a_{30}&a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\a_{11}\\a_{21}\\a_{31}\end{pmatrix}=a_{11}\mathbf{e}_1+a_{21}\mathbf{e}_2+a_{31}\mathbf{e}_3,$$

$$\alpha(\mathbf{e}_2)=\begin{pmatrix}1&0&0&0\\a_{10}&a_{11}&a_{12}&a_{13}\\a_{20}&a_{21}&a_{22}&a_{23}\\a_{30}&a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\a_{12}\\a_{22}\\a_{32}\end{pmatrix}=a_{12}\mathbf{e}_1+a_{22}\mathbf{e}_2+a_{32}\mathbf{e}_3,$$

$$\alpha(\mathbf{e}_3)=\begin{pmatrix}1&0&0&0\\a_{10}&a_{11}&a_{12}&a_{13}\\a_{20}&a_{21}&a_{22}&a_{23}\\a_{30}&a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\a_{13}\\a_{23}\\a_{33}\end{pmatrix}=a_{13}\mathbf{e}_1+a_{23}\mathbf{e}_2+a_{33}\mathbf{e}_3.$$

The next step is to compute $\alpha(e_i)\circ_t\alpha(e_j)$ for $i,j\in\{0,1,2,3\}$.

$$\begin{aligned}\alpha(\mathbf{e}_0)\circ_t\alpha(\mathbf{e}_0)&=\left(1,2a_{10}\lambda^{t+1},2a_{20}\mu^{t+1},a+\frac{a-aa_{30}^2}{2^{t+1}}+\frac{a_{30}-a}{2^t}\right)\\&=\mathbf{e}_0+2a_{10}\lambda^{t+1}\mathbf{e}_1+2a_{20}\mu^{t+1}\mathbf{e}_2+\left(a+\frac{a-aa_{30}^2}{2^{t+1}}+\frac{a_{30}-a}{2^t}\right)\mathbf{e}_3,\end{aligned}$$

$$\begin{aligned}\alpha(\mathbf{e}_0)\circ_t\alpha(\mathbf{e}_1)&=\left(0,a_{11}\lambda^{t+1},a_{21}\mu^{t+1},\frac{a_{31}-aa_{30}a_{31}}{2^{t+1}}\right)\\&=a_{11}\lambda^{t+1}\mathbf{e}_1+a_{21}\mu^{t+1}\mathbf{e}_2+\frac{a_{31}-aa_{30}a_{31}}{2^{t+1}}\mathbf{e}_3,\end{aligned}$$

$$\begin{aligned}\alpha(\mathbf{e}_0)\circ_t\alpha(\mathbf{e}_2) &= \left(0, a_{12}\lambda^{t+1}, a_{22}\mu^{t+1}, \frac{a_{32}-aa_{30}a_{32}}{2^{t+1}}\right)\\ &= a_{12}\lambda^{t+1}\mathbf{e}_1 + a_{22}\mu^{t+1}\mathbf{e}_2 + \frac{a_{32}-aa_{30}a_{32}}{2^{t+1}}\mathbf{e}_3,\end{aligned}$$

$$\begin{aligned}\alpha(\mathbf{e}_0)\circ_t\alpha(\mathbf{e}_3) &= \left(0, a_{13}\lambda^{t+1}, a_{23}\mu^{t+1}, \frac{a_{33}-aa_{30}a_{33}}{2^{t+1}}\right)\\ &= a_{13}\lambda^{t+1}\mathbf{e}_1 + a_{23}\mu^{t+1}\mathbf{e}_2 + \frac{a_{33}-aa_{30}a_{33}}{2^{t+1}}\mathbf{e}_3,\end{aligned}$$

$$\begin{aligned}\alpha(\mathbf{e}_1)\circ_t\alpha(\mathbf{e}_0) &= \left(0, a_{11}\lambda^{t+1}, a_{21}\mu^{t+1}, \frac{a_{31}-aa_{30}a_{31}}{2^{t+1}}\right)\\ &= a_{11}\lambda^{t+1}\mathbf{e}_1 + a_{21}\mu^{t+1}\mathbf{e}_2 + \frac{a_{31}-aa_{30}a_{31}}{2^{t+1}}\mathbf{e}_3,\end{aligned}$$

$$\alpha(\mathbf{e}_1)\circ_t\alpha(\mathbf{e}_1) = \left(0,0,0,\frac{-aa_{31}^2}{2^{t+1}}\right) = \frac{-aa_{31}^2}{2^{t+1}}\mathbf{e}_3,$$

$$\alpha(\mathbf{e}_1)\circ_t\alpha(\mathbf{e}_2) = \left(0,0,0,\frac{-aa_{31}a_{32}}{2^{t+1}}\right) = \frac{-aa_{31}a_{32}}{2^{t+1}}\mathbf{e}_3,$$

$$\alpha(\mathbf{e}_1)\circ_t\alpha(\mathbf{e}_3) = \left(0,0,0,\frac{-aa_{31}a_{33}}{2^{t+1}}\right) = \frac{-aa_{31}a_{33}}{2^{t+1}}\mathbf{e}_3,$$

$$\begin{aligned}\alpha(\mathbf{e}_2)\circ_t\alpha(\mathbf{e}_0) &= \left(0, a_{12}\lambda^{t+1}, a_{22}\mu^{t+1}, \frac{a_{32}-aa_{30}a_{32}}{2^{t+1}}\right)\\ &= a_{12}\lambda^{t+1}\mathbf{e}_1 + a_{22}\mu^{t+1}\mathbf{e}_2 + \frac{a_{32}-aa_{30}a_{32}}{2^{t+1}}\mathbf{e}_3,\end{aligned}$$

$$\alpha(\mathbf{e}_2)\circ_t\alpha(\mathbf{e}_1) = \left(0,0,0,\frac{-aa_{31}a_{32}}{2^{t+1}}\right) = \frac{-aa_{31}a_{32}}{2^{t+1}}\mathbf{e}_3,$$

$$\alpha(\mathbf{e}_2)\circ_t\alpha(\mathbf{e}_2) = \left(0,0,0,\frac{-aa_{32}^2}{2^{t+1}}\right) = \frac{-aa_{32}^2}{2^{t+1}}\mathbf{e}_3,$$

$$\alpha(\mathbf{e}_2)\circ_t\alpha(\mathbf{e}_3) = \left(0,0,0,\frac{-aa_{32}a_{33}}{2^{t+1}}\right) = \frac{-aa_{32}a_{33}}{2^{t+1}}\mathbf{e}_3,$$

$$\alpha(\mathbf{e}_3) \circ_t \alpha(\mathbf{e}_0) = \left(0, a_{13}\lambda^{t+1}, a_{23}\mu^{t+1}, \frac{a_{33} - aa_{30}a_{33}}{2^{t+1}}\right)$$
$$= a_{13}\lambda^{t+1}\mathbf{e}_1 + a_{23}\mu^{t+1}\mathbf{e}_2 + \frac{a_{33} - aa_{30}a_{33}}{2^{t+1}}\mathbf{e}_3,$$

$$\alpha(\mathbf{e}_3) \circ_t \alpha(\mathbf{e}_1) = \left(0,0,0, \frac{-aa_{31}a_{33}}{2^{t+1}}\right) = \frac{-aa_{31}a_{33}}{2^{t+1}}\mathbf{e}_3,$$

$$\alpha(\mathbf{e}_3) \circ_t \alpha(\mathbf{e}_2) = \left(0,0,0, \frac{-aa_{32}a_{33}}{2^{t+1}}\right) = \frac{-aa_{32}a_{33}}{2^{t+1}}\mathbf{e}_3,$$

$$\alpha(\mathbf{e}_3) \circ_t \alpha(\mathbf{e}_3) = \left(0,0,0, \frac{-aa_{33}^2}{2^{t+1}}\right) = \frac{-aa_{33}^2}{2^{t+1}}\mathbf{e}_3.$$

Hence, the multiplications of α of basis vectors are reported in Tables (6.1) and (6.2). Finally, one can compute $\alpha(\mathbf{e}_i \circ_t \mathbf{e}_j)$ for $i, j \in \{0,1,2,3\}$. The results are presented in Table (6.3).

Table 6.1: Multiplication of α of the basis vectors - part (a)

$\circ_t$	$\alpha(e_0)$	$\alpha(e_1)$
$\alpha(\mathbf{e}_0)$	$\mathbf{e}_0 + 2a_{10}\lambda^{t+1}\mathbf{e}_1 + 2a_{20}\mu^{t+1}\mathbf{e}_2 + \left(a + \frac{a-aa_{30}^2}{2^{t+1}} + \frac{a_{30}-a}{2^t}\right)\mathbf{e}_3$	$a_{11}\lambda^{t+1}\mathbf{e}_1 + a_{21}\mu^{t+1}\mathbf{e}_2 + \frac{a_{31}-aa_{30}a_{31}}{2^{t+1}}\mathbf{e}_3$
$\alpha(\mathbf{e}_1)$	$a_{11}\lambda^{t+1}\mathbf{e}_1 + a_{21}\mu^{t+1}e_2 + \frac{a_{31}-aa_{30}a_{31}}{2^{t+1}}\mathbf{e}_3$	$\frac{-aa_{31}^2}{2^{t+1}}\mathbf{e}_3$
$\alpha(\mathbf{e}_2)$	$a_{12}\lambda^{t+1}\mathbf{e}_1 + a_{22}\mu^{t+1}\mathbf{e}_2 + \frac{a_{32}-aa_{30}a_{32}}{2^{t+1}}\mathbf{e}_3$	$\frac{-aa_{31}a_{32}}{2^{t+1}}\mathbf{e}_3$
$\alpha(\mathbf{e}_3)$	$a_{13}\lambda^{t+1}\mathbf{e}_1 + a_{23}\mu^{t+1}\mathbf{e}_2 + \frac{a_{33}-aa_{30}a_{33}}{2^{t+1}}\mathbf{e}_3$	$\frac{-aa_{31}a_{33}}{2^{t+1}}\mathbf{e}_3$

Table 6.2: Multiplication of α of the basis vectors - part (b)

$\circ_t$	$\alpha(e_2)$	$\alpha(e_3)$
$\alpha(\mathbf{e}_0)$	$a_{12}\lambda^{t+1}\mathbf{e}_1 + a_{22}\mu^{t+1}\mathbf{e}_2 + \frac{a_{32}-aa_{30}a_{32}}{2^{t+1}}\mathbf{e}_3$	$a_{13}\lambda^{t+1}\mathbf{e}_1 + a_{23}\mu^{t+1}\mathbf{e}_2 + \frac{a_{33}-aa_{30}a_{33}}{2^{t+1}}\mathbf{e}_3$
$\alpha(\mathbf{e}_1)$	$\frac{-aa_{31}a_{32}}{2^{t+1}}\mathbf{e}_3$	$\frac{-aa_{31}a_{33}}{2^{t+1}}\mathbf{e}_3$
$\alpha(\mathbf{e}_2)$	$\frac{-aa_{32}^2}{2^{t+1}}\mathbf{e}_3$	$\mathbf{0}$
$\alpha(\mathbf{e}_3)$	$\frac{-aa_{32}a_{33}}{2^{t+1}}\mathbf{e}_3$	$\frac{-aa_{32}a_{33}}{2^{t+1}}\mathbf{e}_3$

Table 6.3: $\alpha(\mathbf{e}_i \circ_t \mathbf{e}_j)$ for $i, j \in \{0,1,2,3\}$

i/j	0	1	2	3
0	$\mathbf{e}_0 + \sum_{i=1}^3 (a_{i0} + aa_{i3}\xi_0)\mathbf{e}_i$	$\lambda^{t+1}\sum_{i=1}^3 a_{i1}\mathbf{e}_i$	$\mu^{t+1}\sum_{i=1}^3 a_{i2}\mathbf{e}_i$	$\frac{1}{2^{t+1}}\sum_{i=1}^3 a_{i3}\mathbf{e}_i$
1	$\lambda^{t+1}\sum_{i=1}^3 a_{i1}\mathbf{e}_i$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
2	$\mu^{t+1}\sum_{i=1}^3 a_{i2}\mathbf{e}_i$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
3	$\frac{1}{2^{t+1}}\sum_{i=1}^3 a_{i3}\mathbf{e}_i$	$\mathbf{0}$	$\mathbf{0}$	$\frac{-a}{2^{t+1}}\sum_{i=1}^3 a_{i3}\mathbf{e}_i$

Now, two main cases will be considered, namely, $a=0$ and $a \neq 0$.

(I) Let $a=0$. Comparing Tables (6.1) and (6.2) one yields the following set of equations:

$$\begin{aligned}
&a_{10}\left(-1+2\lambda^{t+1}\right)=0, a_{20}\left(-1+2\mu^{t+1}\right)=0,\\
&a_{30}\left(-1+\frac{1}{2^t}\right)=0, a_{21}\left(-\lambda^{1+t}+\mu^{1+t}\right)=0,\\
&a_{31}\left(\frac{1}{2^{t+1}}-\lambda^{t+1}\right)=0, a_{12}\left(-\lambda^{1+t}+\mu^{1+t}\right)=0,\\
&a_{32}\left(\frac{1}{2^{t+1}}-\mu^{t+1}\right)=0, a_{13}\left(\frac{-1}{2^{t+1}}+\lambda^{1+t}\right)=0,\\
&a_{23}\left(\frac{-1}{2^{t+1}}+\mu^{1+t}\right)=0.
\end{aligned}$$

In addition, note that $\det(\alpha) \neq 0$. Hence, one has the following cases.

Case 1.1 Let $a=0, \lambda=\mu=\frac{1}{2}, t=0$. Then, $\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}$.

Case 1.2 Let $a=0, \mu=\frac{1}{2}, t \neq 0, |\lambda| < \frac{1}{2}$. Then,

$$a_{10}=a_{30}=a_{31}=a_{13}=a_{20}=a_{21}=a_{12}=0,$$

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{22} & a_{23} \\ 0 & 0 & a_{32} & a_{33} \end{pmatrix}.$$

Case 1.3 Let $a=0, \lambda=\frac{1}{2}, t \neq 0, |\mu| < \frac{1}{2}$. Then,

$$a_{10}=a_{20}=a_{30}=a_{21}=a_{12}=a_{32}=a_{23}=0,$$

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & a_{13} \\ 0 & 0 & a_{22} & 0 \\ 0 & a_{31} & 0 & a_{33} \end{pmatrix}.$$

Case 1.4 Let $a = 0, \lambda \neq \frac{1}{2}, t \neq 0, \mu \neq \frac{1}{2}, \lambda \neq \mu, |\lambda| < \frac{1}{2}, |\mu| < \frac{1}{2}$. Then,

$$a_{10} = a_{20} = a_{30} = a_{21} = a_{31} = a_{12} = a_{32} = a_{13} = a_{23} = 0,$$

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{22} & 0 \\ 0 & 0 & 0 & a_{33} \end{pmatrix}, \det(\alpha) \neq 0.$$

Case 1.5 Let $a = 0, t \neq 0, \lambda = \mu, |\lambda| < \frac{1}{2}, |\mu| < \frac{1}{2}$. Then,

$$a_{10} = a_{20} = a_{30} = a_{31} = a_{32} = a_{13} = a_{23} = 0,$$

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ 0 & 0 & 0 & a_{33} \end{pmatrix}.$$

Case 1.6 Let $a = 0, t \neq 0, \lambda = \mu = \frac{1}{2}$. Then,

$$a_{10} = a_{20} = a_{30} = a_{31} = a_{32} = a_{13} = a_{23} = 0,$$

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ 0 & 0 & 0 & a_{33} \end{pmatrix}.$$

Case 1.7 Let $a=0, t=0, \mu=\frac{1}{2}, |\lambda|<\frac{1}{2}$. Then,

$$a_{10}=a_{21}=a_{31}=a_{12}=a_{13}=0,$$

$$\alpha=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{20} & 0 & a_{22} & a_{23} \\ a_{30} & 0 & a_{32} & a_{33} \end{pmatrix}.$$

Case 1.8 Let $a=0, t=0, \lambda=\frac{1}{2}, |\mu|<\frac{1}{2}$. Then,

$$a_{20}=a_{21}=a_{12}=a_{32}=a_{23}=0,$$

$$\alpha=\begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{10} & a_{11} & 0 & a_{13} \\ 0 & 0 & a_{22} & 0 \\ a_{30} & a_{31} & 0 & a_{33} \end{pmatrix}.$$

Case 1.9 Let $a=0, t=0, \lambda\neq\mu, |\lambda|<\frac{1}{2}, |\mu|<\frac{1}{2}$. Then,

$$a_{10}=a_{20}=a_{21}=a_{31}=a_{12}=a_{32}=a_{13}=a_{23}=0,$$

$$\alpha=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{22} & 0 \\ a_{30} & 0 & 0 & a_{33} \end{pmatrix}.$$

Case 1.10 Let $a=0, t=0, \lambda=\mu, |\lambda|<\frac{1}{2}, |\mu|<\frac{1}{2}$. Then,

$$a_{10}=a_{20}=a_{31}=a_{32}=a_{13}=a_{23}=0,$$

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ a_{30} & 0 & 0 & a_{33} \end{pmatrix}.$$

It is wroth mention that the following condition is used in the previous cases.

$$0 \le \max\{|\lambda|, |\mu|\} \le \frac{\sqrt{1-|a|}}{2} = \frac{1}{2}.$$

(II) Let $a \neq 0$. Then, the following equations can be generated.

$$\begin{aligned}
&-aa_{13}\xi_0 + a_{10}(-1+2\lambda^{t+1}) = 0, -aa_{23}\xi_0 + a_{20}(-1+2\mu^{t+1}) = 0,\\
&-a_{30} + \frac{a_{30}}{2^{t+1}}(2-aa_{30}) + a\xi_0 - aa_{33}\xi_0 = 0, a_{21}(-\lambda^{1+t} + \mu^{t+1}) = 0,\\
&\frac{1}{2^{1+t}}(a_{31} - aa_{30}a_{31}) - a_{31}\lambda^{1+t} = 0, \frac{-1}{2^{1+t}}aa_{31}^2 = 0,\\
&\frac{-1}{2^{1+t}}aa_{31}a_{32} = 0, \frac{-1}{2^{1+t}}aa_{32}^2 = 0,\\
&\frac{-1}{2^{1+t}}aa_{13} = 0, a_{12}(-\lambda^{1+t} + \mu^{t+1}) = 0,\\
&\frac{1}{2^{t+1}}(a_{32} - aa_{30}a_{32}) - a_{32}\mu^{1+t} = 0, a_{13}\left(\frac{-1}{2^{t+1}} + \lambda^{1+t}\right) = 0,\\
&a_{23}\left(\frac{-1}{2^{t+1}} + \mu^{1+t}\right) = 0, \frac{1}{2^{t+1}}aa_{30}a_{33} = 0,\\
&\frac{-1}{2^{t+1}}aa_{31}a_{33} = 0, \frac{-1}{2^{t+1}}aa_{32}a_{33} = 0,\\
&\frac{-1}{2^{t+1}}aa_{23} = 0, \frac{-1}{2^{t+1}}a_{33}(1+a_{33}) = 0,
\end{aligned}$$

where $\xi_0 = 1 + \frac{1}{2^{t+1}} - \frac{1}{2^t}$. Thus, $a_{13} = a_{23} = a_{32} = a_{31} = 0$. Hence, the previous system of equations can be reduce to the following seven equations.

$$
\begin{aligned}
&a_{10}(-1+2\lambda^{1+t})=0, \frac{-1}{2^{t+1}}aa_{30}a_{33}=0,\\
&-a_{30}+\frac{a_{30}}{2^{t+1}}(2-aa_{30})+a\xi_0-aa_{33}\xi_0=0,\\
&\frac{-1}{2^{t+1}}aa_{33}(1+a_{33})=0, a_{20}(-1+2\mu^{1+t})=0,\\
&a_{21}(-\lambda^{1+t}+\mu^{1+t})=0, a_{12}(-\lambda^{1+t}+\mu^{1+t})=0.
\end{aligned}
$$

Thus, the following five cases can be reported.

Case 2.1 Let $a_{33}=0, \lambda=\mu, t\neq 0$. Then,

$$a_{10}=a_{20}=0, a_{30}=\frac{2-2^{1+t}\pm\sqrt{(-2+2^{1+t})^2+2^{3+t}a^2\xi_0}}{2a},$$

$$\alpha=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & a_{11} & a_{12} & 0\\ 0 & a_{21} & a_{22} & 0\\ a_{30} & 0 & 0 & 0\end{pmatrix}.$$ Due to the fact that $\det(\alpha)\neq 0$ and since $\det(\alpha)=0$, this case can not be considered.

Case 2.2 Let $a_{33}=0, \lambda\neq\mu, \lambda=\frac{1}{2}, t\neq 0$. Then,

$$a_{20}=a_{21}=a_{12}=0, a_{30}=\frac{2-2^{1+t}\pm\sqrt{(-2+2^{1+t})^2+2^{3+t}a^2\xi_0}}{2a},$$

$$\alpha=\begin{pmatrix} 1 & 0 & 0 & 0\\ a_{10} & a_{11} & 0 & 0\\ 0 & 0 & a_{22} & 0\\ a_{30} & 0 & 0 & 0\end{pmatrix}.$$

Due to the fact that $\det(\alpha)\neq 0$ and since $\det(\alpha)=0$, this case can not be considered.

Case 2.3 Let $a_{33} = 0, \lambda \neq \mu, \mu = \frac{1}{2}, t = 0$. Then,

$$a_{10} = a_{21} = a_{12} = 0, a_{30} = \frac{2 - 2^{1+t} \pm \sqrt{(-2 + 2^{1+t})^2 + 2^{3+t} a^2 \xi_0}}{2a},$$

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{20} & 0 & a_{22} & 0 \\ a_{30} & 0 & 0 & 0 \end{pmatrix}.$$

Due to the fact that $\det(\alpha) \neq 0$ and since $\det(\alpha) = 0$, this case can not be considered.

Case 2.4 Let $a_{33} = 0, \lambda \neq \mu, \lambda \neq \frac{1}{2}, \mu \neq \frac{1}{2}, t = 0$. Then,

$$a_{10} = a_{20} = a_{21} = a_{12} = 0, a_{30} = \frac{2 - 2^{1+t} \pm \sqrt{(-2 + 2^{1+t})^2 + 2^{3+t} a^2 \xi_0}}{2a},$$

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{22} & 0 \\ a_{30} & 0 & 0 & 0 \end{pmatrix}.$$

Due to the fact that $\det(\alpha) \neq 0$ and since $\det(\alpha) = 0$, this case can not be considered.

Case 2.5 Let $a_{33} \neq 0$. Then,

$$a_{13} = a_{23} = a_{32} = a_{31} = 0 = a_{30}.$$

Thus, the following system of equations can be generated.

$$a_{10}(-1+2\lambda^{1+t})=0, a\xi_0 - aa_{33}\xi_0 = 0, \frac{-1}{2^{t+1}}aa_{33}(1+a_{33})=0,$$
$$a_{20}(-1+2\mu^{1+t})=0, a_{21}(-\lambda^{1+t}+\mu^{1+t})=0, a_{12}(-\lambda^{1+t}+\mu^{1+t})=0.$$

Since $t \geq 0$ and $\xi_0 = 1-\frac{1}{2^{t+1}} \neq 0$, $a\xi_0 - aa_{33}\xi_0 = 0$ yields that $a_{33} = 1$. Thus, $\frac{-1}{2^{t+1}}aa_{33}(1+a_{33})=0$ implies that $a=0$ which is a contradiction. Thus, this case is impossible.

Finally, the following possibilities for α can be obtained.

1. Let $a=0, \lambda=\mu=\frac{1}{2}, t=0$. Then,
$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}.$$
2. Let $a=0, \mu=\frac{1}{2}, t\neq 0, |\lambda|<\frac{1}{2}$. Then,
$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{22} & a_{23} \\ 0 & 0 & a_{32} & a_{33} \end{pmatrix}.$$
3. Let $a=0, \lambda=\frac{1}{2}, t\neq 0, |\mu|<\frac{1}{2}$. Then,
$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & a_{13} \\ 0 & 0 & a_{22} & 0 \\ 0 & a_{31} & 0 & a_{33} \end{pmatrix}.$$
4. Let $a=0, \lambda\neq\frac{1}{2}, t\neq 0, \mu\neq\frac{1}{2}, \lambda\neq\mu, |\lambda|<\frac{1}{2}, |\mu|<\frac{1}{2}$. Then,

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{22} & 0 \\ 0 & 0 & 0 & a_{33} \end{pmatrix}.$$

5. Let $a = 0, t \neq 0, \lambda = \mu, |\lambda| < \frac{1}{2}, |\mu| < \frac{1}{2}$. Then,

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ 0 & 0 & 0 & a_{33} \end{pmatrix}.$$

6. Let $a = 0, t \neq 0, \lambda = \mu = \frac{1}{2}$. Then,

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ 0 & 0 & 0 & a_{33} \end{pmatrix}.$$

7. Let $a = 0, t = 0, \mu = \frac{1}{2}, |\lambda| < \frac{1}{2}$. Then,

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{20} & 0 & a_{22} & a_{23} \\ a_{30} & 0 & a_{32} & a_{33} \end{pmatrix}.$$

8. Let $a = 0, t = 0, \lambda = \frac{1}{2}, |\mu| < \frac{1}{2}$. Then,

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{10} & a_{11} & 0 & a_{13} \\ 0 & 0 & a_{22} & 0 \\ a_{30} & a_{31} & 0 & a_{33} \end{pmatrix}.$$

9. Let $a = 0, t = 0, \lambda \neq \mu, |\lambda| < \frac{1}{2}, |\mu| < \frac{1}{2}$. Then,

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{22} & 0 \\ a_{30} & 0 & 0 & a_{33} \end{pmatrix}.$$

10. Let $a = 0, t = 0, \lambda = \mu, |\lambda| < \frac{1}{2}, |\mu| < \frac{1}{2}$. Then,

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ a_{30} & 0 & 0 & a_{33} \end{pmatrix}.$$

6.3 Positivity of automorphisms of $(A_t, \circ_t)$

Assume that

$$\mathbb{T} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \mathbf{a} = \begin{pmatrix} a_{10} \\ a_{20} \\ a_{30} \end{pmatrix},$$

and $\widetilde{\mathbb{T}} = \mathbf{a} + \mathbb{T}(\mathbf{f})$. Let

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

It is noted that α is positive if and only if $\alpha(S) \subset S$. Take $\varphi \in S$, then $\mathbf{f} = (1, f_1, f_2, f_3)$ and

$$\alpha(\varphi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$$= (1, a_{10} + \Sigma_{j=1}^{3} a_{1j} f_j, a_{20} + \Sigma_{j=1}^{3} a_{2j} f_j, a_{30} + \Sigma_{j=1}^{3} a_{3j} f_j$$

$$= (1, a + \mathbb{T}(\mathbf{f})).$$

The positivity condition for α is equivalent to

$$\|\mathbf{a}+\mathbb{T}(\mathbf{f})\| \leq 1, \forall \|\mathbf{f}\| \leq 1. \tag{6.2}$$

Now, some sufficient conditions to satisfy Equation (6.2) will be provided.

Theorem 6.3.1 *If* $\|\mathbf{a}+\mathbb{T}(\mathbf{f})\| \leq 1$*, then*

$$\|\mathbb{T}\|^2 + \|\mathbf{a}\|^2 \leq 1. \tag{6.3}$$

Proof. Due to the parallelogram equality and $\|\mathbf{a}+\mathbb{T}(\mathbf{f})\| \leq 1, \|\mathbf{a}-\mathbb{T}(\mathbf{f})\| \leq 1, \forall \mathbf{f}, \|\mathbf{f}\| \leq 1$, then $2(\|\mathbf{a}\|^2 + \|\mathbb{T}\mathbf{f}\|^2) \leq 2$ which implies that $\|\mathbb{T}\mathbf{f}\|^2 + \|\mathbf{a}\|^2 \leq 1$. Hence, $\|\mathbb{T}\mathbf{f}\|^2 \leq 1 - \|\mathbf{a}\|^2, \forall \mathbf{f}, \|\mathbf{f}\| \leq 1$. Thus, $\|\mathbb{T}\|^2 \leq 1 - \|\mathbf{a}\|^2$ which gives $\|\mathbb{T}\|^2 + \|\mathbf{a}\|^2 \leq 1$. □

Remark We stress that condition (6.3) is necessary condition but not sufficient. Indeed, consider the following example. Let

$$\mathbb{T} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \mathbf{f} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{a} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

Then,

$$\|\mathbb{T}\|^2 + \|\mathbf{a}\|^2 = \left(\max\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}\right)^2 + \left(\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}\right)^2 = 1 \leq 1$$

but

$$\|\mathbf{a}+\mathbb{T}\mathbf{f}\|^2 = \left\| \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\|^2 = \frac{3}{2} > 1.$$

Corollary 6.3.2 *If* $\mathbf{a} = \mathbf{0}$*, then,* $\|\mathbf{a}+\mathbb{T}\mathbf{f}\| \leq 1$ *if and only if* $\|\mathbb{T}\mathbf{f}\| \leq 1, \forall \|\mathbf{f}\| \leq 1$ *which is equivalent to* $\|\mathbb{T}\| \leq 1$*. Hence, Theorem (6.3.1) will be necessary and sufficient statement.*

Theorem 6.3.3 *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with* $a, b, c, d \in \mathbb{R}$*. Then,*

$$\|A\| = \frac{1}{\sqrt{2}} \sqrt{a^2+b^2+c^2+d^2+\sqrt{(a^2+b^2+c^2+d^2)^2+4(ad-bc)^2}}.$$

Proof. It is clear that:

$$A^T A = \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix}$$

is symmetric matrix and its eigenvalues are nonnegative real numbers. Thus,

$$0 = \det(A^T A - \lambda I_2) = \det \begin{pmatrix} a^2+c^2-\lambda & ab+cd \\ ab+cd & b^2+d^2-\lambda \end{pmatrix}$$
$$= \lambda^2 - (a^2+b^2+c^2+d^2)\lambda + (a^2+c^2)(b^2+d^2) - (ab+cd)^2.$$

Hence,

$$\lambda_1, \lambda_2 = \frac{a^2+b^2+c^2+d^2 \pm \sqrt{(a^2+b^2+c^2+d^2)^2+4(ad-bc)^2}}{2}, \lambda_1 \geq \lambda_2.$$

Thus,

$$\|A\| = \frac{1}{\sqrt{2}}\sqrt{a^2+b^2+c^2+d^2+\sqrt{(a^2+b^2+c^2+d^2)^2+4(ad-bc)^2}}.$$

□

Theorem 6.3.4 *Let $f:\Omega\to\mathbb{R}$ be a function that is defined by*

$$f(u,v) = Au + Bv + C$$

where $C \geq 0$ and $\Omega = \{(u,v)\in\mathbb{R} : u \geq 0, v \geq 0, u+v \leq 1\}$. Then, the maximum value of f on Ω is

$$\max_{(u,v)\in\Omega} f(u,v) = \max\{A+C, B+C, C\}.$$

Proof. Since $\frac{\partial f}{\partial u} = A$ and $\frac{\partial f}{\partial v} = B$, then the critical numbers of f is all $(u,v)\in\Omega$ if $A = B = 0$. If $|A|+|B| \neq 0$, then f does not have critical point in the interior of Ω. Hence, the maximum of f is on the boundary. Three cases should be considered.

Case 1 Let $v = 0$. Then, $f(u,v) = Au + C, 0 \leq u \leq 1$ which yields that

$$\max_{\substack{(u,v)\in\Omega \\ v=0}} f(u,v) = \max\{A+C, C\}.$$

Case 2 Let $u = 0$. Then, $f(u,v) = Bv + C, 0 \leq v \leq 1$ which gives that

$$\max_{\substack{(u,v)\in\Omega \\ u=0}} f(u,v) = \max\{B+C, C\}.$$

Case 3 Let $u + v = 1$. Then, $f(u,v) = g(u) = Au + (1-u)B + C, 0 \leq u \leq 1$. Hence,

$g'(u) = A - B$. Thus,

$$\max_{\substack{(u,v)\in\Omega \\ u+v=1}} f(u,v) = \begin{cases} \max\{g(0), g(1)\} & A \neq B, \\ \max\{g(0), g(1), A+C\} & A = B, \end{cases}$$

or

$$\max_{\substack{(u,v)\in\Omega \\ u+v=1}} f(u,v) = \begin{cases} \max\{B+C, A+C\} & A \neq B, \\ \max\{B+C, A+C\} & A = B. \end{cases}$$

Combining all cases, one can get:

$$\max_{(u,v)\in\Omega} f(u,v) = \max\{A+C, B+C, C\}.$$

□

Now, the positivity of the ten matrices in the previous section will be investigated. Now, using Theorem 6.3.1 with $\mathbf{a} = 0$, one provides necessary and sufficient conditions for the positivity of C.

Case 1.2 Let $\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{22} & a_{23} \\ 0 & 0 & a_{32} & a_{33} \end{pmatrix}$. Then, it follows from Theorem (6.3.1) that α is positive if

$$\left\| \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \right\|^2 + |a_{11}|^2 \leq 1$$

which yields by Theorem (6.3.3),

$$\frac{a_{22}^2 + a_{23}^2 + a_{32}^2 + a_{33}^2 + \sqrt{(a_{22}^2 + a_{23}^2 + a_{32}^2 + a_{33}^2)^2 + 4(a_{22}a_{33} - a_{23}a_{32})^2}}{2} + |a_{11}|^2 \leq 1.$$

Case1.3 Let $\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & a_{13} \\ 0 & 0 & a_{22} & 0 \\ 0 & a_{31} & 0 & a_{33} \end{pmatrix}$. Then, it follows from Theorem (6.3.1) that α is positive if

$$\left\| \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} \right\|^2 + |a_{22}|^2 \leq 1$$

which yields by Theorem (6.3.3),

$$\frac{a_{11}^2 + a_{13}^2 + a_{31}^2 + a_{33}^2 + \sqrt{(a_{11}^2 + a_{13}^2 + a_{31}^2 + a_{33}^2)^2 + 4(a_{11}a_{33} - a_{13}a_{32})^2}}{2} + |a_{22}|^2 \leq 1.$$

Case 1.4 Let $\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{22} & 0 \\ 0 & 0 & 0 & a_{33} \end{pmatrix}$. Then, α is positive if

$$\max\{|a_{11}|, |a_{22}|, |a_{33}|\} \leq 1.$$

Case 1.5 Let $\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ 0 & 0 & 0 & a_{33} \end{pmatrix}$. Then, it follows from Theorem (6.3.1) that α is positive if

$$\left\| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right\|^2 + |a_{33}|^2 \leq 1$$

which yields by Theorem (6.3.3),

$$\frac{a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 + \sqrt{(a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2)^2 + 4(a_{11}a_{22} - a_{12}a_{21})^2}}{2} + |a_{33}|^2 \leq 1.$$

Case 1.6 Let $\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ 0 & 0 & 0 & a_{33} \end{pmatrix}$. Then, it follows from Theorem (6.3.1) that α is positive if $\left\| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right\|^2 + |a_{33}|^2 \leq 1$ which yields by Theorem (6.3.3),

$$\frac{a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 + \sqrt{(a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2)^2 + 4(a_{11}a_{22} - a_{12}a_{21})^2}}{2} + |a_{33}|^2 \leq 1.$$

In the next cases, by using (6.2), the positivity of α will be investigated.

Case 1.7 Let $\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{20} & 0 & a_{22} & a_{23} \\ a_{30} & 0 & a_{32} & a_{33} \end{pmatrix}$. Then, α is positive if

$\|\mathbf{a} + \mathbb{T}\mathbf{f}\|^2 \leq 1$, where $\mathbb{T} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$ and $\mathbf{a} = \begin{pmatrix} 0 \\ a_{20} \\ a_{30} \end{pmatrix}$.

To find sufficient conditions for the positivity of α, one should examine $\|\mathbf{a} + \mathbb{T}\mathbf{f}\|^2$. Thus,

$$\begin{aligned}
\|\mathbf{a} + \mathbb{T}\mathbf{f}\|^2 &= a_{11}^2 f_1^2 + (|a_{20}| + |a_{22}|f_2 + |a_{23}|f_3)^2 + (|a_{30}| + |a_{32}|f_2 + |a_{33}|f_3)^2 \\
&= a_{11}^2 f_1^2 + a_{20}^2 + a_{22}^2 f_2^2 + a_{23}^2 f_3^2 + 2|a_{20}||a_{22}|f_2 + 2|a_{20}||a_{23}|f_3 + 2|a_{22}||a_{23}|f_2 f_3 \\
&\quad + a_{30}^2 + 2|a_{30}||a_{32}|f_2 + 2|a_{30}||a_{33}|f_3 + a_{32}^2 f_2^2 + 2|a_{32}||a_{33}|f_2 f_3 + a_{33}^2 f_3^2 \\
&\leq a_{11}^2 f_1^2 + a_{22}^2 f_2^2 + 2(|a_{20}||a_{22}| + |a_{30}||a_{32}|)|f_2| + 2(|a_{30}||a_{33}|)|f_3| \\
&\quad + 2(|a_{22}||a_{23}| + |a_{32}||a_{33}|)|f_2||f_3| + (a_{23}^2 + a_{33}^2)f_3^2 + (a_{20}^2 + a_{30}^2) \\
&\quad + a_{32}^2 f_2^2.
\end{aligned}$$

Since $2|f_2||f_3| \leq f_2^2 + f_3^2$ and $2|f_2| \leq 1 + f_2^2$,

$$\|\mathbf{a} + \mathbb{T}\mathbf{f}\|^2 \leq a_{11}^2 f_1^2 + (a_{22}^2 + a_{32}^2 + |a_{22}||a_{23}| + |a_{32}||a_{33}|) f_2^2$$
$$+(|a_{20}||a_{22}| + |a_{30}||a_{32}|)(1 + f_2^2) + (a_{23}^2 + a_{33}^2 + |a_{22}||a_{23}| + |a_{32}||a_{33}|) f_3^2$$
$$+2|a_{30}||a_{33}||f_3| + (a_{22}^2 + a_{30}^2)$$
$$= a_{11}^2 f_1^2 + (a_{22}^2 + a_{32}^2 + |a_{22}||a_{23}| + |a_{32}||a_{33}| + (|a_{20}||a_{22}| + |a_{30}||a_{32}|) f_2^2$$
$$+(a_{23}^2 + a_{33}^2 + |a_{22}||a_{23}| + |a_{32}||a_{33}|) f_3^2$$
$$+2|a_{30}||a_{33}||f_3| + (a_{22}^2 + a_{30}^2 + |a_{20}||a_{22}| + |a_{30}||a_{32}|).$$

Let

$$\beta = \max\{a_{11}^2, a_{22}^2 + a_{32}^2 + |a_{22}||a_{23}| + |a_{32}||a_{33}| + |a_{20}||a_{22}| + |a_{30}||a_{32}|\}.$$

Then,

$$\|\mathbf{a} + \mathbb{T}\mathbf{f}\|^2 \leq \beta(f_1^2 + f_2^2) + (a_{23}^2 + a_{33}^2 + |a_{22}||a_{23}| + |a_{32}||a_{33}|) f_3^2$$
$$+2|a_{30}||a_{33}||f_3| + (a_{22}^2 + a_{30}^2 + |a_{20}||a_{22}| + |a_{30}||a_{32}|)$$
$$\leq \beta(1 - f_3^2) + (a_{23}^2 + a_{33}^2 + |a_{22}||a_{23}| + |a_{32}||a_{33}|) f_3^2$$
$$+2|a_{30}||a_{33}||f_3| + (a_{22}^2 + a_{30}^2 + |a_{20}||a_{22}| + |a_{30}||a_{32}|)$$
$$= Au^2 + Bu + C \leq 1,$$

where

$$A = a_{23}^2 + a_{33}^2 + |a_{22}||a_{23}| + |a_{32}||a_{33}| - \beta,$$
$$B = 2|a_{30}||a_{33}|, u = |f_3|,$$
$$C = \beta + a_{22}^2 + a_{30}^2 + |a_{20}||a_{22}| + |a_{30}||a_{32}|.$$

Hence,

$$(1-C) - Bu - Au^2 \geq 0.$$

Let $a = -A, b = -B$, and $c = 1-C$. Then, $f(u) = au^2 + bu + c \geq 0$ for all $u \in [0,1]$. By Lemma 3.2.1, α is positive if

$$1 - \beta - a_{22}^2 - a_{30}^2 - |a_{20}||a_{22}| - |a_{30}||a_{32}| \geq 0, \tag{6.4}$$

and

$$\begin{aligned} &-a_{23}^2 - a_{33}^2 - |a_{22}||a_{23}| - |a_{32}||a_{33}| + \beta - 2|a_{30}||a_{33}| \\ &+1 - \beta - a_{22}^2 - a_{30}^2 - |a_{20}||a_{22}| - |a_{30}||a_{32}| \geq 0 \end{aligned}$$

which gives that

$$\begin{aligned} &1 - a_{23}^2 - a_{33}^2 - a_{22}^2 - a_{30}^2 - |a_{22}||a_{23}| - |a_{32}||a_{33}| - 2|a_{30}||a_{33}| \\ &-|a_{20}||a_{22}| - |a_{30}||a_{32}| \geq 0. \end{aligned} \tag{6.5}$$

In addition, either conditions (6.6)-(6.9) or (6.10) hold.

$$\beta - a_{23}^2 - {}_{33}^2 - |a_{22}||a_{23}| - |a_{32}||a_{33}| > 0 \tag{6.6}$$

$$2a_{30}a_{33} < 0 \tag{6.7}$$

$$|a_{30}||a_{33}| + a_{23}^2 + a_{33}^2 + |a_{22}||a_{23}| + |a_{32}||a_{33}| - \beta > 0 \tag{6.8}$$

$$\begin{aligned} &4a_{30}^2 + a_{33}^2 + 4(a_{23}^2 + a_{33}^2 + |a_{22}||a_{23}| + |a_{32}||a_{33}| - \beta)(1 - \beta \\ &-a_{22}^2 - a_{30}^2 + a_{20}a_{22} + a_{30}a_{32}) \leq 0 \end{aligned} \tag{6.9}$$

or

$$a_{23}^2 + a_{33}^2 + |a_{22}||a_{23}| + |a_{32}||a_{33}| - \beta > 0. \tag{6.10}$$

and one of the following conditions is satisfied. Finally, the following conditions are founded.

1. $1-\beta-a_{22}^2-a_{30}^2-|a_{20}||a_{22}|-|a_{30}||a_{32}|\geq 0,$

2. $$1-a_{23}^2-a_{33}^2-a_{22}^2-a_{30}^2-|a_{22}||a_{23}|-|a_{32}||a_{33}|-2|a_{30}||a_{33}|$$
$$-|a_{20}||a_{22}|-|a_{30}||a_{32}|\geq 0.$$

and one of the following conditions is satisfied.

I. $\beta > a_{23}^2+a_{33}^2+|a_{22}||a_{23}|+|a_{32}||a_{33}|.$

a. $a_{30}a_{33}<0$

b. $|a_{30}||a_{33}|+a_{23}^2+a_{33}^2+|a_{22}||a_{23}|+|a_{32}||a_{33}|>\beta$

c. $$4a_{30}^2a_{33}^2+4(a_{23}^2+a_{33}^2+|a_{22}||a_{23}|+|a_{32}||a_{33}|-\beta)(1-\beta$$
$$-a_{22}^2-a_{30}^2-a_{20}a_{22}-a_{30}a_{32})\leq 0.$$

II. $\beta < a_{23}^2+a_{33}^2+|a_{22}||a_{23}|+|a_{32}||a_{33}|.$

Case 1.8 Let $\alpha=\begin{pmatrix}1&0&0&0\\a_{10}&a_{11}&0&a_{13}\\0&0&a_{22}&0\\a_{30}&a_{31}&0&a_{33}\end{pmatrix}$. Then, $\mathbf{a}=\begin{pmatrix}a_{10}\\0\\a_{30}\end{pmatrix}$ and

$$\mathbb{T}=\begin{pmatrix}a_{11}&0&a_{13}\\0&a_{22}&0\\a_{30}&0&a_{33}\end{pmatrix}.$$

Now, the goal is to reduce this case to case (1.7). Define the matrix U by

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, $U^{-1} = U, \widetilde{\mathbf{f}} = U \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} f_2 \\ f_1 \\ f_3 \end{pmatrix}$, and

$$\widetilde{\mathbb{T}}_7 = U\mathbb{T}U^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{30} & 0 & a_{33} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} a_{22} & 0 & 0 \\ 0 & a_{11} & a_{13} \\ 0 & a_{31} & a_{33} \end{pmatrix}.$$

Thus, $\mathbb{T} = U^{-1}\widetilde{\mathbb{T}}_7 U$. Then,

$$\|\mathbf{a} + \mathbb{T}\mathbf{f}\| = \|U^{-1}U\mathbf{a} + U^{-1}\widetilde{\mathbb{T}}_7 U\mathbf{f}\| = \|U^{-1}(U\mathbf{a} + \widetilde{\mathbb{T}}_7 U\mathbf{f})\| = \|\widetilde{\mathbf{a}} + \widetilde{\mathbb{T}}_7\widetilde{\mathbf{f}}\|$$

where $\widetilde{\mathbf{a}} = U\mathbf{a} = \begin{pmatrix} 0 \\ a_{10} \\ a_{30} \end{pmatrix}$. Therefore, from Case (1.7), one obtains the following conditions.

1. $1 - \beta - a_{11}^2 - a_{30}^2 - |a_{10}||a_{11}| - |a_{30}||a_{31}| \geq 0$,

2. $$1 - a_{13}^2 - a_{33}^2 - a_{11}^2 - a_{30}^2 - |a_{11}||a_{13}| - |a_{31}||a_{33}| - 2|a_{30}||a_{33}|$$
$$-|a_{10}||a_{11}| - |a_{30}||a_{31}| \geq 0.$$

and one of the following conditions is satisfied.

I. $\beta > a_{13}^2 + a_{33}^2 + |a_{11}||a_{13}| + |a_{31}||a_{33}|$.

a. $a_{30}a_{33} < 0$.

b. $|a_{30}||a_{33}| + a_{13}^2 + a_{33}^2 + |a_{11}||a_{13}| + |a_{31}||a_{33}| > \beta$.

c.

$$4a_{30}^2a_{33}^2 + 4(a_{13}^2 + a_{33}^2 + |a_{11}||a_{13}| + |a_{31}||a_{33}| - \beta)(1 - \beta$$
$$-a_{11}^2 - a_{30}^2 - a_{10}a_{11} - a_{30}a_{31}) \leq 0.$$

II. $\beta < a_{13}^2 + a_{33}^2 + |a_{11}||a_{13}| + |a_{31}||a_{33}|$.

Case 1.9 Let $\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{22} & 0 \\ a_{30} & 0 & 0 & a_{33} \end{pmatrix}$. Then, α is positive if and only if

$$\max\{|a_{30}|^2 + |a_{11}|^2 + |a_{22}|^2 + |a_{33}|^2\} \leq 1.$$

Let us check conditions of case (1.7). Then,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{22} & 0 \\ a_{30} & 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} 1 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 1 \\ a_{11}f_1 \\ a_{22}f_2 \\ a_{30} + a_{33}f_3 \end{pmatrix}.$$

Then,

$$I = a_{11}^2 f_1^2 + a_{22}^2 f_2^2 + (|a_{30}| + |a_{33}|f_3)^2$$
$$\leq a_{11}^2 f_1^2 + a_{22}^2 f_2^2 + a_{30}^2 + a_{33}^2 f_3^2 + 2|a_{30}||a_{33}||f_3|.$$

Let $\beta = \max\{a_{11}^2, a_{22}^2\}$. Then,

$$
\begin{aligned}
I &\le \beta(f_1^2 + f_2^2) + a_{33}^2 f_3^2 + 2|a_{30}||a_{33}||f_3| + a_{30}^2 \\
&\le \beta(1 - f_3^2) + a_{33}^2 f_3^2 + 2|a_{30}||a_{33}||f_3| + a_{30}^2 \\
&= (a_{33}^2 - \beta) f_3^2 + 2|a_{30}||a_{33}||f_3| + \beta + a_{30}^2 \le 1.
\end{aligned}
$$

Then, $f(u) = au^2 + bu + c, u \in [0,1]$, where $u = |f_3|, a = \beta - a_{33}^2, b = -2|a_{30}||a_{33}|$, and $c = 1 - \beta - a_{30}^2$. By Lemma (3.2.1), α is positive if

$$1 - \beta - a_{30}^2 \ge 0 \tag{6.11}$$

$$1 - a_{33}^2 - 2|a_{30}||a_{33}| - a_{30}^2 \ge 0 \tag{6.12}$$

and either conditions (6.13-6.16) or (6.17) hold.

$$\beta - a_{33}^2 > 0 \tag{6.13}$$

$$|a_{30}||a_{33}| < 0 \tag{6.14}$$

$$2|a_{30}||a_{33}| + 2(a_{33}^2 - \beta) > 0 \tag{6.15}$$

$$4a_{30}^2 a_{33}^2 + 4(a_{33}^2 - \beta)(1 - \beta - a_{30}^2) \le 0 \tag{6.16}$$

or

$$a_{33}^2 - \beta > 0. \tag{6.17}$$

Since condition (6.13) can be satisfied, so the conditions become

$$1 - \beta - a_{30}^2 \ge 0 \tag{6.18}$$

$$1 - a_{33}^2 - 2|a_{30}||a_{33}| - a_{30}^2 \ge 0 \tag{6.19}$$

$$a_{33}^2 - \beta > 0. \tag{6.20}$$

Case 1.10 Let $\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ a_{30} & 0 & 0 & a_{33} \end{pmatrix}$. Then, $\mathbf{a} = \begin{pmatrix} 0 \\ 0 \\ a_{30} \end{pmatrix}$ and

$$\mathbb{T} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}.$$

Now, the goal is to reduce this case to case (1.7). Define the matrix U by

$$U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then, $U^{-1} = U$, $\widetilde{\widetilde{\mathbf{f}}} = U \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} f_3 \\ f_2 \\ f_1 \end{pmatrix}$, and

$$\widetilde{\widetilde{\mathbb{T}}}_7 = U\mathbb{T}U^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{33} & 0 & 0 \\ 0 & a_{22} & a_{21} \\ 0 & a_{12} & a_{11} \end{pmatrix}.$$

Thus, $\mathbb{T} = U^{-1}\widetilde{\mathbb{T}}_7 U$. Then,

$$\|\mathbf{a} + \mathbb{T}\mathbf{f}\| = \|U^{-1}U\mathbf{a} + U^{-1}\widetilde{\widetilde{\mathbb{T}}}_7 U\mathbf{f}\| = \|U^{-1}(U\mathbf{a} + \widetilde{\widetilde{\mathbb{T}}}_7 U\mathbf{f})\| = \|\widetilde{\widetilde{\mathbf{a}}} + \widetilde{\widetilde{\mathbb{T}}}_7 \widetilde{\widetilde{\mathbf{f}}}\|$$

where $\widetilde{\widetilde{\mathbf{a}}} = U\mathbf{a} = \begin{pmatrix} a_{30} \\ 0 \\ 0 \end{pmatrix}$. Therefore, from Case (1.7), one obtains the following conditions.

1. $1 - \beta - a_{22}^2 \geq 0$,

2. $$1 - a_{21}^2 - a_{11}^2 - a_{22}^2 - |a_{22}||a_{21}| - |a_{12}||a_{11}| \geq 0.$$

and one of the following conditions is satisfied.

I. $\beta > a_{21}^2 + a_{11}^2 + |a_{22}||a_{21}| + |a_{12}||a_{11}|$.

a. $a_{21}^2 + a_{11}^2 + |a_{22}||a_{21}| + |a_{12}||a_{11}| > \beta$.

b. $$4(a_{21}^2 + a_{11}^2 + |a_{22}||a_{21}| + |a_{12}||a_{11}| - \beta)(1 - \beta - a_{22}^2) \leq 0.$$

II. $\beta < a_{21}^2 + a_{11}^2 + |a_{22}||a_{21}| + |a_{12}||a_{11}|$.

Let us check conditions of case (1.7). Then,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ a_{30} & 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} 1 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 1 \\ a_{11}f_1 + a_{12}f_2 \\ a_{21}f_1 + a_{22}f_2 \\ a_{30} + a_{33}f_3 \end{pmatrix}.$$

Then,

$$
\begin{aligned}
I &= (|a_{11}|f_1 + |a_{12}|f_2)^2 + (|a_{21}|f_1 + |a_{22}|f_2)^2 + (|a_{30}| + |a_{33}|f_3)^2 \\
&\leq a_{11}^2 f_1^2 + a_{12}^2 f_2^2 + 2|a_{11}||a_{12}||f_1||f_2| + a_{21}^2 f_1^2 + a_{22}^2 f_2^2 \\
&+ 2|a_{21}||a_{22}|f_1 f_2 + a_{30}^2 + a_{33}^2 f_3^2 + 2|a_{30}||a_{33}||f_3| \\
&\leq a_{11}^2 f_1^2 + a_{12}^2 f_2^2 + |a_{11}||a_{12}|(f_1^2 + f_2^2) + a_{21}^2 f_1^2 + a_{22}^2 f_2^2 \\
&+ |a_{21}||a_{22}|(f_1^2 + f_2^2) + a_{30}^2 + a_{33}^2 f_3^2 + 2|a_{30}||a_{33}||f_3| \\
&= (a_{11}^2 + |a_{11}||a_{12}| + a_{21}^2 + |a_{21}||a_{22}|)f_1^2 + (a_{12}^2 + |a_{11}||a_{12}| + a_{22}^2 + |a_{21}||a_{22}|)f_2^2 \\
&+ a_{33}^2 f_3^2 + 2|a_{30}||a_{33}|f_3 + a_{30}^2.
\end{aligned}
$$

Let $\beta = \max\{a_{11}^2 + |a_{11}||a_{12}| + a_{21}^2 + |a_{21}||a_{22}|, a_{12}^2 + |a_{11}||a_{12}| + a_{22}^2 + |a_{21}||a_{22}|\}$.
Then,

$$
\begin{aligned}
I &\leq \beta(f_1^2 + f_2^2) + a_{33}^2 f_3^2 + 2|a_{30}||a_{33}||f_3| + a_{30}^2 \\
&\leq \beta(1 - f_3^2) + a_{33}^2 f_3^2 + 2|a_{30}||a_{33}||f_3| + a_{30}^2 \leq 1.
\end{aligned}
$$

Then, $f(u) = au^2 + bu + c, u \in [0,1]$, where $u = |f_3|, a = \beta - a_{33}^2, b = -2|a_{30}||a_{33}|$, and $c = 1 - \beta - a_{30}^2$. By Lemma (3.2.1), α is positive if

$$1 - \beta - a_{30}^2 \geq 0 \tag{6.21}$$

$$1 - a_{33}^2 - 2|a_{30}||a_{33}| - a_{30}^2 \geq 0 \tag{6.22}$$

and either conditions (6.23)-(6.26) or (6.27) hold.

$$\beta - a_{33}^2 > 0 \tag{6.23}$$

$$|a_{30}||a_{33}| < 0 \tag{6.24}$$

$$2|a_{30}||a_{33}| + 2(a_{33}^2 - \beta) > 0 \tag{6.25}$$

$$4a_{30}^2 a_{33}^2 + 4(a_{33}^2 - \beta)(1 - \beta - a_{30}^2) \leq 0 \tag{6.26}$$

or

$$a_{33}^2 - \beta > 0. \tag{6.27}$$

Since condition (6.27) can be satisfied, so the conditions become:

$$1 - \beta - a_{30}^2 \geq 0 \tag{6.28}$$

$$1 - a_{33}^2 - 2|a_{30}||a_{33}| - a_{30}^2 \geq 0 \tag{6.29}$$

$$a_{33}^2 - \beta > 0. \tag{6.30}$$

From these calculations, one can summarize the previous work in the following table.

Table 6.4: Conditions for the positivity of α - part (a)

Cases	α	Conditions for positivity
1.1	$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}$	$\Vert\mathbf{a}+\mathbb{T}(\mathbf{f})\Vert \leq 1, \forall \Vert\mathbf{f}\Vert \leq 1$
1.2	$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{22} & a_{23} \\ 0 & 0 & a_{32} & a_{33} \end{pmatrix}$	$\frac{1}{2}\Big(a_{22}^2+a_{23}^2+a_{32}^2+a_{33}^2$ $+\sqrt{(a_{22}^2+a_{23}^2+a_{32}^2+a_{33}^2)^2+4(a_{22}a_{33}-a_{23}a_{32})^2}\Big)$ $+\vert a_{11}\vert^2 \leq 1$
1.3	$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & a_{13} \\ 0 & 0 & a_{22} & 0 \\ 0 & a_{31} & 0 & a_{33} \end{pmatrix}$	$\frac{1}{2}\Big(a_{11}^2+a_{13}^2+a_{31}^2+a_{33}^2$ $+\sqrt{(a_{11}^2+a_{13}^2+a_{31}^2+a_{33}^2)^2+4(a_{11}a_{33}-a_{13}a_{32})^2}\Big)$ $+\vert a_{22}\vert^2 \leq 1$
1.4	$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{22} & 0 \\ 0 & 0 & 0 & a_{33} \end{pmatrix}$	$\max\{\vert a_{11}\vert, \vert a_{22}\vert, \vert a_{33}\vert\} \leq 1$
1.5	$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ 0 & 0 & 0 & a_{33} \end{pmatrix}$	$\frac{1}{2}\Big(a_{11}^2+a_{12}^2+a_{21}^2+a_{22}^2$ $+\sqrt{(a_{11}^2+a_{12}^2+a_{21}^2+a_{22}^2)^2+4(a_{11}a_{22}-a_{12}a_{21})^2}\Big)$ $+\vert a_{33}\vert^2 \leq 1$
1.6	$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ 0 & 0 & 0 & a_{33} \end{pmatrix}$	$\frac{1}{2}\Big(a_{11}^2+a_{12}^2+a_{21}^2+a_{22}^2$ $+\sqrt{(a_{11}^2+a_{12}^2+a_{21}^2+a_{22}^2)^2+4(a_{11}a_{22}-a_{12}a_{21})^2}\Big)$ $+\vert a_{33}\vert^2 \leq 1$
1.7	$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{20} & 0 & a_{22} & a_{23} \\ a_{30} & 0 & a_{32} & a_{33} \end{pmatrix}$	1. $1-\beta-a_{22}^2-a_{30}^2-\vert a_{20}\vert\vert a_{22}\vert-\vert a_{30}\vert\vert a_{32}\vert \geq 0$, 2. $1-a_{23}^2-a_{33}^2-a_{22}^2-a_{30}^2-\vert a_{22}\vert\vert a_{23}\vert-\vert a_{32}\vert\vert a_{33}\vert$ $-2\vert a_{30}\vert\vert a_{33}\vert-\vert a_{20}\vert\vert a_{22}\vert-\vert a_{30}\vert\vert a_{32}\vert \geq 0$ and one of the following conditions is satisfied. I. $\beta > a_{23}^2+a_{33}^2+\vert a_{22}\vert\vert a_{23}\vert+\vert a_{32}\vert\vert a_{33}\vert$. a) $a_{30}a_{33} < 0$ b) $\vert a_{30}\vert\vert a_{33}\vert+a_{23}^2+a_{33}^2+\vert a_{22}\vert\vert a_{23}\vert+\vert a_{32}\vert\vert a_{33}\vert > \beta$ c) $4a_{30}^2a_{33}^2+4(a_{23}^2+a_{33}^2+\vert a_{22}\vert\vert a_{23}\vert$ $+\vert a_{32}\vert\vert a_{33}\vert-\beta)(1-\beta-a_{22}^2-a_{30}^2$ $-a_{20}a_{22}-a_{30}a_{32}) \leq 0$ II. $\beta < a_{23}^2+a_{33}^2+\vert a_{22}\vert\vert a_{23}\vert+\vert a_{32}\vert\vert a_{33}\vert$.
1.8	$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{10} & a_{11} & 0 & a_{13} \\ 0 & 0 & a_{22} & 0 \\ a_{30} & a_{31} & 0 & a_{33} \end{pmatrix}$	1. $1-\beta-a_{11}^2-a_{30}^2-\vert a_{10}\vert\vert a_{11}\vert-\vert a_{30}\vert\vert a_{31}\vert \geq 0$, 2. $1-a_{13}^2-a_{33}^2-a_{11}^2-a_{30}^2-\vert a_{11}\vert\vert a_{13}\vert-\vert a_{31}\vert\vert a_{33}\vert$ $-2\vert a_{30}\vert\vert a_{33}\vert-\vert a_{10}\vert\vert a_{11}\vert-\vert a_{30}\vert\vert a_{31}\vert \geq 0$ and one of the following conditions is satisfied. I. $\beta > a_{13}^2+a_{33}^2+\vert a_{11}\vert\vert a_{13}\vert+\vert a_{31}\vert\vert a_{33}\vert$. a) $a_{30}a_{33} < 0$ b) $\vert a_{30}\vert\vert a_{33}\vert+a_{13}^2+a_{33}^2+\vert a_{11}\vert\vert a_{13}\vert+\vert a_{31}\vert\vert a_{33}\vert > \beta$ c) $4a_{30}^2a_{33}^2+4(a_{13}^2+a_{33}^2+\vert a_{11}\vert\vert a_{13}\vert$ $+\vert a_{31}\vert\vert a_{33}\vert-\beta)(1-\beta-a_{11}^2-a_{30}^2$ $-a_{10}a_{11}-a_{30}a_{31}) \leq 0$ II. $\beta < a_{13}^2+a_{33}^2+\vert a_{11}\vert\vert a_{13}\vert+\vert a_{31}\vert\vert a_{33}\vert$.

Table 6.5: Conditions for the positivity of α - part (b)

Cases	α	Conditions for positivity
1.9	$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{22} & 0 \\ a_{30} & 0 & 0 & a_{33} \end{pmatrix}$	$1-\beta-a_{30}^2 \geq 0$ $1-a_{33}^2-2\lvert a_{30}\rvert\lvert a_{33}\rvert-a_{30}^2 \geq 0$ $a_{33}^2-\beta > 0$
1.10	$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ a_{30} & 0 & 0 & a_{33} \end{pmatrix}$	$1-\beta-a_{30}^2 \geq 0$ $1-a_{33}^2-2\lvert a_{30}\rvert\lvert a_{33}\rvert-a_{30}^2 \geq 0$ $a_{33}^2-\beta > 0$

Chapter 7: Conclusions

In this thesis, a class of flow quantum Lotka-Volterra genetic algebras (FQLVG-A) is investigated. The structures of this class of FQLVG-A are presented. Also, the derivation of a class of FQLVG-A are described. In addition to the automorphisms of a class of FQLVG-A and their positivity are presented and proven.

In chapter two, the basic preliminaries which are used in this thesis are given. The space of all linear operators on n-dimensional Hilbert space $\mathbb{C}^n$ is defined. Then, every linear operator is represent as $n \times n$ matrix. Several conditions that characteristic the positive matrices are given. In addition, special types of maps are presented such as linear, positive, unital, and completely positive and their properties are given. Pauli matrices and their properties are investigated. Then, positive, trace preserving and unital operators on $M_2(\mathbb{C})$ are described. The quadratic stochastic operators are defined. At the end of this chapter, some properties of quantum quadratic stochastic operators on $M_2(\mathbb{C})$ are recalled.

In Chapter 3, symmetric commutative q.q.o.s on the commutative algebra $DM_2(\mathbb{C})$ are described and formulated. Interesting results that are equivalent to the symmetric quasi q.q.o. are proven. Moreover, a quantum analogue of Lotka-Volterra operators on $M_2(\mathbb{C})$ are defined and some properties of these operators are presented.

In chapter 4, a flow of quantum genetic Lotka-Volterra algebras are defined. Moreover, the necessary and sufficient conditions for the associativity and alternatively of FQGLV-A are derived. In addition, the idempotent elements in FQGLV-A are found.

In Chapter 5, the derivation of FQGLV-A are investigated in details . First, the definition of the derivation and its properties in $M_4(\mathbb{C})$ are discussed. Then, fourteen derivations on $M_4(\mathbb{C})$ are derived and the necessary conditions to guarantee that these are derivations are given and proven.

In Chapter 6, automorphisms of FQGLV-A are studied. Ten types of automorphisms are derived and necessary conditions are obtained. Their positivity are discussed in details. Necessary and sufficient conditions for their positivity are presented and proven.

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