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REPRODUCING KERNEL METHOD FOR SOLVING FUZZY INITIAL VALUE PROBLEMS

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This thesis is submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

Under the Supervision of Prof. Muhammed I. Syam

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Declaration of Original Work

I, Qamar Kamel Dallashi , the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled *"Reproducing Kernel Method for Solving Fuzzy Initial Value Problems"*, hereby, solemnly declare that this thesis is my own original research work that has been done and prepared by me under the supervision of Professor Muhammed I. Syam, in the College of Science at UAEU. This work has not previously been presented or published, or formed the basis for the award of any academic degree, diploma or a similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my thesis have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this thesis.

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Abstract

In this thesis, numerical solution of the fuzzy initial value problem will be investigated based on the reproducing kernel method. Problems of this type are either difficult to solve or impossible, in some cases, since they will produce a complicated optimized problem. To overcome this challenge, reproducing kernel method will be modified to solve this type of problems. Theoretical and numerical results will be presented to show the efficiency of the proposed method.

Keywords: Fuzzy initial value problems, Convergence, Reproducing kernel method.

Title and Abstract (in Arabic)

طريقة توليد المصغرلحل المسائل الضبابية ذات القيم البدائية

الملخص

في هذه الأطروحة ، سيتم دراسة طريقة عددية لحل المسائل البدائية الضبابية بالإعتماد على طريقة توليد المصغر. المسائل من هذا النوع إما صعب حلها أو غير ممكن في بعض الحالات لأنها تولد مسائل قصوى معقدة. من أجل هذة المشكلة تم عرض بعض النظريات والحلول العددية لإثبات فعالية الطريقة المقترحة.

مفاهيم البحث الرئيسية: مسائل بدائية ضبابية ، تقارب ، طريقة توليد المصغر.

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Dedication

To my precious parents and teachers

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Chapter 1: Fuzzy Logic

1.1 Introduction

The term fuzzy means things that are not very clear or vague. In real life, a situation might come across us, and it cannot be decided whether this statement is true or false. At that time, fuzzy logic offers valuable flexibility for reasoning. Also, it considers the uncertainties of any situation. There are many applications for deal with fuzzy logic in Mathematics. The fuzzy logic algorithm helps to solve a problem after considering all available data. Then, it takes the best possible decision for the given input. This fuzzy logic imitates the human decision-making process, which considers the possibilities between true and false digital values. Although the notion of fuzzy logic has been studied since the 1920s, the term fuzzy logic was first used by Dr. Lotfi Zadeh, a professor at UC Berkeley, in 1965. He noted that conventional computer logic was not capable of manipulating data representative of subjective or unclear human ideas. In the 19th century, George Boole created a system of algebra and set theory, known as fuzzy set, which could mathematically calculate two-valued logic, mapping true and false as 1 and 0, respectively. Then, in the early 20^{th} century, Jan Lukasiewicz proposed a three-valued logic: true, possible, false. However, this idea did not gain widespread acceptance. In recent years, fuzzy logic has become attractive to many researchers due to its potential application in various fields. Fuzzy logic has been applied to various fields, such as in computer science, information science, mathematics, engineering, economic, business, and finance. Fuzzy logic and fuzzy set are powerful mathematical tools in modeling entropic systems, for example in industry, nature, and the humanities.

Several researchers have studied the fuzzy boundary value problems. For example, Sanchez et al. [1] discussed the fuzzy solution for nonlinear fuzzy boundary value problem with Gaussian fuzzy numbers as boundary values. Gong [2] illustrated discontinuous fuzzy initial value problems and two kinds of fuzzy Volterra integral equations with use of the fuzzy Laplace transform. Zhou et al. [3] illustrated numerous duality outcomes for fuzzy number quadratic programming problems with fuzzy coefficients. Tapaswini et al. [4] explained a new technique based on the Galerkin method for solving n^{th} order fuzzy boundary value problem. Gumah et al [5] studied a numerical method for certain hybrid fuzzy differential equations with an application of a reproducing kernel technique for fuzzy differential equations. Patel and Desai [6] illustrated a fuzzy Laplace transform to solve fuzzy initial value problem under a strongly generalized differentiability concept. Diniz et al. [7] investigated conditions to solve a fuzzy variational problem using Zadeh's extension. Wu and Feng [8] used a mixed fuzzy boundary control problem within a class of nonlinear coupled systems explained by an Ordinary Differential Equation (ODE) and boundary-disturbed uncertain beam equation. Suhhiem [9] introduced a modified method for solving second order fuzzy differential equations. Niu et al. [10] proposed Simplified Reproducing Kernel Method (SRKM) and Least Squares Method (LSM) for solving nonlinear singular boundary value problems. Shah and Wang [11] developed a powerful method for the numerical solution of Boundary Value Problems (BVPs) of Fractional Order Differential Equations (FDEs). Pradip [12] implemented a computational technique for the efficient solution of a class of singular boundary value problems. Wasques et al. [13] illustrated a numerical solution for an n-dimensional initial-value problem where the initial conditions are given by interactive fuzzy numbers. Wasques et al. [14] explained numerical solutions for fuzzy initial value problems, where the initial conditions are given by interactive fuzzy numbers. Jeyaraj and Rajan [15] used the explicit Runge-Kutta method of order four with Butcher table to solve the fuzzy initial value problems. Al-Refai et al. [16] used the implicit hybrid block method to solve fuzzy initial value problems.

The reproducing kernel method was first presented in the 1907 work of Stanisław Zaremba doubting boundary value problems for harmonic and biharmonic functions. James Mercer simultaneously tested functions which satisfy the reproducing property in the theory of integral equations. The concept of the reproducing kernel remained not used for nearly twenty years until it came up in the dissertations of Gabor Szego, Stefan Bergman, and Salomon Bochner. The subject was eventually systematically improved in the early 1950s by Nachman Aronszajn and Stefan Bergman. These spaces have wide applications, including complex analysis, harmonic analysis, and quantum mechanics. RKM are particularly essential in the field of statistical learning theory because of the celebrated explained theorem which states that every function in reproducing kernel methods that minimize an empirical risk functional can be written as a linear combination of the kernel function studied at the training points. This is a practically useful result as it effectively simplifies the empirical risk minimization problem from an infinite dimensional to a finite dimensional optimization problem.

Several researchers study the reproducing kernel method such that, Kashkari and Syam [17] used the RKM for solving Fredholm Integrodifferential equation. Du et al. [18] developed a reproducing kernel method for solving Fredholm integro-differential equations with weakly singular kernels in reproducing kernel Hilbert space. Arqub [10] applied reproducing kernel Hilbert space for the solutions of systems of first-order, two-point boundary value problems for ordinary differential equations. Akgül [20] applied the reproducing kernel method to fractional differential equations with non-local and non-singular kernel. Akgül [21] investigated the boundary layer flow of a Powell-Eyring non-Newtonian fluid over a stretching sheet by a reproducing kernel method. The meshfree interpolation functions are derived from the RKM. A singular kernel is introduced to impose the essential boundary conditions by Sadamoto et al. [22] Gholami et al. [23] discussed the fuzzy inner product space and the fuzzy Hilbert space. Mei and Lin [24] simplified RKM to solve the linear Volterra integral equations with variable coefficients. Geng and Qian [25] devoted to the numerical treatment of a class of singularly perturbed delay boundary value problems with a left layer using Kernel RKM. Li and Wu [26] constructed and applied reproducing kernels with polynomial to solve variable order fractional functional boundary value problems. Moradi and Javadi [27] introduced a semi-analytical technique for the numerical solution of nonlinear oscillators under the damping effect by using the reproducing kernel Hilbert space method. Alvandi and Paripour [28] applied the reproducing kernel method to Volterra nonlinear integro-differential equations. Arqub et al. [29] discussed the numerical solution of Fredholm integro–differential equation in reproducing RKM. Saadeh et al. [30] implemented a relatively recent analytical technique, called Iterative Reproducing Kernel Method (IRKM), to obtain a computational solution for fuzzy two-point boundary value problem based on a generalized differentiability concept. Qi et al. [31] introduced a RKM for solving nonlocal fractional boundary value problems with uncertainty.

The purpose of this thesis is to find numerical solution of the fuzzy initial value problems of first and second order. Suggested methods which are given a high order of precision to the exact solutions will be implemented even when it is impossible to find the solution. To reach this target, the reproducing kernel method will be applied in initial value problems. Then, the suggested methods will be elaborated to solve the fuzzy type of these problems using some properties of fuzzy operations. Moreover, convergence of the suggested method will be examined. As well, various examples to represent the logic and precision of the suggested methods are demonstrated and the numerical results are compared with the existing ones in the literature.

Chapter 2: Preliminaries

2.1 Fuzzy Numbers

In this section, the definition of fuzzy number and fuzzy will be illustrated.

Definition 2.1.1. Let $M = [m_1, m_2]$ and $N = [n_1, n_2]$ be two intervals, then the addition and subtraction are defined as

$$M + N = [m_1 + n_1, m_2 + n_2],$$

and

$$M - N = [m_1 - n_2, m_2 - n_1].$$

Definition 2.1.2. A fuzzy number is a function $u : \Re \to [0,1]$ satisfying the following properties:

1. u is normal, i.e; there exists $x_0 \in \Re$ with $u(x_0) = 1$,

2. u is a convex fuzzy set, i.e; $u(\lambda x + (1 - \lambda y)) \geq \min\{u(x), u(y)\}$, $\forall x, y \in \Re, \lambda \in [0,1]$,

3. u is upper semi-continuous on $\Re$,

4. $\overline{\{x \in \Re : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

The set of all fuzzy numbers is denoted by $F_{\Re}$.

Next, the definition of the $\alpha - cut$ of a fuzzy number will be illustrated.

Definition 2.1.3. Let $\beta \in F_{\Re}$. Then the $\alpha-$cut set β_α for $\alpha \in (0,1]$ is

$$\beta_\alpha = \{x \in \Re : \beta(x) \geq \alpha\},$$

and the $0-$cut set is given by

$$\beta_0 = \overline{\{x \in \Re : \beta(x) > 0\}}.$$

Example 2.1.1. let $u : \Re \to [0,1]$ be defined by

$$u(x) = \begin{cases} 1-x, & 0 \leq x \leq 1 \\ x+1, & -1 \leq x \leq 0 \\ 0, & \text{otherwise} \end{cases}$$

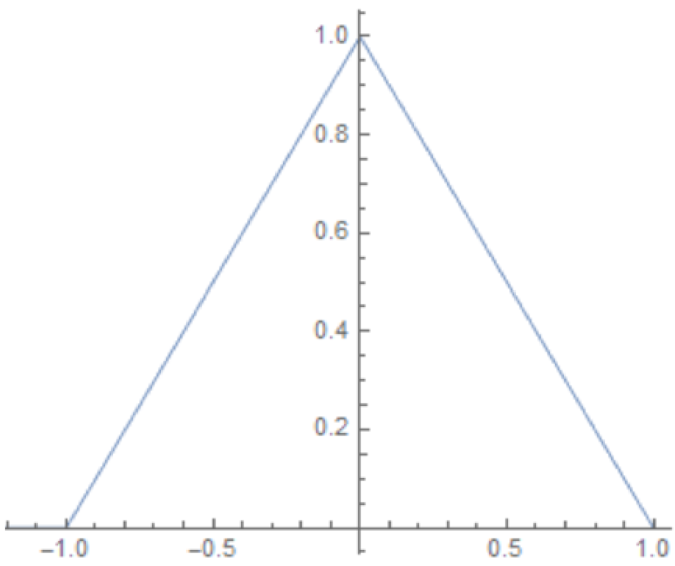

Figure 2.1: Symmetric triangle fuzzy number

For any $w, v \in \Re, \lambda \in [0,1], \lambda w + (1-\lambda)v$ is between w and v.

Then, $u(\lambda w+(1-\lambda)v)$ is between $u(w)$ and $u(v)$. Thus,

$$u(\lambda w+(1-\lambda)v) \geq \min\{u(w),u(v)\}.$$

Hence, u is fuzzy convex. For any $\alpha \in (0,1]$, $\{x \in \Re : \ u(x) \geq \alpha\} = [\alpha-1, 1-\alpha]$ is subset of $\Re$. Thus, u is upper semi-continuous on $\Re$. Finally, the closure of

$$\{x \in \Re : u(x) > 0\}$$

is $[-1,1]$ which is compact. Thus, u is a fuzzy number.

Definition 2.1.4. Let $u, v \in F_{\Re}$ with $u_\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$ and $v_\alpha = [\underline{v}_\alpha, \overline{v}_\alpha]$. Then, for $\lambda \in \Re$ and $\alpha \in [0,1]$, the arithmetic operations in fuzzy numbers are defined by $\alpha - cut$ as

1- Addition

$$(u \oplus v)_\alpha = [\underline{u}_\alpha + \underline{v}_\alpha, \overline{u}_\alpha + \overline{v}_\alpha].$$

2- Subtraction

$$(u \ominus v)_\alpha = [\underline{u}_\alpha - \overline{v}_\alpha, \overline{u}_\alpha - \underline{v}_\alpha].$$

3- Scalar multiplication

$$(\lambda \odot u)_\alpha = \begin{cases} [\lambda \underline{u}_\alpha, \lambda \overline{u}_\alpha], & \lambda \geq 0 \\ [\lambda \overline{u}_\alpha, \lambda \underline{u}_\alpha], & \lambda < 0 \end{cases}.$$

4- Multiplication

$$(u \otimes v)_\alpha = [\min\{\underline{u}_\alpha \underline{v}_\alpha, \underline{u}_\alpha \overline{v}_\alpha, \overline{u}_\alpha \underline{v}_\alpha, \overline{u}_\alpha \overline{v}_\alpha\}, \max\{\underline{u}_\alpha \underline{v}_\alpha, \underline{u}_\alpha \overline{v}_\alpha, \overline{u}_\alpha \underline{v}_\alpha, \overline{u}_\alpha \overline{v}_\alpha\}].$$

Example 2.1.2. Let $u = (0,1,2)$ and $v = (1,2,3)$. Then, $u_\alpha = [\alpha, 2-\alpha]$ and $v_\alpha = [\alpha + 1, 3-\alpha]$,

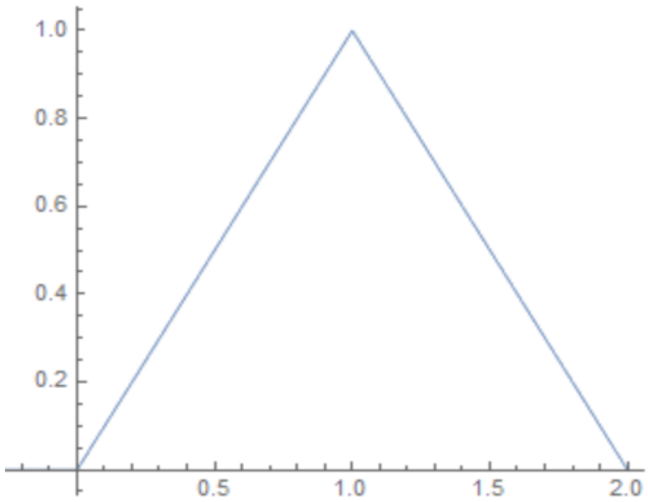

Figure 2.2: The graph of $u = (0,1,2)$

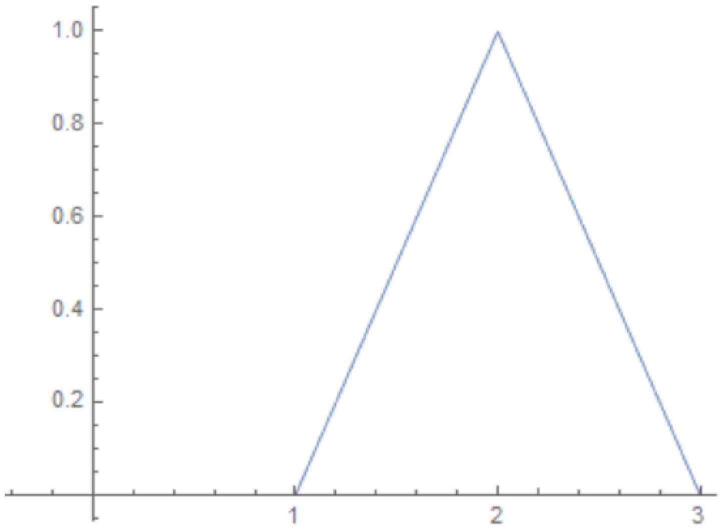

Figure 2.3: The graph of $v = (1,2,3)$

$(u \oplus v)_\alpha = [1+2\alpha, 5-2\alpha]$, $(u \ominus v)_\alpha = [-1,-1] = -1$,

$(2 \odot u)_\alpha = [2\alpha, 4-2\alpha], (-2 \odot u)_\alpha = [-4+2\alpha, -2\alpha]$.

Definition 2.1.5. Triangular fuzzy number is a fuzzy number represented with three points as follow

$$A = (a_1, a_2, a_3).$$

This representation is interpreted as membership function

$$\mu_{(A)}(x) = \begin{cases} 0, & x < a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3 \\ 0, & x > a_3 \end{cases}.$$

Definition 2.1.6. Trapezoidal fuzzy number is a fuzzy number represented with four points as follow

$$A = (a_1, a_2, a_3, a_4).$$

This representation is interpreted as membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ 1, & a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3}, & a_3 \leq x \leq a_4 \\ 0, & x > a_4 \end{cases}.$$

Example 2.1.3. The triangle fuzzy number $A = [1,2,4]$ is given by Figure 2.4.

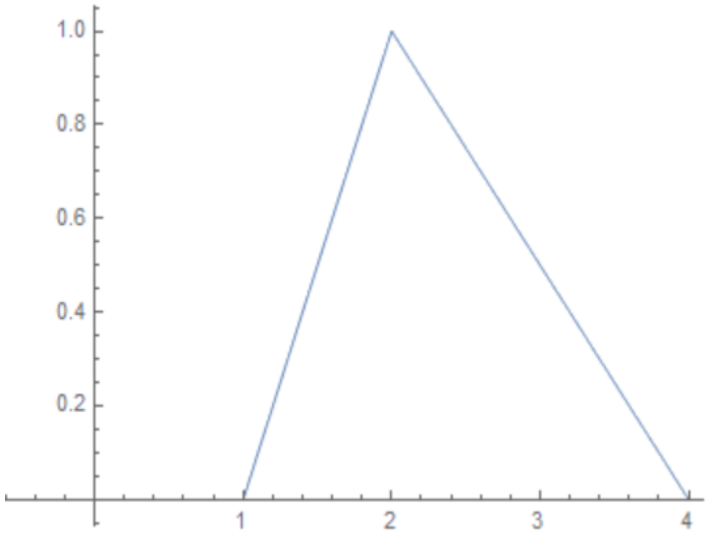

Figure 2.4: Triangle fuzzy number

Example 2.1.4. The trapezoidal fuzzy number $A = [2,3,5,8]$ is given by Figure 2.5.

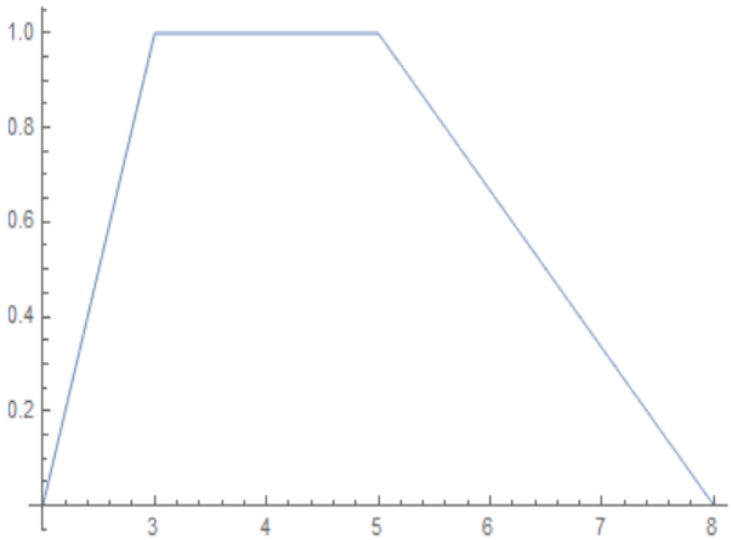

Figure 2.5: Trapezoidal fuzzy number

Definition 2.1.7. Let A and B be two subsets of $\Re$. Then, the Hausdorff metric d_H is defined by

$$d_H(A,B) = \max\left\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\right\}.$$

Then, the metric d_F on $F_{\Re}$ is defined by

$$d_F(u,v) = \sup_{\alpha\in[0,1]} \{d_H(u_\alpha, v_\alpha),\ \ u_\alpha, v_\alpha \in F_{\Re}\}$$

$$= \sup_{\alpha\in[0,1]} max\{\ |\underline{u}_\alpha - \underline{v}_\alpha|\,, |\overline{u}_\alpha - \overline{v}_\alpha|\ \}.$$

Example 2.1.5. Let $A = [-1,2]$ and $B = [0,4]$. Then,

$$d_H(A,B) = \max\left\{ \sup_{x\in A} \inf_{y\in B} \|x-y\|\,, \sup_{y\in B} \inf_{x\in A} \|x-y\| \right\} = 2.$$

Let $u = (0,1,2)$ and $v = (1,2,3)$. Then, $u_\alpha = [\alpha, 2-\alpha]$ and $v_\alpha = [1+\alpha, 3-\alpha]$. Then,

$$d_F(u,v) = \sup_{\alpha\in[0,1]} max\{\ |\alpha-(1+\alpha)|\,, |2-\alpha-(3-\alpha)|\ \}$$

$$= \sup_{\alpha\in[0,1]} max\{1,1\ \} \ \ = 1.$$

Theorem 2.1.1. *$(F_{\Re}, d_F)$ is a complete metric space with the following properties for all $u,v,w,z \in F_{\Re}, \lambda \in \Re$*

$1 - d_F(u\oplus w, v\oplus w) = d_F(u\oplus v)$,

$2 - d_F(\lambda\odot u, \lambda\odot w) = |\lambda|\, d_F(u,w)$,

$3 - d_F(u\oplus v, w\oplus z) \le d_F(u\oplus w) + d_F(v\oplus z)$.

Proof. Simple calculation implies that

$$
\begin{aligned}
d_F(u \oplus w, v \oplus w) &= \sup_{0 \le \alpha \le 1} \max \{ | \, (\underline{u}_\alpha + \underline{w}_\alpha) - (\underline{v}_\alpha + \underline{w}_\alpha) \, |, | \, (\overline{u}_\alpha + \overline{w}_\alpha) - (\overline{v}_\alpha + \overline{w}_\alpha) \, | \} \\
&= \sup_{0 \le \alpha \le 1} \max \{ |\underline{u}_\alpha - \underline{v}_\alpha| \, , |\overline{u}_\alpha - \overline{v}_\alpha| \} \\
&= d_F(u \oplus v).
\end{aligned}
$$

which proves the first part.

Next,

$$
\begin{aligned}
\{ |\lambda \underline{u}_\alpha - \lambda \underline{w}_\alpha| \, , |\lambda \overline{u}_\alpha - \lambda \overline{w}_\alpha| \} &= |\lambda|_{0 \le \alpha \le 1} \sup_{0 \le \alpha \le 1} \max \{ |\underline{u}_\alpha - \underline{w}_\alpha| \, , |\overline{u}_\alpha - \overline{w}_\alpha| \} \\
&= |\lambda| d_F(u, w)
\end{aligned}
$$

which completes the proof of the second part. Using the triangle inequality, we have

$$
\left| (\underline{u}_\alpha + \underline{v}_\alpha) - (\underline{w}_\alpha + \underline{z}_\alpha) \right| \le |\underline{u}_\alpha - \underline{w}_\alpha| + \left| \underline{v}_\alpha - \underline{z}_\alpha \right|
$$

and

$$
|(\overline{u}_\alpha + \overline{v}_\alpha) - (\overline{w}_\alpha + \overline{z}_\alpha)| \le |\overline{u}_\alpha - \overline{w}_\alpha| + |\overline{v}_\alpha - \overline{z}_\alpha|
$$

which implies that

$$
\begin{gathered}
\max\{ \left| (\underline{u}_\alpha + \underline{v}_\alpha) - (\underline{w}_\alpha + \underline{z}_\alpha) \right| , |(\overline{u}_\alpha + \overline{v}_\alpha) - (\overline{w}_\alpha + \overline{z}_\alpha)| \} \\
\le [\max\{ |\underline{u}_\alpha - \underline{w}_\alpha| \, , |\overline{u}_\alpha - \overline{w}_\alpha| \} + \max\{ \left| \underline{v}_\alpha - \underline{z}_\alpha \right| , |\overline{v}_\alpha - \overline{z}_\alpha| \}].
\end{gathered}
$$

Hence,

$$d_F(u \oplus v, w \oplus z) \leq d_F(u \oplus w) + d_F(v \oplus z).$$

which is the proof of the third part. □

Theorem 2.1.2. *1-Let*

$$\beta(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

Then, $\beta \in F_{\Re}$ is identity element with respect to $\oplus$.

2-None of $u \in F_{\Re} - \Re$ has inverse in $F_{\Re}$ with respect to $\oplus$.

3-For any $x, y \geq 0$ or $x, y \leq 0$ and any $u \in F_{\Re}$, we have $(x+y) \odot u = x \odot u \oplus y \odot u$.

The result is not true in general.

4-For any $\lambda \in \Re$ and any $u, v \in F_{\Re}$, we have $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v$.

5-For any $\lambda, \mu \in \Re$ and any $u \in F_{\Re}$, we have $\lambda \odot (\mu \odot u) = (\lambda\mu) \odot u$.

Proof. 1- Let $u \in F_{\Re}$. Then, for any $\alpha \in [0,1]$,

$$(u \oplus \beta)_\alpha = [\underline{u}_\alpha + 0, \overline{u}_\alpha + 0] = [\underline{u}_\alpha, \overline{u}_\alpha] = u_\alpha$$

and

$$(\beta \oplus u)_\alpha = [0 + \underline{u}_\alpha, 0 + \overline{u}_\alpha] = [\underline{u}_\alpha, \overline{u}_\alpha] = u_\alpha.$$

Thus, $u \oplus \beta = \beta \oplus u = u$.

2- Let $u \in F_{\Re} - \Re$ and $v \in F_{\Re}$ be such that

$$(u \oplus v)_\alpha = [\underline{u}_\alpha + \underline{v}_\alpha, \overline{u}_\alpha + \overline{v}_\alpha] = [0,0].$$

Then, $\underline{v}_\alpha = -\underline{u}_\alpha$ and $\overline{v}_\alpha = -\overline{u}_\alpha$. Since $\underline{u}_\alpha \leq \overline{u}_\alpha$ and $\underline{v}_\alpha = -\underline{u}_\alpha \leq \overline{v}_\alpha = -\overline{u}_\alpha$, $\underline{u}_\alpha = \overline{u}_\alpha$ for $\alpha \in [0,1]$. Thus, $u(x) = v(x)$ for all $x \in \Re$. Hence, $u \in \Re$ which is a contradiction.

3- For any $x, y \geq 0$ and any $u \in F_{\Re}$, we have

$$\begin{aligned} ((x+y) \odot u)_\alpha &= [(x+y)\underline{u}_\alpha, (x+y)\overline{u}_\alpha] = [x\underline{u}_\alpha, x\overline{u}_\alpha] \oplus [y\underline{u}_\alpha, y\overline{u}_\alpha] \\ &= (x \odot u \oplus y \odot u)_\alpha \end{aligned}$$

and for any $x, y \leq 0$ and any $u \in F_{\Re}$, we have

$$\begin{aligned} ((x+y) \odot u)_\alpha &= [(x+y)\overline{u}_\alpha, (x+y)\underline{u}_\alpha] = [x\overline{u}_\alpha, x\underline{u}_\alpha] \oplus [y\overline{u}_\alpha, y\underline{u}_\alpha] \\ &= (x \odot u \oplus y \odot u)_\alpha \end{aligned}$$

for any $\alpha \in [0,1]$. In general, the result is not true. Let $x = 1$ and $y = -2$.

Let $u : \Re \to [0,1]$ be defined by

$$u(x) = \left(\frac{1}{1+x^2}\right)^2.$$

Then,

$$((x+y)\odot u)_{0.5}=\left[-\sqrt{\sqrt{2}-1},\sqrt{\sqrt{2}-1}\right]$$

and

$$\begin{aligned}(x\odot u\oplus y\odot u)_{0.5}&=\left[-\sqrt{\sqrt{2}-1},\sqrt{\sqrt{2}-1}\right]+\left[-2\sqrt{\sqrt{2}-1},2\sqrt{\sqrt{2}-1}\right]\\&=\left[-3\sqrt{\sqrt{2}-1},3\sqrt{\sqrt{2}-1}\right].\end{aligned}$$

Thus, $((x+y)\odot u)_{0.5}\neq(x\odot u\oplus y\odot u)_{0.5}$.

4. For any $\lambda\geq 0$ and any $u,v\in F_{\mathfrak{R}}$, we have

$$(\lambda\odot(u\oplus v))_\alpha=[\lambda\left(\underline{u}_\alpha+\underline{v}_\alpha\right),\lambda\left(\overline{u}_\alpha+\overline{v}_\alpha\right)]=[\lambda\underline{u}_\alpha,\lambda\overline{u}_\alpha]\oplus[\lambda\underline{v}_\alpha,\lambda\overline{v}_\alpha]$$

$$=(\lambda\odot[\underline{u}_\alpha,\overline{u}_\alpha])\oplus(\lambda\odot[\underline{v}_\alpha,\overline{v}_\alpha])=(\lambda\odot u\oplus\lambda\odot v)_\alpha.$$

For any $\lambda<0$ and any $u,v\in F_{\mathfrak{R}}$, we have

$$(\lambda\odot(u\oplus v))_\alpha=[\lambda\left(\overline{u}_\alpha+\overline{v}_\alpha\right),\lambda\left(\underline{u}_\alpha+\underline{v}_\alpha\right)]=[\lambda\overline{u}_\alpha,\lambda\underline{u}_\alpha]\oplus[\lambda\overline{v}_\alpha,\lambda\underline{v}_\alpha]$$

$$=(\lambda\odot[\overline{u}_\alpha,\underline{u}_\alpha])\oplus(\lambda\odot[\overline{v}_\alpha,\underline{v}_\alpha])=(\lambda\odot u\oplus\lambda\odot v)_\alpha.$$

5. For any $\lambda, \mu \in \Re$ and any $u \in F_{\Re}$, we have

$$(\lambda \odot (\mu \odot u))_\alpha = \begin{cases} \lambda \odot [\mu \underline{u}_\alpha, \mu \overline{u}_\alpha], & \mu \geq 0 \\ \lambda \odot [\mu \overline{u}_\alpha, \mu \underline{u}_\alpha], & \mu < 0 \end{cases}$$

$$= \begin{cases} [\lambda \mu \underline{u}_\alpha, \lambda \mu \overline{u}_\alpha], & \mu \geq 0, \lambda \geq 0 \\ [\lambda \mu \overline{u}_\alpha, \lambda \mu \underline{u}_\alpha], & \mu \geq 0, \lambda < 0 \\ [\lambda \mu \overline{u}_\alpha, \lambda \mu \underline{u}_\alpha], & \mu < 0, \lambda \geq 0 \\ [\lambda \mu \underline{u}_\alpha, \lambda \mu \overline{u}_\alpha], & \mu < 0, \lambda < 0 \end{cases}$$

$$= ((\lambda \mu) \odot u)_\alpha$$

for any $\alpha \in [0, 1]$. □

2.2 Differentiation of Fuzzy Functions

In this section, the differentiation of fuzzy functions will be illustrated.

Definition 2.2.1. Let u and v be two fuzzy numbers such that there exists a fuzzy number w such that

$$w \oplus v = u.$$

Then, w is called Hukuhara difference of u and v and it denoted by $u \ominus_H v$.

Example 2.2.1. Let $u = (-2, 0, 2)$ and $v = (-1, 3, 7)$ be two triangular fuzzy numbers.

Then,

$$u = w \oplus v.$$

where $w = (-1, -3, -5)$.

Then,

$$w = u \ominus_H v.$$

It is worth mentioning that there are two important properties should be satisfied which are

1- $0 = u \ominus_H u$.

2- $((u \oplus v) \ominus_H v)_\alpha = u_\alpha$ for all $\alpha \in [0,1]$.

Definition 2.2.2. Let B be a subset of $\Re$. A fuzzy function $F : B \to F_{\Re}$ is said to be H-differentiable at $x_0 \in B$ if and only if there exist a fuzzy number $Df(x_0)$ such that the following limits (with respect to metric d_F) exist and

$$\begin{aligned} Df(x_0) &= \lim_{h \to 0^+} \frac{1}{h} \odot (f(x_0 + h) \ominus_H f(x_0)) \\ &= \lim_{h \to 0^+} \frac{1}{h} \odot (f(x_0) \ominus_H f(x_0 - h)). \end{aligned}$$

In this case , $Df(x_0)$ is called Hukuhara derivative of f at x_0. If f is H-differentiable at each $x \in B$, then f is H-differentiable on B.

Example 2.2.2. Let $f : \Re \to F_{\Re}$ be a fuzzy function defined by

$$f(x) = u \odot x$$

where u is a fuzzy number. Then,

$$(f(x+h)\ominus_H f(x))_\alpha = ((u\odot(x+h))\ominus_H (u\odot x))_\alpha$$
$$= [(x+h)\underline{u}_\alpha,(x+h)\overline{u}_\alpha]\ominus_H [x\underline{u}_\alpha, x\overline{u}_\alpha]$$
$$= [h\underline{u}_\alpha, h\overline{u}_\alpha].$$

Thus,

$$\left(\frac{1}{h}\odot(f(x+h)\ominus_H f(x))\right)_\alpha = [\underline{u}_\alpha,\overline{u}_\alpha] = u_\alpha$$

which implies that

$$\lim_{h\to 0^+}\frac{1}{h}\odot(f(x+h)\ominus_H f(x) = u.$$

Similarly, for small $h>0$, we have $x-h>0$ and

$$(f(x+h)\ominus_H f(x))_\alpha = ((u\odot x)\ominus_H (u\odot(x-h)))_\alpha$$
$$= [x\underline{u}_\alpha, x\overline{u}_\alpha]\ominus_H [(x-h)\underline{u}_\alpha,(x-h)\overline{u}_\alpha]$$
$$= [h\underline{u}_\alpha, h\overline{u}_\alpha].$$

Thus,

$$\left(\frac{1}{h}\odot(f(x)\ominus_H f(x-h))\right)_\alpha = [\underline{u}_\alpha,\overline{u}_\alpha] = u_\alpha$$

which implies that

$$\lim_{h\to 0^+}\frac{1}{h}\odot(f(x)\ominus_H f(x-h) = u.$$

Thus, $Df(x) = u$. For $x < 0$, $x+h < 0$ for small $h > 0$. Thus,

$$(f(x+h) \ominus_H f(x))_\alpha = ((u \odot (x+h)) \ominus_H (u \odot x))_\alpha$$

$$= [(x+h)\overline{u}_\alpha, (x+h)\underline{u}_\alpha] \ominus_H [x\overline{u}_\alpha, x\underline{u}_\alpha]$$

$$= [h\overline{u}_\alpha, h\underline{u}_\alpha].$$

However, $h\overline{u}_\alpha \nleq h\underline{u}_\alpha$ for $\alpha \in [0,1]$. Thus, Hukuhara difference does not exist which means $f(x)$ is not H-differentiable when $x < 0$. When $x = 0$, we have

$$(f(0) \ominus_H f(0-h))_\alpha = ((u \odot 0) \ominus_H (u \odot (-h)))_\alpha = [h\overline{u}_\alpha, h\underline{u}_\alpha].$$

Thus,

$$\left(\frac{1}{h} \odot (f(0) \ominus_H f(0-h))\right)_\alpha = [\overline{u}_\alpha, \underline{u}_\alpha]$$

which implies that

$$\lim_{h \to +} \frac{1}{h} \odot (f(0) \ominus_H f(0-h)) = [\overline{u}, \underline{u}].$$

Also

$$(f(0+h) \ominus_H f(0))_\alpha = ((u \odot h) \ominus_H (u \odot 0))_\alpha = [h\underline{u}_\alpha, h\overline{u}_\alpha].$$

Thus,

$$\left(\frac{1}{h} \odot (f(0+h) \ominus_H f(0))\right)_\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$$

which implies that

$$\lim_{h \to 0^+} \frac{1}{h} \odot (f(0) \ominus_H f(0-h)) = [\underline{u}, \overline{u}].$$

Thus, $f(x)$ is not H-differentiable at $x = 0$. Therefore, $f(x)$ is H-differentiable when $x > 0$ and $Df(x) = u$.

Example 2.2.3. Let $f:\Re\to F_{\Re}$ be a fuzzy function defined by

$$f(x)=u\odot x^2$$

where u is a fuzzy number. Then,

$$\begin{aligned}\frac{1}{h}\odot(f(x+h)\ominus_H f(x))_\alpha &= \frac{1}{h}\odot((u\odot(x+h)^2)\ominus_H(u\odot x^2))_\alpha\\ &= [(2x+h)\underline{u}_\alpha,(2x+h)\overline{u}_\alpha].\end{aligned}$$

which implies that

$$\lim_{h\to 0^+}\frac{1}{h}\odot(f(x+h)\ominus_H f(x))=\begin{cases}2x\odot u, & x\geq 0\\ DNE, & x<0\end{cases}.$$

Similarly,

$$\lim_{h\to 0^+}\frac{1}{h}\odot(f(x)\ominus_H f(x-h))=\begin{cases}2x\odot u, & x\geq 0\\ DNE, & x<0\end{cases}.$$

Then, $f'(x)=2x\odot u$ if $x\geq 0$.

Theorem 2.2.1. *Let $f(x):I\to F_{\Re}$ be a fuzzy function defined by*

$$f(x)=u\odot g(x),$$

where u is a fuzzy number and $I=(u,v)\subset\Re$. Let $g:I\to\Re_+$ be differentiable function at $x_0\in I$. If $g'(x_0)>0$, then

1. *Hukuhara differences of f exist at x_0,*

2. f is H-differentiable at x_0,

3. $f'(x_0) = u \odot g'(x_0)$.

Example 2.2.4. a) let $f : \Re_+ \to F_{\Re}$ be a fuzzy function defined by

$$f(x) = u \odot x^3.$$

where u is a fuzzy number. Then, $g(x) = x^3$. Thus, $g'(x) = 3x^2$. Hence, $g(x)$ and $g^{'}(x)$ are positive when $x > 0$. Thus f is H-differentiable on $(0, \infty)$ and

$$f'(x) = u \odot 3x^2.$$

b) Let $f : (0, \infty) \to F_{\Re}$ be a fuzzy function defined by $f(x) = u \odot \cosh x$ where u is a fuzzy number. Then, $g(x) = \cosh x$. Thus, $g'(x) = \sinh x$. Hence, $g(x)$ and $g'(x)$ are positive when $x > 0$. Thus, f is H-differentiable on $(0, \infty)$ and $f'(x) = u \odot \sinh x$

Also,

$$g(x) = \cosh x > 0,$$

$$g'(x) = \sinh x > 0,$$

$$g''(x) = \cosh x > 0,$$

$$g'''(x) = \sinh x > 0,$$

and so on, so $f(x)$ is n-times H-differentiable on $(0, \infty)$ and

$$f^{(n)}(x) = \begin{cases} u \odot \sinh x, & n \text{ is odd} \\ u \odot \cosh x & n \text{ is even} \end{cases}.$$

Remark: Its clear that theorem 2.1.1 gives the necessary condition to the H-differentiations but not sufficient condition. For example, $f(x) = u \odot 1$ is H-differentiable $f'(x) = u \odot 0$

but does not satisfy conditions Theorem 2.2.1.

Theorem 2.2.2. *Let* $f : M \to F_{\Re}$ *be a H-differentiable at* x_0 *with derivative* $f^{'}(x_0)$ *where* $M \subset \Re$ *and* $x_0 \in M$. *Then,* $f'_{\alpha}(x_0) = \left[\underline{f}^{'}(x_0), \bar{f}'(x_0)\right]$ *and* $\underline{f}(x), \overline{f}(x)$ *are differentiable at* x_0 *for all* $\alpha \in [0,1]$

Definition 2.2.3. Given two fuzzy numbers $u, v \in F_{\Re}$, the gH-difference is the fuzzy number w, if exists, such that

$$u \ominus_{gH} v = w \text{ iff either } u = v + w \text{ or } v = u - w.$$

We note that

$$(u \ominus_{gH} v)_{\alpha} = [\min\{\underline{u}_{\alpha} - \underline{v}_{\alpha}, \overline{u}_{\alpha} - \overline{v}_{\alpha}\}, \max\{\underline{u}_{\alpha} - \underline{v}_{\alpha}, \overline{u}_{\alpha} - \overline{v}_{\alpha}\}],$$

and if H-difference exists, then $u \ominus_{gH} v = u \ominus_{H} v$. Hence, $u \ominus_{gH} v = w$ exists if one of the following holds

1- $\underline{w}_{\alpha} = \underline{u}_{\alpha} - \underline{v}_{\alpha}$ and $\overline{w}_{\alpha} = \overline{u}_{\alpha} - \overline{v}_{\alpha}$ with $\underline{w}_{\alpha}$ is increasing and $\overline{w}_{\alpha}$ is decreasing with $\underline{w}_{\alpha} \leq \overline{w}_{\alpha}$ for all $\alpha \in [0,1]$.

2- $\underline{w}_{\alpha} = \overline{u}_{\alpha} - \overline{v}_{\alpha}$ and $\overline{w}_{\alpha} = \underline{u}_{\alpha} - \underline{v}_{\alpha}$ with $\underline{w}_{\alpha}$ is increasing and $\overline{w}_{\alpha}$ is decreasing with $\underline{w}_{\alpha} \leq \overline{w}_{\alpha}$ for all $\alpha \in [0,1]$.

Example 2.2.5. Let $u = (-2, 0, 2)$ and $v = (6, 8, 10)$ be two fuzzy triangles. Then,

$$\left(\underline{u \ominus_{gH} v}\right)_{\alpha} = \min\{2\alpha - 2 - (2\alpha + 6), -2\alpha + 2 - (-2\alpha + 10)\} = -8$$

and

$$\left(\overline{u \ominus_{gH} v}\right)_{\alpha} = \max\{2\alpha - 2 - (2\alpha + 6), -2\alpha + 2 - (-2\alpha + 10)\} = -8$$

Conditions (1) and (2) hold. Thus, $u \ominus_{gH} v$ exists.

b) Let $u = (0, 3, 6)$ and $v = (0, 2, 3, 4)$ be fuzzy triangle and trapezoidal. Then,

$$\left(\underline{u \ominus_{gH} v}\right)_1 = 3 - 2 = 1$$

and

$$\left(\overline{u \ominus_{gH} v}\right)_1 = 3 - 3 = 0.$$

Then, $\left(\underline{u \ominus_{gH} v}\right)_1 \nleq \left(\overline{u \ominus_{gH} v}\right)_1$. Thus, $u \ominus_{gH} v$ does not exist.

Definition 2.2.4. Let B be an interval of $\Re$. Let $x_0, x_0 + h \in B$. A fuzzy function $f : B \to F_R$ is said to be gH-differentiable at x_0 if and only if there exists a fuzzy number $f'_{gH}(x_0)$ such that the following limit (with respect to metric d_F) exists

$$f'_{gH}(x_0) = \lim_{h \to 0} \frac{1}{h} f(x_0 + h) \ominus_{gH} f(x_0).$$

In this case, $F'_{gH}(x_0)$ is called gH-derivative of f at x_0. If f is gH-differentiable at $x \in B$, then f is gH-differentiable f over B.

Example 2.2.6. a) Let $f : \Re \to F_{\Re}$ be a fuzzy function defined by

$$f(x) = u \odot x$$

where u is a fuzzy number. Then for $h > 0$,

$$(f(x+h) \ominus_{gH} f(x))_\alpha = [\min\left\{\underline{f(x+h)}_\alpha - \underline{f(x)}_\alpha, \overline{f(x+h)}_\alpha - \overline{f(x)}_\alpha\right\},$$
$$\max\left\{f(x)_\alpha - \underline{f(x)}_\alpha, \overline{f(x+h)}_\alpha - \overline{f(x)}_\alpha\right\}] = [\underline{u}_\alpha h, \overline{u}_\alpha h]$$

and

$$(f(x) \ominus_{gH} f(x-h))_\alpha = [\min\left\{\underline{f(x)}_\alpha - \underline{f(x-h)}_\alpha, \overline{f(x)}_\alpha - \overline{f(x-h)}_\alpha\right\},$$
$$\max\left\{\underline{f(x)}_\alpha - \underline{f(x-h)}_\alpha, \overline{f(x)}_\alpha - \overline{f(x-h)}_\alpha\right\}] = [\underline{u}_\alpha h, \overline{u}_\alpha h]$$

for all $\alpha \in [0,1]$. Thus,

$$f'_{gH}(x) = \lim_{h\to 0} \frac{1}{h} f(x+h) \ominus_{gH} f(x) = [\underline{u}, \overline{u}] = u.$$

Thus, f is gH-differentiable on $(-\infty, \infty)$ and $f'_{gH}(x) = u$.

b) Let $f : \Re \to F_{\Re}$ be a fuzzy function defined by

$$f(x) = u$$

where u is a fuzzy number. Then, $\underline{f}(x) = \underline{u}$ and $\overline{f}(x) = \overline{u}$ are differentiable. Then using the same argument as in part (a), f is gH-differentiable on $(-\infty, \infty)$ and $f'_{gH}(x) = 0$.

c) Let $f : \Re_+ \to F_{\Re}$ be a fuzzy function defined by

$$f(x) = u \odot x^2$$

where u is a fuzzy number. Then, $\underline{f}(x) = \underline{u}x^2$ and $\bar{f}(x) = \overline{u}x^2$ are differentiable, then using the same argument as in part (a), f is gH-differentiable on $(-\infty, \infty)$ and $f'_{gH}(x) = u \odot 2x$.

d) Let $f:(0,\infty)\to F_{\mathfrak{R}}$ be a fuzzy function defined by

$$f(x)=u\odot\sinh x$$

where u is a fuzzy number. Then, $\underline{f}(x)=\underline{u}\sinh x$ and $\overline{f}(x)=\overline{u}\sinh x$ are differentiable, then using the same argument as in part (a), f is gH-differentiable on $(-\infty,\infty)$ and $f'_{gH}(x)=u\odot\cosh x$.

Theorem 2.2.3. *If $f,g:A\to F_{\mathfrak{R}}$ are H-differentiable at $x_0\in A\subseteq\mathfrak{R}$ and $\gamma\in\mathfrak{R}$, then $f\oplus g$ and $\gamma\odot f$ are H-differentiable at x_0 and*

$$(f\oplus g)'(x_0)=f'(x_0)\oplus g'(x_0),\quad (\gamma\odot f)'(x_0)=\gamma\odot f'(x_0).$$

Also, $f\in C^n(A,F_{\mathfrak{R}})$ if $\left(f^{(i)}(x)\right)_\alpha=\left[(\underline{f}(x))^{(i)}_\alpha,(\overline{f}(x))^{(i)}_\alpha\right]$ for $i=0,1,\ldots,n$, and $\alpha\in[0,1]$.

Example 2.2.7. The fuzzy function $f(x)=u_n\odot x^n\oplus u_{n-1}\odot x^{n-1}\oplus\ldots\oplus u_1\odot x,\ n>0$. Then, f are H-differentiable on $(0,\infty)$ and

$$f'(x)=u_n\odot nx^{n-1}\oplus u_{n-1}\odot(n-1)x^{n-2}\oplus\ldots\oplus u_1\odot 1.$$

Theorem 2.2.4. *Let $u\in F_{\mathfrak{R}}$ and $g:I^n\to\mathfrak{R}_+$ and $I=(v,w)\subset\mathfrak{R}_+$ be differentiable at $x_0\in I^n$. Let $f:I^n\to F_{\mathfrak{R}}$ be defined by $f(x)=u\odot g(x)$. If $\frac{\partial g(x_0)}{\partial x_i}>0$, for $i=1,2,\ldots,n$, then the partial derivative exists at x_0 and $\frac{\partial\hat{f}(x_0)}{\partial x_i}=\hat{a}\odot\frac{\partial g(x_0)}{\partial x_i}$ for $i=1,2,\ldots,n$.*

Example 2.2.8. The fuzzy function $f(x,y)=u\odot e^{3x+4y}$. Then, $g(x,y)=e^{3x+4y}>0$.

Since

$$\frac{\partial g}{\partial x} = 3e^{3x+4y} > 0, \frac{\partial g}{\partial y} = 4e^{3x+4y} > 0, \frac{\partial^2 g}{\partial x^2} = 9e^{3x+4y} > 0,$$

$$\frac{\partial^2 g}{\partial y^2} = 16e^{3x+4y} > 0, \frac{\partial^2 g}{\partial x \partial y} = 12e^{3x+4y} > 0,$$

then

$$\frac{\partial f}{\partial x} = u \odot 3e^{3x+4y} > 0, \frac{\partial f}{\partial y} = u \odot 4e^{3x+4y} > 0, \frac{\partial^2 f}{\partial x^2} = u \odot 9e^{3x+4y} > 0,$$

$$\frac{\partial^2 f}{\partial y^2} = u \odot 16e^{3x+4y} > 0, \frac{\partial^2 f}{\partial x \partial y} = u \odot 12e^{3x+4y} > 0.$$

It is easy to see that $f \in C^{\infty}(\Re, F_{\Re})$.

Chapter 3: Fuzzy Initial Value Problems

3.1 Direct Method for Solving Fuzzy Boundary Value Problems

In this section, the concept of the direct method to solve boundary and initial value problems will be illustrated. The trickiness of using this method will be explained. To clarify the idea of this method, consider the following fuzzy differential equation of the form

$$y'' = f\left(t, y, y'\right), \quad a < t < b \tag{3.1}$$

If equation (3.1) is linear problem, then it can be written as

$$a(t)y'' + b(t)y' + c(t)y = g(t), \quad a < t < b.$$

where a, b, c, *and* g are fuzzy functions. Since the functions are fuzzy, then the linear fuzzy problem can be written in the α-cut format as

$$\begin{aligned}&\left[\underline{a}_\alpha(t), \overline{a}_\alpha(t)\right]\left[\underline{y}''(t), \overline{y}''(t)\right] + \left[\underline{b}_\alpha(t), \overline{b}_\alpha(t)\right]\left[\underline{y}'_\alpha(t), \overline{y}'_\alpha(t)\right]\\ &+ \left[\underline{c}_\alpha(t), \overline{c}_\alpha(t)\right]\left[\underline{y}_\alpha(t), \overline{y}_\alpha(t)\right] = \left[\underline{g}_\alpha(t), \overline{g}_\alpha(t)\right].\end{aligned}$$

Thus, we will get two complicated optimization problems

$$\begin{aligned}&\min\left\{\underline{a}_\alpha \underline{y''}, \underline{a}_\alpha \overline{y}'', \overline{a}_\alpha \underline{y}'', \overline{a}_\alpha \overline{y}''\right\} + \min\left\{\underline{b}_\alpha \underline{y}', \underline{b}_\alpha \overline{y}'_\alpha, \overline{\mathrm{b}}_\alpha \underline{y}'_\alpha, \overline{b}_\alpha \overline{y}'_\alpha\right\}\\ &\quad + \min\left\{\underline{c}_\alpha \underline{y}_\alpha, \underline{c}_\alpha \overline{y}_\alpha, \overline{c}_\alpha \underline{y}_\alpha, \overline{c}_\alpha \overline{y}_\alpha\right\} = \underline{g}_\alpha(t).\end{aligned}$$

and

$$\begin{aligned}&\max\left\{\underline{a}_\alpha \underline{y''}, \underline{a}_\alpha \overline{y}'', \overline{a}_\alpha \underline{y}'', \overline{a}_\alpha \overline{y}''\right\} + \max\left\{\underline{b}_\alpha \underline{y}', \underline{b}_\alpha \overline{y}'_\alpha, \overline{b}_\alpha \underline{y}'_\alpha, \overline{b}_\alpha \overline{y}'_\alpha\right\}\\ &\quad + \max\left\{\underline{c}_\alpha \underline{y}_\alpha, \underline{c}_\alpha \overline{y}_\alpha, \overline{c}_\alpha \underline{y}_\alpha, \overline{c}_\alpha \overline{y}_\alpha\right\} = \overline{g}_\alpha(t).\end{aligned}$$

The above min-max problems are difficult to solve and sometimes not possible. To explain this idea, let's consider the following simple examples.

Example 3.1.1. Consider the following second order fuzzy initial value problem

$$y'' = -y(x)\ , \qquad 0 < x < 1.$$

with

$$y(0) = 0, \qquad y'(0) = [0.9 + 0.1\beta, 1.1 - 0.1\beta].$$

Let

$$y = [y_1, y_2], \qquad y(0) = [0, 0],$$

$$y'' = [\,y_1'', y_2''], \qquad y'(0) = [0.9 + 0.1\beta, 1.1 - 0.1\beta].$$

Then,

$$[\,y_1'', y_2''] \quad = [-y_2, -y_1],$$

which gives

$$y_1'' = -y_2\ , \qquad y_1(0) = 0\ , \qquad y_1'(0) = \ 0.9 + 0.1\beta,$$

$$y_2'' = -y_1\ , \qquad y_2(0) = 0\ , \qquad y_2'(0) = 1.1 - 0.1\beta.$$

Since

$$y_1'' = -y_2,$$

then,

$$y_1'''' = -y_2'' = -\,(-y_1) = y_1$$

which leads to

$$y_1'''' - y_1 = 0.$$

Let $y = e^{rx}$. Then, $e^{rx}(r^4 - 1) = 0$ with roots $r_1 = 1$, $r_2 = -1$, $r_3 = i$, $r_4 = -i$.

Then,

$$y_1(x) = C_1 \cosh x + C_2 \sinh x + C_3 \cos x + C_4 \sin x.$$

Thus,

$$y_2(x) = -y_1'' = -C_1 \cosh x - C_2 \sinh x + C_3 \cos x + C_4 \sin x.$$

After substituting the initial conditions, the solution for this problem will be

$$y(x) = [(-0.1 + 0.1\beta) \sinh x + \sin x \, , (-0.1 + 0.1\beta) \sinh x + \sin x].$$

Example 3.1.2. Consider the following fuzzy boundary value problem

$$y'' = y(x), \qquad 0 < x < 1,$$

with boundary conditions

$$y(0) = [0.75 + 0.25\beta, 1.125 - 0.125\beta],$$

and

$$y(1) = [(0.75 + 0.25\beta)\cosh 1 + (1.125 - 0.125\beta)\sinh 1,$$

$$(1.125 - 0.125\beta)cosh1 + (1.125 - 0.125\beta)\sinh 1].$$

Let

$$y = [y_1, y_2]$$

and

$$y'' = [y_1'', y_2''].$$

Then,

$$y_1'' = y_1, \quad y_2'' = y_2.$$

Hence, the solution of

$$y_1'' - y_1 = 0$$

is

$$y_1(x) = C_1 \cosh x + C_2 \sinh x.$$

Since

$$y_1(0) = C_1 = 0.75 + 0.25\beta,$$

and

$$y_1(1) = (0.75 + 0.25\beta)\cosh 1 + (1.25 - 0.125\beta)\sinh 1,$$

$$= C_1 \cosh 1 + C_2 \sinh 1.$$

Then,

$$y_1(x) = (0.75 + 0.25\beta)\cosh x + (1.25 - 0.125\beta)\sinh x.$$

Similarly the solution of

$$y_2'' = y_2$$

with

$$y_2(0) = 1.125 - 0.125\beta$$

and

$$y_2(1) = (1.125 - 0.125\beta)\cosh 1 + (1.125 - 0.125\beta)\sinh 1$$

is

$$y_2(x) = (1.125 - 0.125\beta)\cosh x + (1.125 - 0.125\beta)\sinh x.$$

Therefore, the general solution is

$$y(x) = [(0.75 + 0.25\beta)\cosh x + (1.125 - 0.125\beta)\sinh x,$$

$$(1.125 - 0.125\beta)\cosh x + (1.125 - 0.125\beta)\sinh x].$$

Example 3.1.3. Consider the following first order initial value problem

$$y_1' = -2y_2 + \alpha, \qquad y_1(0) = 0$$

$$y_2' = -2y_1 + \alpha + 2, \qquad y_2(0) = 2\alpha.$$

Then,

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \alpha \\ \alpha+2 \end{bmatrix}, \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 2\alpha \end{bmatrix}.$$

Thus,

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -2 \\ -2 & -\lambda \end{bmatrix} = \lambda^2 - 4$$

which implies that

$$\lambda_1, \lambda_2 = \pm 2.$$

For $\lambda_1 = -2,$ the augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ -2 & 2 & 0 \end{array}\right].$$

Multiply the first row by $\left(\frac{1}{2}\right)$ to get

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -2 & 2 & 0 \end{array}\right].$$

Multiply first row by 2 then add the result to the second row to get

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus,

$$x_1 - x_2 = 0 \quad \text{or} \quad x_1 = x_2.$$

Hence, the corresponding eigenvector is

$$V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 2,$ using similar method to find the second eigenvector

$$V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Therefore, the homogenous solution is

$$y_h = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_1 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Let the particular solution be $y_p = B$. Then,

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} B + \begin{bmatrix} \alpha \\ \alpha+2 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} B = \begin{bmatrix} -\alpha \\ -\alpha-2 \end{bmatrix}.$$

Hence,

$$B = -\frac{1}{4}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} -\alpha \\ -\alpha-2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} -\alpha \\ -\alpha-2 \end{bmatrix} = \begin{bmatrix} \frac{\alpha+2}{2} \\ \frac{\alpha}{2} \end{bmatrix}.$$

Therefore, the general solution is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_g = C_1 e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{\alpha+2}{2} \\ \frac{\alpha}{2} \end{bmatrix}.$$

Then,

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 2\alpha \end{bmatrix} = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{\alpha+2}{2} \\ \frac{\alpha}{2} \end{bmatrix}.$$

which implies that

$$\begin{bmatrix} C_1 - C_2 \\ C_1 + C_2 \end{bmatrix} = \begin{bmatrix} -\frac{\alpha+2}{2} \\ 2\alpha - \frac{\alpha}{2} \end{bmatrix} = \begin{bmatrix} -\frac{\alpha+2}{2} \\ \frac{3\alpha}{2} \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -\frac{\alpha+2}{2} \\ \frac{3\alpha}{2} \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{\alpha+2}{2} \\ \frac{3\alpha}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{3\alpha-2}{2} \\ \frac{4\alpha+2}{2} \end{bmatrix}.$$

which gives that

$$C_1 = \frac{\alpha - 1}{2} \text{ and } C_2 = \alpha + \frac{1}{2}.$$

Then,

$$y_1(t) = \left(\frac{\alpha - 1}{2}\right) e^{-2t} - \left(\alpha + \frac{1}{2}\right) e^{2t} + \left(\frac{\alpha + 2}{2}\right)$$

and

$$y_2(t) = \left(\frac{\alpha - 1}{2}\right) e^{-2t} - \left(\alpha + \frac{1}{2}\right) e^{2t} + \left(\frac{\alpha}{2}\right).$$

It is clear that the previous fuzzy initial value problems are difficult to solve.

Example 3.1.4. Consider the following fuzzy initial value problem

$$y'' = -(y')^2, \quad 0 < x < 1,$$

with

$$y(0) = [\beta, 2-\beta],$$

and

$$y'(0) = [1+\beta, 3-\beta].$$

Let

$$y(x) = [y_1, y_2],$$

$$y'(x) = [y_1', y_2'],$$

and

$$y''(x) = [y_1'', y_2''].$$

Then,

$$[y_1'', y_2''] = [-\max\{(y_1')^2, y_1'y_2', (y_2')^2\}, -\min\{(y_1')^2, y_1'y_2', (y_2')^2\}].$$

Hence,

$$y_1'' = -\max\{(y_1')^2, y_1'y_2', (y_2')^2\},$$

with

$$y_1(0) = \beta, \qquad y_1'(0) = 1+\beta,$$

and

$$y_2'' = -\min\{(y_1')^2, y_1'y_2', (y_2')^2\},$$

with

$$y_2(0) = 2-\beta, \qquad y_2'(0) = 3-\beta.$$

If the same technique was implemented to solve the nonlinear case, this will result complicated optimization problem which are not possible to solve explicitly in most of the cases. For this reason, reproducing kernel method will modified to solve such problems.

Chapter 4: Reproducing Kernel Method for Solving Initial Value Problems

4.1 Reproducing Kernel Spaces

In this section, Reproducing Kernel Space (RKS) will be defined and explained by several examples.

Definition 4.1.1. Let $F \neq \emptyset$. A function $K : F \times F \to \mathbb{C}$ is called a reproducing kernel function of the Hilbert space H if and only if

a) $K(\cdot,t) \in H$ for all $t \in F$.

b) $\langle \varphi, K(\cdot,t) \rangle = \varphi(t)$ For all $t \in F$ and all $\varphi \in H$.

The last condition is called " the reproducing property" as the value of the function φ at the point t is reproduced by the inner product of φ with the kernel $K(\cdot,t)$. A Hilbert space which possesses a reproducing kernel is called a reproducing kernel Hilbert space.

Definition 4.1.2. Let $W_2^1[0,1] = \{u : u$ is absolutely continues real valued function on $[0,1], u^{'} \in L^2[0,1]\}$.

The inner product in $W_2^1[0,1]$ is defined as

$$(u(x),v(x))_{W_2^1[0,1]} = u(0)v(0) + \int_0^1 u'(x)v'(x)dx, \tag{4.1}$$

and it's norm is defined as

$$\|u\|_{W_2^1[0,1]} = \sqrt{(u(x),u(x))_{W_2^1[0,1]}} \tag{4.2}$$

where $u, v \in W_2^1[0,1]$.

Theorem 4.1.1. *The Hilbert space $W_2^1[0,1]$ is a reproducing kernel and it's reproducing kernel function $R_y(x)$ can be defined by*

$$R_y(x) = \begin{cases} 1+x, & x \leqslant y \\ 1+y, & x > y \end{cases}.$$

Proof. Let

$$u(y) = (u(x), R_y(x)) = u(0)R_y(0) + \int_0^1 u'(x)R_y'(x)dx. \tag{4.3}$$

Using integration by parts, we get

$$u(y) = u(0)R_y(0) + R_y'(1)u(1) - R_y'(0)u(0) - \int_0^1 u(x)R_y^{(2)}(x)dx.$$

Therefore,

$$R_y(0) - R_y'(0) = 0, \tag{4.4}$$

$$R_y'(1) = 0. \tag{4.5}$$

Thus,

$$u(y) = \langle u(x), R_y(x) \rangle = -\int_0^1 u(x)R_y^{(2)}(x)dx.$$

Hence,

$$-R_y^{(2)}(x) = \delta(x-y) = \begin{cases} 1, \text{ if } x = y \\ 0, \text{ if } x \neq y \end{cases}.$$

Hence,

$$R_y(x) = \begin{cases} C_1(y) + C_2(y)x, & y \geq x \\ d_1(y) + d_2(y)x, & y < x \end{cases}.$$

Since $-R_y^{(2)}(x) = \delta(x-y)$, then

$$R_y(y^+) = R_y(y^-), \tag{4.6}$$

and

$$\frac{\partial R_y(y^+)}{\partial y} - \frac{\partial R_y(y^-)}{\partial y} = -1 \tag{4.7}$$

For simplicity, let $C_i(y) = C_i$ and $d_i(y) = d_i$ for $i = 1,2$. Solve system (4.4) - (4.7) to get

$$C_1 - C_2 = 0,$$

$$C_1 + C_2 y = d_1 + d_2 y,$$

$$d_2 - C_2 = -1,$$

$$d_2 = 0.$$

One can get

$$C_1(y) = 1, \;\; C_2(y) = 1, \;\; d_1(y) = 1+y, \;\; d_2(y) = 0.$$

Then,

$$R_y(x) = \begin{cases} 1+x, & y \geqslant x \\ 1+y, & x > y \end{cases}.$$

□

Definition 4.1.3. Let $W_2^2[0,1] = \{u : u, u'$ are absolutely continuous real valued function on $[0,1], u^{(2)} \in L^2[0,1], u(0) = 0\}$. Define the inner product by

$$< u(x), v(x) >_{W_2^2[0,1]} = u(0)v(0) + u'(0)v'(0) + \int_0^1 u^{(2)}(x)v^{(2)}(x)dx.$$

and it's norm is defined as

$$\|u\|_{W_2^2[0,1]} = \sqrt{(u(x), u(x))_{W_2^2[0,1]}} \tag{4.8}$$

where $u, v \in W_2^2[0,1]$.

Theorem 4.1.2. *The Hilbert space $W_2^2[0,1]$ is a reproducing kernel space and it's reproducing kernel function $R_y(x)$ can be defined by*

$$R_y(x) = \begin{cases} yx + \frac{y}{2}x^2 - \frac{1}{6}x^3, & x \leq y \\ -\frac{y^3}{6} + (y + \frac{y^2}{2})x, & x > y \end{cases}.$$

Proof. Let $R_y(x)$ be a reproducing kernel function. Then,

$$u(y) = \langle u(x), R_y(x) \rangle = u(0)R_y(0) + u'(0)R'_y(0) + \int_0^1 u^{(2)}(x)R_y^{(2)}(x)dx.$$

Using integration by parts two times, we get

$$\langle u(x), R_y(x) \rangle = u(0)R_y(0) + u'(0)R'_y(0) + u'(1)R_y^{(2)}(1) - u'(0)R_y^{(2)}(0)$$
$$-u(1)R_y^{(3)}(1) + u(0)R_y^{(3)}(0) + \int_0^1 u(x)R_y^{(4)}(x)dx.$$

Substitute the condition $u(0) = 0$. Then,

$$\langle u(x), R_y(x) \rangle = u'(0)R'_y(0) + u'(1)R_y^{(2)}(1) - u'(0)R_y^{(2)}(0) - u(1)R_y^{(3)}(1) + \int_0^1 u(x)R_y^{(4)}(x)dx.$$

Thus, the following conditions are obtained

$$R'_y(0) - R_y^{(2)}(0) = 0, \tag{4.9}$$

$$R_y^{(3)}(1) = 0, \tag{4.10}$$

$$R_y^{(2)}(1) = 0. \tag{4.11}$$

Hence,

$$u(y) = \int_0^1 R_y^{(4)}(x)u(x)dx$$

which implies that

$$R_y^{(4)}(x) = \delta(x-y), \quad \delta(x-y) = \begin{cases} 1, \text{ if } x = y \\ 0, \text{ if } x \neq y \end{cases}.$$

Hence, $R_y(x)$ is a polynomial of degree 3 when $x < y$ and polynomial of degree 3 when $x > y$. Thus,

$$R_y(x) = \begin{cases} \sum_{i=1}^4 C_i(y)x^{i-1}, & x \leqslant y \\ \sum_{i=1}^4 d_i(y)x^{i-1}, & x > y \end{cases}.$$

Since

$$R_y^{(4)}(x) = \delta(x-y),$$

then

$$\frac{\partial^k R_y(y^+)}{\partial y^k} = \frac{\partial^k R_y(y^-)}{\partial y^k}, \quad \text{for } k = 0,1,2 \tag{4.12}$$

and

$$\frac{\partial^3 R_y(y^+)}{\partial^3 y} - \frac{\partial^3 R_y(y^-)}{\partial^3 y} = 1. \tag{4.13}$$

Since $R_y(x) \in W_2^2[0,1]$,

$$R_y(0) = 0. \tag{4.14}$$

Thus,

$$R_y(x) = \begin{cases} C_1 + C_2 x + C_3 x^2 + C_4 x^3, & x \le y \\ d_1 + d_2 x + d_3 x^2 + d_4 x^3, & x > y \end{cases},$$

where

$$\begin{aligned}
&C_2 - 2C_3 = 0, \\
&6d_4 = 0, \\
&2d_3 + 6d_4 = 0, \\
&C_1 + C_2 y + C_3 y^2 + C_4 y^3 = d_1 + d_2 y + d_3 y^2 + d_4 y^3, \\
&C_2 + 2C_3 y + 3C_4 y^2 = d_2 + 2d_3 y + 3d_4 y^2, \\
&2C_3 + 6C_4 y = 2d_3 + 6d_4 y, \\
&6d_4 - 6C_4 = 1, \\
&C_1 = 0.
\end{aligned}$$

One can get

$$\begin{aligned}
&C_1 = 0, \quad C_2 = y, \quad C_3 = \frac{y}{2}, \quad C_4 = -\frac{1}{6}, \\
&d_1 = -\frac{y^3}{6}, \quad d_2 = y + \frac{y^2}{2}, \quad d_3 = 0, \quad d_4 = 0.
\end{aligned}$$

Therefore,

$$R_y(x) = \begin{cases} yx + \frac{y}{2}x^2 - \frac{1}{6}x^3, & x \le y \\ -\frac{y^3}{6} + (y + \frac{y^2}{2})x, & x > y \end{cases}.$$

□

Definition 4.1.4. Let $W_2^3[0,1] = \{u : u, u', u^{(2)}$ are absolutely continuous real valued functions on $[0,1], u^{(3)} \in L^2[0,1], u(0) = u'(0) = 0\}$ with inner product

$$\langle u(x), v(x)\rangle = u(0)v(0) + u'(0)v'(0) + u(1)v(1) + \int_0^1 u^{(3)}(x)v^{(3)}(x)dx, \;\; u, v \in W_2^3[0,1]$$

and the norm

$$\|u\|_{W_2^3} = \sqrt{\langle u, u\rangle_{W_2^3}}.$$

Theorem 4.1.3. *The Hilbert space $W_2^3[0,1]$ is a reproducing kernel space and it's reproducing kernel function $R_y(x)$ can be defined by*

$$R_y(x) = \begin{cases} -\frac{x^2}{y^2}, & x \le y \\ \frac{1}{120}y^5 - \frac{1}{24}xy^4 + \frac{y^5 - 12}{12y^2}x^2 - \frac{1}{12}x^3y^2 & \\ +\frac{1}{24}x^4y^2 - \frac{1}{120}x^5y^2, & x > y \end{cases}.$$

Proof. Let

$$u(y) = \langle u(x), R_y(x)\rangle = u(0)R_y(0) + u'(0)R'_y(0) + u(1)R_y(1) + \int_0^1 u^{(3)}(x)R_y^{(3)}(x)dx.$$

Integrating by parts three times to get

$$\langle u, R_y \rangle = u(0)R_y(0) + u'(0)R_y'(0) + u(1)R_y(1) + u^{(2)}(1)R_y^{(3)}(1) - u^{(2)}(0)R_y^{(3)}(0)$$
$$-u'(1)R_y^{(4)}(1) + u'(0)R_y^{(4)}(0) + u(1)R_y^{(5)}(1) - u(0)R_y^{(5)}(0) - \int_0^1 u(x)R_y^{(6)}(x)dx.$$

Substitute the conditions $u(0) = u'(0) = 0$, to get

$$u(y) = u(1)R_y(1) + u^{(2)}(1)R_y^{(3)}(1) - u^{(2)}(0)R_y^{(3)}(0) - u'(1)R_y^{(4)}(1)$$
$$+u(1)R_y^{(5)}(1) - \int_0^1 u(x)R_y^{(6)}(x)dx.$$

Let

$$R_y(1) - R_y^{(5)}(1) = 0, \tag{4.15}$$

$$R_y^{(3)}(1) = 0, \tag{4.16}$$

$$R_y^{(3)}(0) = 0, \tag{4.17}$$

$$R_y^{(4)}(1) = 0. \tag{4.18}$$

Thus, under these conditions, we get

$$u(y) = -\int_0^1 u(x)R_y^{(6)}(x)dx. \tag{4.19}$$

This implies that

$$-R_y^{(6)}(x) = \delta(x-y)$$

where

$$\delta(x-y)=\begin{cases} 1 \text{ if } x=y \\ 0 \text{ if } \ x\neq y \end{cases}.$$

Thus,

$$R_y(x)=\begin{cases} \sum_{i=1}^{6} C_1(y)x^{i-1} & x\leqslant y \\ \sum_{i=1}^{6} d_i(y)x^{i-1} & x>y \end{cases}.$$

Since

$$-R_y^{(6)}(x)=\delta(x-y),$$

then

$$\frac{\partial^k R_y(y^+)}{\partial y^k}=\frac{\partial^K R_y(y^-)}{\partial y^k}, \quad \text{for} \quad k=0,1,2,3,4 \tag{4.20}$$

and

$$\frac{\partial^5 R_y(y^+)}{\partial y^5}-\frac{\partial^5 R_y(y^-)}{\partial y^5}=-1. \tag{4.21}$$

Since $R_y\in W_2^3[0,1]$, then

$$R_y(0)=0, \tag{4.22}$$

and

$$R_y'(0)=0. \tag{4.23}$$

Then, we get the following system

$$6C_4 = 0,$$

$$4!d_5 + 5!d_6 = 0,$$

$$6d_4 + 24d_5y + 60d_6y^2 = 0,$$

$$d_1 + d_2y + d_3y^2 + d_4y^3 + d_5y^4 + d_6y^5 - 5!d_6 = 0,$$

$$C_1 + C_2y + C_3y^2 + C_4y^3 + C_5y^4 + C_6y^5 = d_1 + d_2y + d_3y^2 + d_4y^3 + d_5y^4 + d_6y^5,$$

$$C_2 + 2C_3y + 3C_4y^2 + 4C_5y^3 + 5C_6y^4 = d_2 + 2d_3y + 3d_4y^2 + 4d_5y^3 + 5d_6y^4,$$

$$2C_3 + 6C_4y_+ 12C_5y^2 + 20C_6y^3 = 2d_3 + 6d_4y + 12d_5y^2 + 20d_6y^3,$$

$$6C_4 + 24C_5y + 60C_6y^2 = 6d_4 + 24d_5y + 60d_6y^2,$$

$$24C_5 + 120C_6y = 24d_5 + 120d_6y,$$

$$120d_6 - 120C_6 = -1,$$

$$C_1 = 0,$$

$$C_2 = 0.$$

It follows

$$C_1(y) = 0,\ \ C_2(y) = 0,\ \ C_3(y) = \tfrac{-1}{y^2},\ \ C_4(y) = 0,$$

$$C_5(y) = 0,\ \ C_6(y) = 0,\ \ d_1(y) = \tfrac{1}{120}y^5,$$

$$d_2(y) = -\tfrac{1}{24}y^4,\ \ d_3(y) = \tfrac{y^5-12}{12y^2},\ \ d_4(y) = -\tfrac{1}{12}y^2,$$

$$d_5(y) = \tfrac{1}{24}y^2,\ \ d_6(y) = -\tfrac{1}{120}y^2.$$

Then,

$$R_y(x) = \begin{cases} -\frac{x^2}{y^2}, & x \leq y \\ \frac{1}{120}y^5 - \frac{1}{24}xy^4 + \frac{y^5-12}{12y^2}x^2 - \frac{1}{12}x^3y^2 & \\ +\frac{1}{24}x^4y^2 - \frac{1}{120}x^5y^2, & x > y \end{cases}.$$

□

Definition 4.1.5. Let $W_2^4[0,1] = \{u : u', u'', u'''$ are absolutely continues real valued functions, on $u^{(4)} \in L^2[0,1], u(0) = u'(0) = u(1) = 0\}$. with inner product in $W_2^4[0,1]$ is defined as

$$\begin{aligned}(u(x), v(x))_{W_2^4[0,1]} = \; & u(0)v(0) + u'(0)v'(0) + u^{(2)}(0)v^{(2)}(0) + u^{(3)}(0)v^{(3)}(0) \\ & + \int_0^1 u^{(4)}(x)v^{(4)}(x)dx\end{aligned}$$

and the norm

$$\|u\|_{W_2^4[0,1]} = \sqrt{(u(x), u(x))_{W_2^4[0,1]}}.$$

Theorem 4.1.4. *The Hilbert space $W_2^4[0,1]$ is a reproducing kernel space and reproducing kernel function $R_y(x)$ can be defined by*

$$R_y(x) = \begin{cases} \left(\frac{2y^2}{71} - \frac{7y^3}{284} - \frac{7y^4}{1136} + \frac{21y^5}{5680} - \frac{7y^6}{5680} + \frac{y^7}{5680}\right)x^2 \\ + \left(-\frac{7y^2}{284} + \frac{16y^3}{639} - \frac{7y^4}{10224} + \frac{7y^5}{17040} - \frac{7y^6}{51120} + \frac{y^7}{51120}\right)x^3 \\ + \left(-\frac{7y^2}{1136} + \frac{4y^3}{639} - \frac{7y^4}{40896} + \frac{7Y^5}{68160} - \frac{7y^6}{204480} + \frac{y^7}{204480}\right)x^4 \\ + \left(-\frac{y^2}{2130} + \frac{7y^3}{17040} + \frac{7y^4}{68160} - \frac{7y^5}{113600} + \frac{7y^6}{340800} - \frac{y^7}{340800}\right)x^5 \\ + \left(\frac{y}{720} + \frac{-1260y^2 - 140y^3 - 35y^4 + 21y^5 - 7y^6 + y^7}{1022400}\right)x^6 \\ + \left(-\frac{1}{5040} + \frac{1260y^2 + 140y^3 + 35y^4 - 21y^5 + 7y^6 - y^7}{7156800}\right)x^7 & , \quad y \le x \\ -\frac{y^7}{5040} + \left(\frac{y^6}{7200}\right)x + \left(\frac{2y^2}{71} - \frac{7y^3}{284} - \frac{7y^4}{1136} - \frac{y^5}{2130} - \frac{7y^6}{5680} + \frac{y^7}{5680}\right)x^2 \\ + \left(-\frac{7y^2}{284} + \frac{16y^3}{639} + \frac{4y^4}{639} + \frac{7y^5}{17040} - \frac{7y^6}{51120} + \frac{y^7}{51120}\right)x^3 \\ + \left(\frac{-1260y^2 - 140y^3 - 35y^4 + 21y^5 - 7y^6 + y^7}{204480}\right)x^4 \\ + \left(\frac{1260y^2 + 140y^3 + 35y^4 - 21y^5 + 7y^6 - y^7}{340800}\right)x^5 \\ + \left(\frac{-1260y^2 - 140y^3 - 35y^4 + 21y^5 - 7y^6 + y^7}{1022400}\right)x^6 \\ + \left(\frac{1260y^2 + 140y^3 + 35y^4 - 21y^5 + 7y^6 - y^7}{7156800}\right)x^7 & , \quad y > x \end{cases}.$$

Proof. Let

$$u(y) = \langle u(x), R_y(x) \rangle = u(0)R_y(0) + u'(0)R_y^{'}(0) + u^{(2)}(0)R_y^{(2)}(0)$$
$$+u^{(3)}(0)R_y^{(3)}(0) + \int_0^1 u^{(4)}(x)R_y^{(4)}(x)dx.$$

Integration by parts four times to get

$$\langle u(x), R_y(x) \rangle = u(0)R_y(0) + u'(0)R_y^{'}(0) + u^{(2)}(0)R_y^{(2)}(0) + u^{(3)}(0)R_y^{(3)} + u^{(3)}(1)R_y^{(4)}(1)$$
$$-u^{(3)}(0)R_y^{(4)}(0) - u^{(2)}(1)R_y^{(5)}(1) + u^{(2)}(0)R_y^{(5)}(0) + u'(1)R_y^{(6)}(1)$$
$$-u'(0)R_y^{(6)}(0) - u(1)R_y^{(7)}(1) + u(0)R_y^{(7)}(0)\int_0^1 u(x)R_y^{(8)}(x)dx.$$

Substitute the conditions $u(0) = u'(0) = u(1) = 0$, to get

$$u(y) = u^{(2)}(0)R_y^{(2)}(0) + u^{(2)}(0)R_y^{(2)}(0) + u^{(3)}(1)R_y^{(4)}(1) - u^{(3)}(0)R_y^{(4)}(0) - u^{(2)}(1)R_y^{(5)}(1)$$
$$+u^{(2)}(0)R_y^{(5)}(0) + u'(1)R_y^{(6)}(1) + \int_0^1 u(x)R_y^{(8)}(x)dx.$$

Let

$$R_y^{(3)}(0) - R_y^{(4)}(0) = 0, \tag{4.24}$$

$$R_y^{(2)}(0) + R_y^{(5)}(0) = 0, \tag{4.25}$$

$$R_y^{(4)}(1) = 0, \tag{4.26}$$

$$R_y^{(5)}(1) = 0, \tag{4.27}$$

$$R_y^{(6)}(1) = 0. \tag{4.28}$$

Thus, under the conditions, we get

$$u(y) = \int_0^1 R_y^{(8)}(x)u(x)dx.$$

This implies that

$$R_y^{(8)}(x) = \delta(x-y)$$

where

$$\delta(x-y) = \begin{cases} 1, \text{ if } x = y \\ 0, \text{ if } \ x \neq y \end{cases}.$$

Thus,

$$R_y(x) \begin{cases} \sum_{i=1}^{8} c_i(y)x^{i-1}, & x \leqslant y \\ \sum_{i=1}^{8} d_i(y)x^{i-1}, & x > y \end{cases}.$$

Since

$$R_y^{(8)}(x) = \delta(x-y),$$

then

$$\frac{\partial^k R_y(y^+)}{\partial y^k} = \frac{\partial^K R_y(y^-)}{\partial y^k} \quad \text{for} \quad k = 0,1,2,3,4,5,6 \tag{4.29}$$

and

$$\frac{\partial^{(7)} R_y(y^+)}{\partial y^7} - \frac{\partial^{(7)} R_y(y^-)}{\partial y^7} = 1. \tag{4.30}$$

Since $R_y(x) \in W_2^4[0,1]$, then

$$R_y(0) = 0, \tag{4.31}$$

$$R'_y(0) = 0, \tag{4.32}$$

and

$$R_y(1) = 0. \tag{4.33}$$

Solve systems (4.24)-(4.33), we get

$$3!C_4 - 4!C_5 = 0,$$

$$2C_3 + 5!C_6 = 0,$$

$$4!d_5 + 5!d_6 + 360d_7 + 840d_8 = 0,$$

$$5!d_6 + 6!d_7 + 2520d_8 = 0,$$

$$6!d_7 + 7!d_8 = 0,$$

$$\sum_{i=1}^{8} d_i y^{i-1} = \sum_{i=1}^{8} C_i y^{i-1},$$

$$\sum_{i=2}^{8} (i-1) d_i y^{i-2} = \sum_{i=2}^{8} (i-1) C_i y^{i-2},$$

$$\sum_{i=3}^{8} (i-1)(i-2) d_i y^{i-3} = \sum_{i=3}^{8} (i-1)(i-2) C_i y^{i-3},$$

$$\sum_{i=4}^{8} (i-1)(i-2)(i-3) d_i y^{i-4} = \sum_{i=4}^{8} (i-1)(i-2)(i-3) C_i y^{i-4},$$

$$24d_5 + 120d_6 y_+ 360d_7 y^2 + 840d_8 y^3 = 24C_5 + 120C_6 y + 360C_7 y^2 + 840C_8 y^3,$$

$$120d_6 + 720d_7 y + 2520d_8 y^2 = 120C_6 + 720C_7 y + 2520C_8 y^2,$$

$$720d_7 + 5040d_8 y = 720C_7 + 5040C_8 y,$$

$$5040d_8 - 5040C_8 = 1,$$

$$C_1 = 0,$$

$$d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 + d_8 = 0,$$

$$C_2 = 0.$$

Solve the last system, it follows

$$C_1 = 0,\ C_2 = 0,\ C_3 = \frac{2y^2}{71} - \frac{7y^3}{284} - \frac{7y^4}{1136} + \frac{21y^5}{5680} - \frac{7y^6}{5680} + \frac{y^7}{5680},$$
$$C_4 = -\frac{7y^2}{284} + \frac{16y^3}{639} - \frac{7y^4}{10224} + \frac{7y^5}{17040} - \frac{7y^6}{51120} + \frac{y^7}{51120},$$
$$C_5 = -\frac{7y^2}{1136} + \frac{4y^3}{639} - \frac{7y^4}{40896} + \frac{7Y^5}{68160} - \frac{7y^6}{204480} + \frac{y^7}{204480},$$
$$C_6 = -\frac{y^2}{2130} + \frac{7y^3}{17040} + \frac{7y^4}{68160} - \frac{7y^5}{113600} + \frac{7y^6}{340800} - \frac{y^7}{340800},$$
$$C_7 = \frac{y}{720} + \frac{-1260y^2 - 140y^3 - 35y^4 + 21y^5 - 7y^6 + y^7}{1022400},$$
$$C_8 = -\frac{1}{5040} + \frac{1260y^2 + 140y^3 + 35y^4 - 21y^5 + 7y^6 - y^7}{7156800},\ d_1 = -\frac{y^7}{5040},$$
$$d_2 = \frac{y^6}{720},\ d_3 = \frac{2y^2}{71} - \frac{7y^3}{284} - \frac{7y^4}{1136} - \frac{y^5}{2130} - \frac{7y^6}{5680} + \frac{y^7}{5680},$$
$$d_4 = -\frac{7y^2}{284} + \frac{16y^3}{639} + \frac{4y^4}{639} + \frac{7y^5}{17040} - \frac{7y^6}{51120} + \frac{y^7}{51120},$$
$$d_5 = \frac{-1260y^2 - 140y^3 - 35y^4 + 21y^5 - 7y^6 + y^7}{204480},$$
$$d_6 = \frac{1260\mathrm{y}^2 + 140\mathrm{y}^3 + 35\mathrm{y}^4 - 21\mathrm{y}^5 + 7\mathrm{y}^6 - \mathrm{y}^7}{340800},$$
$$d_7 = \frac{-1260y^2 - 140y^3 - 35y^4 + 21y^5 - 7y^6 + y^7}{1022400},$$
$$d_8 = \frac{1260\mathrm{y}^2 + 140\mathrm{y}^3 + 35\mathrm{y}^4 - 21\mathrm{y}^5 + 7\mathrm{y}^6 - \mathrm{y}^7}{7156800}.$$

Then,

$$R_y(x)=\begin{cases}\begin{array}{c}\left(\frac{2y^2}{71}-\frac{7y^3}{284}-\frac{7y^4}{1136}+\frac{21y^5}{5680}-\frac{7y^6}{5680}+\frac{y^7}{5680}\right)x^2\\ +\left(-\frac{7y^2}{284}+\frac{16y^3}{639}-\frac{7y^4}{10224}+\frac{7y^5}{17040}-\frac{7y^6}{51120}+\frac{y^7}{51120}\right)x^3\\ +\left(-\frac{7y^2}{1136}+\frac{4y^3}{639}-\frac{7y^4}{40896}+\frac{7Y^5}{68160}-\frac{7y^6}{204480}+\frac{y^7}{204480}\right)x^4\\ +\left(-\frac{y^2}{2130}+\frac{7y^3}{17040}+\frac{7y^4}{68160}-\frac{7y^5}{113600}+\frac{7y^6}{340800}-\frac{y^7}{340800}\right)x^5\\ +\left(\frac{y}{720}+\frac{-1260y^2-140y^3-35y^4+21y^5-7y^6+y^7}{1022400}\right)x^6\\ +\left(-\frac{1}{5040}+\frac{1260y^2+140y^3+35y^4-21y^5+7y^6-y^7}{7156800}\right)x^7\end{array}, & y\le x\\ \begin{array}{c}-\frac{y^7}{5040}+\left(\frac{y^6}{7200}\right)x+\left(\frac{2y^2}{71}-\frac{7y^3}{284}-\frac{7y^4}{1136}-\frac{y^5}{2130}-\frac{7y^6}{5680}+\frac{y^7}{5680}\right)x^2\\ +\left(-\frac{7y^2}{284}+\frac{16y^3}{639}+\frac{4y^4}{639}+\frac{7y^5}{17040}-\frac{7y^6}{51120}+\frac{y^7}{51120}\right)x^3\\ +\left(\frac{-1260y^2-140y^3-35y^4+21y^5-7y^6+y^7}{204480}\right)x^4\\ +\left(\frac{1260\mathrm{y}^2+140\mathrm{y}^3+35\mathrm{y}^4-21\mathrm{y}^5+7\mathrm{y}^6-\mathrm{y}^7}{340800}\right)x^5\\ +\left(\frac{-1260y^2-140y^3-35y^4+21y^5-7y^6+y^7}{1022400}\right)x^6\\ +\left(\frac{1260\mathrm{y}^2+140\mathrm{y}^3+35\mathrm{y}^4-21\mathrm{y}^5+7\mathrm{y}^6-\mathrm{y}^7}{7156800}\right)x^7\end{array}, & y>x\end{cases}.$$

□

Consider the following second order initial value problem

$$u^{''}+p(x)u'+q(x)u=f(x),\qquad 0\le x\le 1. \tag{4.34}$$

with

$$u(0)=\alpha,\quad u^{'}(0)=\beta \tag{4.35}$$

when $p,q\in C^2(a,b)$ and $f\in L^2[0,1]$. Let α and β be real constant. Transform problem (4.34-4.35) into the operator form as

$$(Lu)(x)=f(x),\qquad 0\le x\le 1 \tag{4.36}$$

with

$$u(0) = u^{'}(0) = 0, \tag{4.37}$$

where

$$Lu = \frac{d^2u}{dx^2} + p(x)\frac{du}{dx} + q(x)(u), \tag{4.38}$$

and

$$L : W_2^3[0,1] \to W_2^1[0,1]. \tag{4.39}$$

Then,

$$\begin{aligned} L(\mu u + v)(x) &= \frac{d^2}{dx^2}(\mu u + v)(x) + p(x)\frac{d}{dx}(\mu u + v)(x) + q(x)(\mu u + v)(x) \\ &= \mu \frac{d^2}{dx^2}u(x) + \mu p(x)\frac{d}{dx}u(x) + \mu q(x)u(x) + \frac{d^2}{dx^2}v(x) \\ &\quad + p(x)\frac{d}{dx}v(x) + q(x)v(x) \\ &= \mu L(u)(x) + L(v)(x) \end{aligned}$$

where μ is constant. Then, L is linear operator.

Theorem 4.1.5. *The linear operator L is a bounded linear operator.*

Proof. We only need to prove $\|Lu\|^2_{W_2^1} \leq M\|u\|^2_{W_2^3}$, where $M > 0$ is a positive constant. By equations (4.1) and (4.2), given that

$$\|Lu\|^2_{W_2^1} = \langle Lu, Lu \rangle_{W_2^1} = \int_0^1 \left[Lu'(x)\right]^2 \,\mathrm{d}x + [Lu(0)]^2.$$

By Theorem (4.1.3), we have

$$u(x) = \langle u(\cdot), R_x(\cdot) \rangle_{W_2^3}$$

and

$$Lu(x) = \langle u(\cdot), LR_x(\cdot)\rangle_{W_2^1}.$$

By Cauchy–Schwarz inequality,

$$|Lu(x)| \leq \|u\|_{W_2^3} \|LR_x\|_{W_2^1} = M_1 \|u\|_{W_2^3}$$

where $M_1 > 0$ is a positive constant. Thus,

$$[(Lu)(0)]^2 \leq M_1^2 \|u\|_{W_2^3}^2.$$

Since

$$(Lu)'(x) = \left\langle u(\cdot), (LR_x)'(\cdot)\right\rangle_{W_2^1},$$

then

$$\left|(Lu)'(x)\right| \leq \|u\|_{W_2^3} \left\|(LR_x)'\right\|_{W_2^3} = M_2 \|u\|_{W_2^3}$$

where $M_2 > 0$ is a positive constant. We have

$$\left[(Lu)'(t)\right]^2 \leq M_2^2 \|u\|_{W_2^3}^2$$

and

$$\int_0^1 \left[(Lu)'(x)\right]^2 \, \mathrm{d}x \leq M_2^2 \|u\|_{W_2^3}^2$$

which implies that

$$\|Lu\|_{W_2^1}^2 \leq M \|u\|_{W_2^3}^2$$

where $M = M_1^2 + M_2^2$ $\square$

4.2 Structure of the Solution

Let $\{x_i\}_{i=1}^{\infty}$ be a countable dense subset of $[0,1]$. Let $\varphi_i(x) = R_{x_i}(x)$ and $\psi_i(x) = L^*\varphi_i(x)$, where L^* is conjugate operator of L. The orthonormal system $\left\{\widehat{\Psi}_i(x)\right\}_{i=1}^{\infty}$ of $W_2^3[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$ by

$$\widehat{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad \beta_{ii} > 0, i = 1,2,\ldots.$$

Theorem 4.2.1. *If $u(x)$ is the exact solution of*

$$u'' = f(x,u,u'), \quad 0 < x < 1. \tag{4.40}$$

with

$$u(0) = u_0, \quad u'(0) = u_1 \tag{4.41}$$

then

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k,u_k)\,\widehat{\Psi}_i(x). \tag{4.42}$$

where $\{x_i\}_{i=1}^{\infty}$ is dense in $[0,1]$.

Proof. Using the (4.1) and uniqueness of solution of (4.34),

$$
\begin{aligned}
u(x) &= \sum_{i=1}^{\infty} \left\langle u(x), \widehat{\Psi}_i(x) \right\rangle_{W_2^3} \widehat{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle u(x), \Psi_k(x) \right\rangle_{W_2^3} \widehat{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle u(x), L^* \varphi_k(x) \right\rangle_{W_2^3} \widehat{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle Lu(x), \varphi_k(x) \right\rangle_{W_2^1} \widehat{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle f(x,u), T_{x_k} \right\rangle_{W_2^1} \widehat{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k, u_k) \widehat{\Psi}_i(x).
\end{aligned}
$$

This completes the proof. □

Theorem 4.2.2. *The approximate solution of a problem* $4.34-4.35$ *can be obtained and written as*

$$
u_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} f(x_k, u_k) \widehat{\Psi}_i(x).
$$

Similar argument can be used for the first order initial value problem. To explain this idea of the solution, let us consider the following example.

Example 4.2.1. Consider

$$
u'(x) = 2 = f(x,u), \quad 0 \leqslant x \leqslant 1
$$

such that

$$
u(0) = 0.
$$

The exact solution is $u(x) = 2x$.

Let

$$R_y(x) = \begin{cases} yx + \frac{yx^2}{2} - \frac{1}{6}x^3, & x \leqslant y \\ -\frac{y^3}{6} + \left(y + \frac{y^2}{2}\right)x, & x > y \end{cases}.$$

Then,

$$R_x(y) = \begin{cases} xy + \frac{xy^2}{2} - \frac{y^3}{6}, & x \geqslant y \\ -\frac{x^3}{6} + xy + \frac{x^2y}{2}, & x < y \end{cases}.$$

After differentiation one time, we get

$$\frac{\partial R_x(y)}{\partial y} = \begin{cases} x + xy - \frac{y^2}{2}, & x \geqslant y \\ x + \frac{x^2}{2}, & x < y \end{cases}.$$

Let $x_1 = 0, x_2 = \frac{1}{4}, x_3 = \frac{1}{2}, x_4 = \frac{3}{4}, x_5 = 1$ be a partition of $[0,1]$. Then,

$$\psi_1(x) = L_y R_x(y)\big|_{y=0} = x,$$

$$\psi_2(x) = L_y R_x(y)\big|_{y=\frac{1}{4}} = \begin{cases} -\frac{1}{32} + \frac{5x}{4}, & x \geqslant \frac{1}{4} \\ x + \frac{x^2}{2}, & x < \frac{1}{4} \end{cases},$$

$$\psi_3(x) = L_y R_x(y)\big|_{y=\frac{1}{2}} = \begin{cases} -\frac{1}{8} + \frac{3x}{2}, & x \geqslant \frac{1}{2} \\ x + \frac{x^2}{2}, & x < \frac{1}{2} \end{cases},$$

$$\psi_4(x) = L_y R_x(y)\big|_{y=\frac{3}{4}} = \begin{cases} -\frac{9}{32} + \frac{7x}{4}, & x \geqslant \frac{3}{4} \\ x + \frac{x^2}{2}, & x < \frac{3}{4} \end{cases},$$

$$\psi_5(x) = L_y R_x(y)\big|_{y=1} = x + \frac{x^2}{2}.$$

Now, $\hat{\psi}_i(x)$ for $i = 1:5$ will be generated using Gram Schmidt process where $\beta_{ii} > 0,\ i = 1,2,3,4,5$. Then,

$$\hat{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik} \psi_k(x).$$

Thus,

$$\hat{\psi}_1(x) = \psi_1(x) = x,$$

$$\beta_{11} = 1.$$

Now,

$$\hat{\psi}_2(x) = \psi_2(x) - \frac{\langle \psi_2, \hat{\psi}_1 \rangle}{\|\hat{\psi}_1\|^2} \hat{\psi}_1(x).$$

Thus $\beta_{21} = -1.20361$, $\beta_{22} = 1$, and

$$\hat{\psi}_2(x) = \begin{cases} -\frac{1}{32} + 2.45361x, & x \geqslant \frac{1}{4} \\ 2.20361x + \frac{x^2}{2}, & x < \frac{1}{4} \end{cases}.$$

Also,

$$\hat{\psi}_3(x) = \psi_3(x) - \frac{\langle \psi_3, \hat{\psi}_2 \rangle}{\|\hat{\psi}_2\|^2} \hat{\psi}_2(x) - \frac{\langle \psi_3, \hat{\psi}_1 \rangle}{\|\hat{\psi}_1\|^2} \hat{\psi}_1(x).$$

Then, $\beta_{31} = -1.32031$, $\beta_{32} = -0.548646$, and $\beta_{33} = 1$. Thus,

$$\hat{\psi}_3(x) = \begin{cases} 0.142145 + 3.50612x, & x > \frac{1}{2} \\ -0.0171452 + 3.00612x + \frac{x^2}{2}, & \frac{1}{4} < x < \frac{1}{2} \\ 2.86896x + 0.774323x^2, & x < \frac{1}{4} \end{cases}.$$

Now,

$$\hat{\psi}_4(x) = \psi_4(x) - \frac{\langle \psi_4, \hat{\psi}_3 \rangle}{\|\hat{\psi}_3\|^2} \hat{\psi}_3(x) - \frac{\langle \psi_4, \hat{\psi}_2 \rangle}{\|\hat{\psi}_2\|^2} \hat{\psi}_2(x) - \frac{\langle \psi_4, \hat{\psi}_1 \rangle}{\|\hat{\psi}_1\|^2} \hat{\psi}_1(x).$$

Then, $\beta_{41} = -1.36768$, $\beta_{42} = -0.568405$, $\beta_{43} = -0.414703$, and $\beta_{44} = 1$. Thus,

$$\hat{\psi}_4(x) = \begin{cases} -0.350851 + 4.45024x, & x > \frac{3}{4} \\ -0.0696006 + 3.70024x + \frac{x^2}{2}, & \frac{1}{2} < x < \frac{3}{4} \\ -0.0177626 + 3.49288x + 0.707352x^2, & \frac{1}{4} < x < \frac{1}{2} \\ 3.35078x + 0.991554x^2, & x < \frac{1}{4} \end{cases}.$$

Finally,

$$\hat{\psi}_5(x) = \psi_5(x) - \frac{\langle \psi_5, \hat{\psi}_4 \rangle}{\|\hat{\psi}_4\|^2} \hat{\psi}_4(x) - \frac{\langle \psi_5, \hat{\psi}_3 \rangle}{\|\hat{\psi}_3\|^2} \hat{\psi}_3(x) - \frac{\langle \psi_5, \hat{\psi}_2 \rangle}{\|\hat{\psi}_2\|^2} \hat{\psi}_2(x) - \frac{\langle \psi_5, \hat{\psi}_1 \rangle}{\|\hat{\psi}_1\|^2} \hat{\psi}_1(x).$$

Then, $\beta_{51} = -1.375$, $\beta_{52} = -0.571463$, $\beta_{53} = -0.416957$, $\beta_{54} = -0.347025$, and $\beta_{55} = 1$. Thus,

$$\hat{\psi}_5(x) = \begin{cases} -0.167579 + 4.32206x + \frac{x^2}{2}, & \frac{3}{4} < x \leqslant 1 \\ -0.0699779 + 4.06179x + 0.673512x^2, & \frac{1}{2} < x < \frac{3}{4} \\ -0.0178582 + 3.85331x + 0.881991x^2, & \frac{1}{4} < x < \frac{1}{2} \\ 3.71045x + 1.16772x^2, & x < \frac{1}{4} \end{cases}.$$

Therefore,

$$u_4(x) = \begin{cases} 2\hat{\psi}_1(x) - 0.407227\hat{\psi}_{21}(x) - 1.73792\hat{\psi}_{31}(x) - 2.70157\hat{\psi}_{41}(x) & \\ -3.42089\hat{\psi}_{51}(x), & x \geqslant \frac{3}{4} \\ 2\hat{\psi}_1 - 0.407227\hat{\psi}_{21}(x) - 1.73792\hat{\psi}_{31}(x) - 2.70157\hat{\psi}_{42}(x) & \\ -3.42089\hat{\psi}_{52}(x), & \frac{1}{2} \leq x < \frac{3}{4} \\ 2\hat{\psi}_1(x) - 0.407227\hat{\psi}_{21}(x) - 1.73792\hat{\psi}_{32}(x) - 2.70157\hat{\psi}_{43}(x) & \\ -3.42089\hat{\psi}_{53}(x), & \frac{1}{4} \leq x < \frac{1}{2} \\ 2\hat{\psi}_1(x) - 0.407227\hat{\psi}_{22}(x) - 1.73792\hat{\psi}_{33}(x) - 2.70157\hat{\psi}_{44}(x) & \\ -3.42089\hat{\psi}_{54}(x), & x < \frac{1}{4} \end{cases}.$$

Then $\|u(x) - u_4(x)\| = \sqrt{\int_0^1 (u(x) - u_4(x))^2 dx} = 1.0 \times 10^{-15}$.

Example 4.2.2. Consider $u''(x) = 2 = f(x,u), \quad 0 \leqslant x \leqslant 1$

such that

$$u(0) = 0, \quad u(1) = 0.$$

Then, the exact solution is $u(x) = x^2 - x$. Let $L : W_2^3[0,1] \longrightarrow W_2^1[0,1]$

such that

$$Lu(x) = u''(x) = f(x,u), \quad 0 < x < 1$$

with $u(0) = u(1) = 0$. Then,

$$R_x(y) = \begin{cases} \begin{array}{c} -x^2y + xy - \frac{1}{20}y^2x^5 + \frac{21}{20}x^2y^2 + \frac{1}{24}x^4y^2 - xy^2 - \frac{1}{12}x^3y^2 + \frac{1}{24}x^2y^4 \\ -\frac{1}{24}xy^4 - \frac{1}{120}x^2y^5 + \frac{1}{12}y^5 \end{array}, & y \leq x \\ \begin{array}{c} \frac{x^5}{120} - \frac{1}{24}x^4y + xy - x^2y - \frac{1}{120}x^5y^2 + \frac{21}{20}x^2y^2 + \frac{1}{24}x^4y^2 - xy^2 - \frac{1}{12}x^2y^3 \\ +\frac{1}{24}x^2y^4 - \frac{1}{120}x^2y^5 \end{array}, & y > x \end{cases}.$$

Then

$$\frac{\partial R_x(y)}{\partial y} = \begin{cases} x - x^2 - 2xy + \frac{21}{10}x^2y - \frac{x^3y}{6} + \frac{x^4y}{12} - \frac{x^5y}{60} - \frac{xy^3}{6} + \frac{x^2y^3}{6} + \frac{y^4}{24} - \frac{x^2y^4}{24}, & y \leqslant x \\ x - x^2 - \frac{x^4}{24} - 2xy + \frac{21x^2y}{10} + \frac{x^4y}{12} - \frac{x^5y}{60} - \frac{x^2y^2}{4} + \frac{x^2y^3}{6} - \frac{x^2y^4}{24}, & y > x \end{cases},$$

and

$$L_yR_x(y) = \begin{cases} -2x + \frac{21}{10}x^2 - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{60} - \frac{xy^2}{2} + \frac{x^2y^2}{2} + \frac{y^3}{6} - \frac{x^2y^3}{6}, & y \leqslant x \\ -2x + \frac{21x^2}{10} + \frac{x^4}{12} - \frac{x^5}{60} - \frac{x^2y}{2} + \frac{x^2y^2}{3} - \frac{x^2y^3}{6}, & y > x \end{cases}.$$

Let $x_1 = 0, x_2 = \frac{1}{4}, x_3 = \frac{1}{2}, x_4 = \frac{3}{4}, x_5 = 1$ be a partition of $[0,1]$. Then,

$$\psi_1(x) = L_y R_x(y)\big|_{y=0} = 2x + \frac{21x^2}{10} + \frac{x^4}{12} - \frac{x^5}{60},$$

$$\psi_2(x) = L_y R_x(y)\big|_{y=\frac{1}{4}} = \begin{cases} \frac{1}{384} + \frac{65}{32}x + \frac{4087}{1920}x^2 - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{60}, & x \geqslant \frac{1}{4} \\ -2x + \frac{3847}{1920}x^2 + \frac{x^4}{12} - \frac{x^5}{60}, & x < \frac{1}{4} \end{cases},$$

$$\psi_3(x) = L_y R_x(y)\big|_{y=\frac{1}{2}} = \begin{cases} \frac{1}{48} - \frac{17}{8}x + \frac{529}{240}x^2 - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{60}, & x \geqslant \frac{1}{2} \\ -2x + \frac{469}{240}x^2 + \frac{x^4}{12} - \frac{x^5}{60}, & x < \frac{1}{2} \end{cases},$$

$$\psi_4(x) = L_y R_x(y)\big|_{y=\frac{3}{4}} = \begin{cases} \frac{9}{128} - \frac{73}{32}x + \frac{1479}{640}x^2 - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{60}, & x \geqslant \frac{3}{4} \\ -2x + \frac{1239}{640}x^2 + \frac{x^4}{12} - \frac{x^5}{60}, & x < \frac{3}{4} \end{cases},$$

$$\psi_5(x) = L_y R_x(y)\big|_{y=1} = -2x + \frac{29}{15}x^2 + \frac{x^4}{12} - \frac{x^5}{60}.$$

Now, we want to find $\hat{\psi}_i(x)$ using Gram Schmidt process where $\beta_{ii} > 0$, for $i = 1,2,3,4,5$. Then,

$$\hat{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x).$$

Simple calculations give that

$$\hat{\psi}_1(x) = \psi_1(x) = 2x + \frac{21x^2}{10} + \frac{x^4}{12} - \frac{x^5}{60}, \quad \beta_{11} = 1,$$

and

$$\hat{\psi}_2(x) = \psi_2(x) - \frac{\langle \psi_2, \hat{\psi}_1 \rangle}{\|\hat{\psi}_1\|^2}\hat{\psi}_1(x),$$

which implies that $\beta_{21} = -1.01136,\ \textit{and}\ \ \beta_{22} = 1$, and

$$\hat{\psi}_2(x) = \begin{cases} \frac{1}{384} - 4.05398x + 4.25251x^2 - 0.335227x^3 + 0.167614x^4 - 0.0335227x^5, & x \geqslant \frac{1}{4} \\ -4.02273x + 4.12751x^2 - 0.168561x^3 + 0.16714x^4 - 0.0335227x^5, & x < \frac{1}{4} \end{cases}.$$

Also,

$$\hat{\psi}_3(x) = \psi_3(x) - \frac{\langle \psi_3, \hat{\psi}_2 \rangle}{\|\hat{\psi}_2\|^2}\hat{\psi}_2(x) - \frac{\langle \psi_3, \hat{\psi}_1 \rangle}{\|\hat{\psi}_1\|^2}\hat{\psi}_1(x).$$

Then $\beta_{31} = -1.02976,\ \ \beta_{32} = -0.509099,\ \ \beta_{33} = 1$, and

$$\hat{\psi}_3(x) = \begin{cases} 0.0221591 - 5.45036x^2 - 0.423144x^3 + 0.211572x^4 \\ -0.0423144x^5 & , \quad x \geq \frac{1}{2} \\ 0.00132578 - 5.09364x + 5.20036x^2 + 0.0256477x^3 \\ +0.211572x^4 - 0.0423144x^5 & , \quad \frac{1}{4} \leq x < \frac{1}{2} \\ -5.07773 \times +5.13672x^2 - 0.171627x^3 + 0.211572x^4 \\ -0.0423144x^5 & , \quad x < \frac{1}{4} \end{cases}.$$

In addition,

$$\hat{\psi}_4(x) = \psi_4(x) - \frac{\langle \psi_4, \hat{\psi}_3 \rangle}{\|\hat{\psi}_3\|^2}\hat{\psi}_3(x) - \frac{\langle \psi_4, \hat{\psi}_2 \rangle}{\|\hat{\psi}_2\|^2}\hat{\psi}_2(x) - \frac{\langle \psi_4, \hat{\psi}_1 \rangle}{\|\hat{\psi}_1\|^2}\hat{\psi}_1(x).$$

Hence, $\beta_{41} = -1.04045,\ \ \beta_{42} = -0.514382,\ \ \beta_{43} = -0.404164,\ \ \beta_{44} = 1$, and

$$\hat{\psi}_4(x) = \begin{cases} 0.0800721 - 6.26583x + 6.48165x^2 - 0.493165x^3 \\ +0.246583x^4 - 0.0493165x^5 & , \quad x \geq \frac{3}{4} \\ 0.00975961 - 5.98458x + 6.10665x^2 - 0.326499x^3 \\ +0.246583x^4 - 0.0493165x^5 & , \quad \frac{1}{2} < x < \frac{3}{4} \\ 0.00133954 - 5.93406x + 6.00561x^2 - 0.259138x^3 \\ +0.246583x^4 - 0.0493165x^5 & , \quad \frac{1}{4} < x < \frac{1}{2} \\ -5.91798x + 5.94132x^2 - 0.173408x^3 \\ +0.246583x^4 - 0.0493165x^5 & , \quad x < \frac{1}{4} \end{cases}.$$

Finally

$$\hat{\psi}_5(x) = \psi_5(x) - \frac{\langle \psi_5, \hat{\psi}_4 \rangle}{\|\hat{\psi}_4\|^2}\hat{\psi}_4(x) - \frac{\langle \psi_5, \hat{\psi}_3 \rangle}{\|\hat{\psi}_3\|^2}\hat{\psi}_3(x) - \frac{\langle \psi_5, \hat{\psi}_2 \rangle}{\|\hat{\psi}_2\|^2}\hat{\psi}_2(x) - \frac{\langle \psi_5, \hat{\psi}_1 \rangle}{\|\hat{\psi}_1\|^2}\hat{\psi}_1(x).$$

Thus $\beta_{51} = -1.04231$, $\beta_{52} = -0.515306$, $\beta_{53} = -0.40489$, $\beta_{54} = -0.345464$, $\beta_{55} = 1$,

and

$$\hat{\psi}_5(x) = \begin{cases} 0.03400676 - 6.77982x + 6.90988x^2 - 0.384662x^3 \\ +0.275664x^4 - 0.0551329x^5 & , \quad \frac{3}{4} < x \leqslant 1 \\ -0.0097716 - 6.68266x + 6.78033x^2 - 0.327085x^3 \\ +0.275664x^4 - 0.0551329x^5 & , \quad \frac{1}{2} < x < \frac{3}{4} \\ 0.00134197 - 6.63205x - 6.67911x^2 + 0.275664x^4 \\ -0.0551329x^5 & , \quad \frac{1}{4} < x < \frac{1}{2} \\ -6.61594x + 6.6147x^2 - 0.173713x^3 + 0.275664x^4 \\ -0.0551329x^5 & , \quad x < \frac{1}{4} \end{cases}.$$

Therefore,

$$u_4(x) = \begin{cases} 2\hat{\psi}_1(x) - 0.0227296\hat{\psi}_{21}(x) - 1.07773\hat{\psi}_{31}(x) - 1.91798\hat{\psi}_{41}(x) \\ -2.1594\hat{\psi}_{51}, & x \le \frac{3}{4} \\ 2\hat{\psi}_1(x) - 0.0227296\hat{\psi}_{21}(x) - 1.07773\hat{\psi}_{31}(x) - 1.91798\hat{\psi}_{42}(x) \\ -2.1594\hat{\psi}_{52}, & \frac{1}{2} \le x < \frac{3}{4}, \\ 2\hat{\psi}_1(x) - 0.0227296\hat{\psi}_{21}(x) - 1.07773\hat{\psi}_{32}(x) - 1.91798\hat{\psi}_{43}(x) \\ -2.1594\hat{\psi}_{53}, & \frac{1}{4} \le x < \frac{1}{2} \\ 2\hat{\psi}_1(x) - 0.0227296\hat{\psi}_{21}(x) - 1.07773\hat{\psi}_{33}(x) - 1.91798\hat{\psi}_{44}(x) \\ -2.1594\hat{\psi}_{54}, & x < \frac{1}{4} \end{cases}$$

Then, $\|u(x)-u_4(x)\| = \sqrt{\int_0^1 (u(x)-u_4(x))^2 dx} = 2.1 \times 10^{-15}$.

Chapter 5: First Order Fuzzy Initial Value Problem

In this chapter, linear and nonlinear first order fuzzy initial value problem(FIVP) will be discussed.

5.1 Linear First Order Fuzzy Initial Value Problem

Consider the following linear first order FIVP

$$y' + a(x)y = b(x), \quad 1 \geqslant x > 0 \tag{5.1}$$

subject to

$$y(0) = \hat{\beta} \tag{5.2}$$

where $\hat{\beta}$ is fuzzy number, $a(x)$ is a continuous function on [0,1], and $b(x)$ is fuzzy function. Let the α-levels of $y(x), b(x)$, and $\hat{\beta}$ be given by

$$y_\alpha(x) = [y_{1\alpha}(x), y_{2\alpha}(x)],$$

$$y_\alpha(0) = [\beta_1, \beta_2],$$

and

$$b_\alpha(x) = [b_{1\alpha}(x), b_{2\alpha}(x)].$$

To solve problem $(5.1)-(5.2)$, three cases should be investigated.

Case 1: Let $a(x) \geqslant 0$ for all $x \in [0,1]$. Then,

$$[y'_{1\alpha}(x), y'_{2\alpha}(x)] + a(x) \odot [y_{1\alpha}(x), y_{2\alpha}(x)] = [b_{1\alpha}(x), b_{2\alpha}(x)]$$

which produces the following system

$$y'_{1\alpha}(x) + a(x)y_{1\alpha}(x) = b_{1\alpha}(x), \quad y_1(0) = \beta_1, \tag{5.3}$$

and

$$y'_{2\alpha}(x)+a(x)y_{2\alpha}(x)=b_{2\alpha}(x), \quad y_2(0)=\beta_2. \tag{5.4}$$

Then, the method illustrated in Chapter 4 will be implemented to problems (5.3) and (5.4), separately.

Case 2: Let $a(x)<0$ for all $x\in[0,1]$. Then,

$$\left[y'_{1\alpha}(x),y'_{2\alpha}(x)\right]+a(x)\odot[y_{1\alpha}(x),y_{2\alpha}(x)]=[b_{1\alpha}(x),b_{2\alpha}(x)]$$

which implies that

$$y'_{1\alpha}(x)+a(x)y_{2\alpha}(x)=b_{1\alpha}(x), \quad y_{1\alpha}(0)=\beta_1, \tag{5.5}$$

and

$$y'_{2\alpha}(x)+a(x)y_{1\alpha}(x)=b_{2\alpha}(x), \quad y_{2\alpha}(0)=\beta_2. \tag{5.6}$$

Let

$$Y_\alpha(x)=\begin{pmatrix} y_{1\alpha}(x) \\ y_{2\alpha}(x)\end{pmatrix}, \quad B_\alpha(x)=\begin{pmatrix} b_{1\alpha}(x) \\ b_{2\alpha}(x)\end{pmatrix}, \quad \lambda=\begin{pmatrix}\beta_1 \\ \beta_2\end{pmatrix},$$

$$A(x)=\begin{pmatrix} 0 & a(x) \\ a(x) & 0\end{pmatrix}.$$

Then, equations (5.5) and (5.6) can be written in the matrix form as

$$Y'_\alpha(x)+A(x)Y_\alpha(x)=B_\alpha(x), \quad Y_\alpha(0)=\lambda. \tag{5.7}$$

Implement the RkM which is discussed in Chapter 4 to solve problem (5.7). Then, $y_{1\alpha}(x)$ and $y_{2\alpha}(x)$ can be found.

Case3: Let $c \in (0,1)$ such that either

$$a)\ \ a(x) \geqslant 0 \ \text{ on } \ [0,c] \ \text{ and } \ a(x) < 0 \ \text{ on } \ (c,1]$$

or

$$b)\ \ a(x) < 0 \ \text{ on } \ [0,c) \ \text{ and } \ a(x) \geqslant 0 \ \text{ on } \ [c,1].$$

Without loss of generality, assume that $a(x) \geqslant 0$ on $[0,c]$ and $a(x) < 0$ on $(c,1]$. Implement case 1 to solve the following problem

$$[y'_{1\alpha}(x), y'_{2\alpha}(x)] + a(x) \odot [y_{1\alpha}(x), y_{2\alpha}(x)] = [b_{1\alpha}(x), b_{2\alpha}(x)]$$

on $[0,c]$. Thus,

$$y'_{1\alpha}(x) + a(x)y_{1\alpha}(x) = b_{1\alpha}(x), \quad y_1(0) = \beta_1, \quad 0 \leq x \leq c.$$

and

$$y'_{2\alpha}(x) + a(x)y_{2\alpha}(x) = b_{2\alpha}(x), \quad y_2(0) = \beta_2, \quad 0 \leq x \leq c.$$

Then, implement case 2 to solve the following problem

$$[y'_{3\alpha}(x), y'_{4\alpha}(x)] + a(x) \odot [y_{3\alpha}(x), y_{4\alpha}(x)] = [b_{1\alpha}(x), b_{2\alpha}(x)]$$

on $[c,1]$. Then, we get the following system

$$y'_{3\alpha}(x) + a(x)y_{4\alpha}(x) = b_{1\alpha}(x), \quad y_{3\alpha}(c) = y_{1\alpha}(c), \quad c \leq x \leq 1,$$

and

$$y'_{4\alpha}(x) + a(x)y_{3\alpha}(x) = b_{2\alpha}(x), \quad y_{4\alpha}(c) = y_{2\alpha}(c), \quad c \leq x \leq 1.$$

Let

$$Y_\alpha(x) = \begin{pmatrix} y_{3\alpha}(x) \\ y_{4\alpha}(x) \end{pmatrix}, \quad B_{1\alpha}(x) = \begin{pmatrix} b_{1\alpha}(x) \\ b_{2\alpha}(x) \end{pmatrix}, \quad \lambda = \begin{pmatrix} y_{1\alpha}(x) \\ y_{2\alpha}(x) \end{pmatrix},$$

$$A(x) = \begin{pmatrix} 0 & a(x) \\ a(x) & 0 \end{pmatrix}.$$

Then,

$$Y'_\alpha(x) + A(x)Y_\alpha = B_\alpha(x), \quad Y_\alpha(0) = \lambda, \quad c < x \le 1.$$

Using the RKM, the above system can be solved to find $y_{3\alpha}(x)$ and $y_{4\alpha}(x)$. Then,

$$y_\alpha(x) = [f_{1\alpha}(x), f_{2\alpha}(x)]$$

where

$$f_{1\alpha}(x) = \begin{cases} y_{1\alpha}(x), & 0 \leqslant x \le c \\ y_{3\alpha}(x), & c < x \leqslant 1 \end{cases},$$

$$f_{2\alpha}(x) = \begin{cases} y_{2\alpha}(x), & 0 \leqslant x \leqslant c \\ y_{4\alpha}(x), & c < x \leqslant 1 \end{cases}.$$

If $a(x)$ Changes it's sign at a finite distinct number of points $c_1, c_2, \ldots, c_n \in (0,1)$, Then the problem on each subinterval will be solved using the technique described in case 3.

Example 5.1.1. Consider the following problem

$$y' + xy = \hat{\gamma}x \quad 0 < x \le 1,$$

subject to

$$y(0) = \hat{\beta}$$

where $\hat{\gamma} = [1,2,3]$ and $\hat{\beta} = [0,1,2]$. Let the α-cut of $y, y', \hat{\beta}$, and $\hat{\gamma}$ be given by

$$y_\alpha = [y_{1\alpha}, y_{2\alpha}],$$

$$y'_\alpha = [y'_{1\alpha}, y'_{2\alpha}],$$

$$\hat{\beta} = [\alpha, 2-\alpha],$$

$$\hat{\gamma} = [\alpha+1, 3-\alpha].$$

Then, the problem becomes

$$y'_{1\alpha} + xy_{1\alpha} = (\alpha+1)x, \qquad y_{1\alpha}(0) = \alpha.$$

$$y'_{2\alpha} + xy_{2\alpha} = (3-\alpha)x, \qquad y_{2\alpha}(0) = 2-\alpha.$$

Then, the exact solution is

$$y_\alpha(x) = \left[1+\alpha - e^{-\frac{x^2}{2}}, 3-\alpha - e^{-\frac{x^2}{2}}\right].$$

Let $y_{6\alpha}(x)$ be the approximate solution as described in chapter 4. Then, the absolute errors $|y_\alpha(ih) - y_{6\alpha}(ih)|$ for $i = 0, 1, \ldots, h$ and $h = 0.1$ are given in Table (5.1).

Table 5.1: The absolute errors in Example 5.1.1

x_i	$\lvert y_\alpha(x_i) - y_{1\alpha 6}(x_i)\rvert$	$\lvert y_\alpha(x_i) - y_{2\alpha 6}(x_i)\rvert$
0	0	0
0.1	$1.1 * 10^{-15}$	$1.2 * 10^{-15}$
0.2	$1.2 * 10^{-15}$	$1.3 * 10^{-15}$
0.3	$1.4 * 10^{-15}$	$1.5 * 10^{-15}$
0.4	$1.7 * 10^{-15}$	$1.7 * 10^{-15}$
0.5	$2.0 * 10^{-15}$	$1.9 * 10^{-15}$
0.6	$2.1 * 10^{-15}$	$2.1 * 10^{-15}$
0.7	$2.4 * 10^{-15}$	$2.3 * 10^{-15}$
0.8	$2.6 * 10^{-15}$	$2.5 * 10^{-15}$
0.9	$2.8 * 10^{-15}$	$2.7 * 10^{-15}$
1	$2.9 * 10^{-15}$	$2.8 * 10^{-15}$

Example 5.1.2. Consider the following problem

$$y' + (-x)y = \hat{\gamma}x, \quad 0 \leq x \leq 1$$

subject to

$$y(0) = \hat{\beta}$$

where $\hat{\gamma} = [-1, 0, 1]$ and $\hat{\beta} = [2, 3, 4]$. Let the α-cut of $y, y', \hat{\gamma},$ and $\hat{\beta}$ be given by

$$y_\alpha = [y_{1\alpha}, y_{2\alpha}], \qquad y'_\alpha = \left[y'_{1\alpha}, y'_{2\alpha}\right],$$

$$\hat{\beta} = [\alpha + 2, 4 - \alpha], \qquad \hat{\gamma} = [\alpha - 1, 1 - \alpha].$$

Then,

$$\left[y'_{1\alpha}, y'_{2\alpha}\right] + [-xy_{2\alpha}, -xy_{1\alpha}] = [x(\alpha - 1), (1 - \alpha)x]$$

which implies that

$$y'_{1\alpha} - xy_{2\alpha} = x(\alpha - 1), \qquad y_1(0) = \alpha + 2,$$

$$y'_{2\alpha} - xy_{1\alpha} = (1 - \alpha)x_1, \qquad y_2(0) = 4 - \alpha.$$

Then, the exact solution

$$Y_\alpha(x) = \begin{pmatrix} y_{1\alpha}(x) \\ y_{2\alpha}(x) \end{pmatrix}, \qquad B_\alpha = \begin{pmatrix} (\alpha - 1)x \\ (1 - \alpha)x \end{pmatrix},$$

$$\lambda = \begin{pmatrix} \alpha + 2 \\ 4 - \alpha \end{pmatrix}, \qquad A(x) = \begin{pmatrix} 0 & -x \\ -x & 0 \end{pmatrix}.$$

Then, using the procedure described in chapter 4, the absolute error are reported in Table (5.2).

Table 5.2: The absolute errors in Example 5.1.1

x_i	$\lvert y_\alpha(x_i) - y_{1\alpha 6}(x_i)\rvert$	$\lvert y_\alpha(x_i) - y_{2\alpha 6}(x_i)\rvert$
0	0	0
0.1	$2.1 * 10^{-14}$	$2.2 * 10^{-14}$
0.2	$2.3 * 10^{-14}$	$2.4 * 10^{-14}$
0.3	$2.4 * 10^{-14}$	$2.5 * 10^{-14}$
0.4	$2.6 * 10^{-14}$	$2.7 * 10^{-14}$
0.5	$2.7 * 10^{-14}$	$2.9 * 10^{-14}$
0.6	$2.9 * 10^{-14}$	$3.0 * 10^{-14}$
0.7	$3.2 * 10^{-14}$	$3.2 * 10^{-14}$
0.8	$4.5 * 10^{-14}$	$4.4 * 10^{-14}$
0.9	$5.1 * 10^{-14}$	$5.2 * 10^{-14}$
1	$5.7 * 10^{-14}$	$5.8 * 110^{-14}$

Example 5.1.3. Consider the following problem

$$y' + (\frac{1}{2} - x)y = \hat{\gamma}, \quad 0 \leq x \leq 1$$

subject to

$$y(0) = \hat{\beta}$$

where $\hat{\gamma} = [0,1,2]$ and $\hat{\beta} = [1,3,4]$. Let the α-cut of y, y', $\hat{\beta}_1$ and $\hat{\gamma}$ be given by

$$y_\alpha = \begin{cases} [y_{1\alpha}, y_{2\alpha}], & 0 \leqslant x < \frac{1}{2} \\ [y_{3\alpha}, y_{4\alpha} x], & \frac{1}{2} \leqslant x \leqslant 1 \end{cases},$$

$$y'_\alpha = \begin{cases} \left[y'_{1\alpha}, y'_{2\alpha}\right], & 0 \leqslant x < \frac{1}{2} \\ \left[y'_{3\alpha}, y'_{4\alpha} x\right], & \frac{1}{2} \leqslant x \leqslant 1 \end{cases},$$

$$\hat{\beta} = [2\alpha+1, 4-\alpha], \quad \hat{\gamma} = [\alpha, 2-\alpha].$$

Then, for $0 \leqslant x \leqslant \frac{1}{2}$, we have

$$y'_{1\alpha} + \left(\tfrac{1}{2} - x\right) y_{1\alpha} = \alpha \left(\tfrac{1}{2} - x\right), \qquad y_{1\alpha}(0) = 2\alpha + 1,$$

$$y'_{2\alpha} + \left(\tfrac{1}{2} - x\right) y_{2\alpha} = (2-\alpha) \left(\tfrac{1}{2} - x\right), \qquad y_{2\alpha}(0) = 4 - \alpha,$$

and for $\frac{1}{2} \leqslant x \leqslant 1$, we have

$$y'_{3\alpha} + \left(\frac{1}{2} - x\right) y_{4\alpha} = (2-\alpha)\left(\frac{1}{2} - x\right), \qquad y_{3\alpha}\left(\frac{1}{2}\right) = y_{1\alpha}\left(\frac{1}{2}\right),$$

$$y'_{4\alpha} + \left(\frac{1}{2} - x\right) y_{3\alpha} = \alpha\left(\frac{1}{2} - x\right), \qquad y_{4\alpha}\left(\frac{1}{2}\right) = y_{2\alpha}\left(\frac{1}{2}\right),$$

Then the exact solution is

$$y_\alpha(x) = \begin{cases} [\alpha + (1+\alpha)e^{\frac{1}{2}(x^2-x)},\ 2-\alpha+2e^{\frac{1}{2}(x^2-x)}], & 0 \leq x \leq \frac{1}{2} \\ [\alpha + \frac{1}{2}(\alpha-1)e^{\frac{1}{8}(1-2x)^2} + \frac{1}{2}(3+\alpha)e^{-\frac{1}{8}+\frac{1}{2}(x^2-x)}, & \\ \frac{1}{2}(4-2\alpha-(\alpha-1)e^{\frac{1}{8}(1-2x)^2} + (3+\alpha)e^{-\frac{1}{8}+\frac{1}{2}(x^2-x)}], & \frac{1}{2} \leq x \leq 1 \end{cases}.$$

Then, using the procedure described in chapter 4, the absolute error is defined by

$$\|y_\alpha - y_{\alpha 6}\| = \int_0^{1/2} (y_{1\alpha}(x) - y_{1\alpha 6}(x))^2 dx$$
$$+ \int_0^{1/2} (y_{2\alpha}(x) - y_{2\alpha 6}(x))^2 dx + \int_{1/2}^1 (y_{1\alpha}(x) - y_{3\alpha}(x))^2 dx$$
$$+ \int_{1/2}^1 (y_{2\alpha}(x) - y_{4\alpha 6}(x))^2 dx = 1 \cdot 3 * 10^{-12}.$$

5.2 Nonlinear Fuzzy Initial Value Problem

Consider the following problem

$$y' = f(x,y), \quad 0 \leq x \leq 1,$$

subject to

$$y(0) = \hat{\beta}.$$

Let the $\alpha-$ cut of $y(x), \hat{\beta}$, and the function $f(x,y)$ be given by

$$y_\alpha(x) = [y_{1\alpha}(x), y_{2\alpha}(x)],$$

$$\hat{\beta} = [\beta_{1\alpha}, \beta_{2\alpha}]$$

and

$$f_\alpha(x,y) = [f_{1\alpha}(x, y_\alpha(x)), f_{2\alpha}(x, y_\alpha(x))].$$

Then,

$$\left[y'_{1\alpha}(x), y'_{2\alpha}(x)\right] = [f_{1\alpha}(x, y_\alpha(x)), f_{2\alpha}(x, y_\alpha(x))]$$

which implies that

$$y'_{1\alpha}(x) = f_{1\alpha}(x, y_{1\alpha}(x), y_{2\alpha}(x)), \quad y_{1\alpha}(0) = \beta_{1\alpha}, \;\; 0 \leq x \leq 1$$

and

$$y'_{2\alpha}(x) = f_{2\alpha}(x, y_{1\alpha}(x), y_{2\alpha}(x)), \quad y_{2\alpha}(0) = \beta_{2\alpha}, \;\; 0 \leq x \leq 1.$$

Then,

$$y_{1\alpha n}(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik}\hat{\psi}_i(x) f_{1\alpha}(x_k, y_{\alpha k})$$

and

$$y_{2\alpha n}(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik}\hat{\psi}_i(x) f_{2\alpha}(x_k, y_{\alpha k})$$

where

$$f_{1\alpha}(x_k, y_{\alpha k}) = f_{1\alpha}(x_k, y_\alpha(x_k))$$
$$= min\{f(x_k, w): \ \ w \in [y_{1\alpha}(x_k), y_{2\alpha}(x_k)]\}$$

and

$$f_{2\alpha}(x_k, y_{\alpha k}) = f_{1\alpha}(x_k, y_\alpha(x_k))$$
$$= max\{f(x_k, w): \ \ w \in [y_{1\alpha}(x_k), y_{2\alpha}(x_k)]\}.$$

Example 5.2.1. Consider the following problem

$$y^{'}(x) = y^2(x) + x^2, \qquad 0 \le x \le 1,$$

subject to

$$y(0) = \hat{\gamma}$$

where $\hat{\gamma} = [0.1\alpha - 0.1, 0.1 - 0.1\alpha]$. Then, the $\alpha-$ cut of y and y' are

$$y_\alpha(x) = [y_{1\alpha}(x), y_{2\alpha}(x)] \text{ and } y'_\alpha(x) = \left[y'_{1\alpha}(x), y'_{2\alpha}(x)\right].$$

Let $f(x,y) = y^2(x) + x^2$. Then, the α-cut of f is

$$f_\alpha(x,y) = [f_{1\alpha}(x, y_\alpha(x)), f_{2\alpha}(x, y_\alpha(x))].$$

Thus,

$$y'_{1\alpha}(x) = f_{1\alpha}(x, y_\alpha), \quad y_{1\alpha}(0) = 0\cdot 1\alpha - 0.1,$$

$$y'_{2\alpha}(x) = f_{2\alpha}(x, y_\alpha), \quad y_{2\alpha}(0) = 0.1 - 0.1\alpha.$$

Then, using the technique which described in chapter 4, one gets

$$y_{1\alpha n}(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik}\hat{\psi}_i(x) f_{1\alpha}(x_k, y_{\alpha k})$$

and

$$y_{2\alpha n}(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik}\hat{\psi}_i(x) f_{2\alpha}(x_k, y_{\alpha k})$$

where

$$f_{1\alpha}(x_k, y_{\alpha k}) = \min\{f(x_k, w) : w \in [y_{1\alpha}(x_k), y_{2\alpha}(x_k)]\}$$

and

$$f_{2\alpha}(x_k, y_{\alpha k}) = \max\{f(x_k, w) : w \in [y_{1\alpha}(x_k), y_{2\alpha}(x_k)]\}.$$

Let $n = 8$. Let the absolute error is defined by the residual of $y_{1\alpha 8}$ an $y_{2\alpha 8}$ as

$$E_1(x_k) = \left| y'_{1\alpha 8}(x_k) - y^2_{1\alpha 8}(x_k) - x_k^2 \right|$$

and

$$E_2(x_k) = \left| y'_{2\alpha 8}(x_k) - y^2_{2\alpha 8}(x_k) - x_k^2 \right|.$$

Then, the result are reported in Table (5.3).

Table 5.3: The absolute errors in Example 5.2.1

x_k	$E_1(x_k)$	$E_2(x_k)$
0	0	0
0.1	$1.1*10^{-8}$	$1.2*10^{-8}$
0.2	$1.2*10^{-8}$	$1.4*10^{-8}$
0.3	$1.4*10^{-8}$	$1.7*10^{-8}$
0.4	$1.7*10^{-8}$	$1.8*10^{-8}$
0.5	$1.9*10^{-8}$	$2.1*10^{-8}$
0.6	$2.1*10^{-8}$	$2.3*10^{-8}$
0.7	$2.4*10^{-8}$	$2.6*10^{-8}$
0.8	$2.7*10^{-8}$	$2.8*10^{-8}$
0.9	$2.9*10^{-8}$	$3.1*10^{-8}$
1	$3.3*10^{-8}$	$3.5*10^{-8}$

Example 5.2.2. Consider the following problem

$$y'(x) = e^y + x^4, \qquad 0 \leq x \leq 1$$

subject to

$$y(0) = \hat{\gamma}$$

where $\hat{\gamma}$=(-1,0,1). Then, the α-cut of y , y', and $\hat{\gamma}$ are

$$y_\alpha = [y_{1\alpha}, y_{2\alpha}] \ , \qquad y'_\alpha = [y'_{1\alpha}, y'_{2\alpha}] \ ,$$

$$\hat{\gamma} = [\alpha - 1, 1 - \alpha].$$

Let

$$f_\alpha(x,y) = [f_{1\alpha}(x, y_\alpha(x)), f_{2\alpha}(x, y_\alpha(x))].$$

Then,

$$y'_{1\alpha}(x) = f_{1\alpha}(x,y), \quad y_{1\alpha}(0) = \alpha - 1,$$

$$y'_{2\alpha}(x) = f_{2\alpha}(x,y), \quad y_{2\alpha}(0) = 1 - \alpha.$$

Then, using the technique which described in chapter 4, one gets

$$y_{1\alpha n}(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} \hat{\psi}_i(x) f_{1\alpha}(x_k, y_{\alpha k})$$

and

$$y_{2\alpha n}(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} \hat{\psi}_i(x) f_{2\alpha}(x_k, y_{\alpha k})$$

where

$$f_{1\alpha}(x_k, y_{\alpha k}) = \min\{f(x_k, w) : w \in [y_{1\alpha}(x_k), y_{2\alpha}(x_k)]\}$$

and

$$f_{2\alpha}(x_k, y_{\alpha k}) = \max\{f(x_k, w) : w \in [y_{1\alpha}(x_k), y_{2\alpha}(x_k)]\}.$$

Let $n = 8$. Let the absolute error be defined by the residual of $y_{1\alpha 8}$ an $y_{2\alpha 8}$ as

$$E_1(x_k) = \left| y'_{1\alpha 8}(x_k) - e^{y(x_k)_{1\alpha 8}} - x_k^4 \right|$$

and

$$E_2(x_k) = \left| y'_{2\alpha 8}(x_k) - e^{y(x_k)_{2\alpha 8}} - x_k^4 \right|.$$

Then, the result are reported in Table (5.4).

Table 5.4: The absolute errors in Example 5.2.2

x_k	$E_1(x_k)$	$E_2(x_k)$
0	0	0
0.1	$2.3*10^{-9}$	$2.2*10^{-9}$
0.2	$2.7*10^{-9}$	$2.5*10^{-9}$
0.3	$3.1*10^{-9}$	$2.9*10^{-9}$
0.4	$3.4*10^{-9}$	$3.3*10^{-9}$
0.5	$3.7*10^{-9}$	$3.7*10^{-9}$
0.6	$3.9*10^{-9}$	$4.1*10^{-9}$
0.7	$4.2*10^{-9}$	$4.3*10^{-9}$
0.8	$4.6*10^{-9}$	$4.7*10^{-9}$
0.9	$4.9*10^{-9}$	$5.1*10^{-9}$
1	$5.1*10^{-9}$	$5.3*10^{-9}$

Chapter 6: Second Order Fuzzy Initial Value Problem

In this chapter linear and nonlinear second order fuzzy initial value problem will be discussed.

6.1 Linear Second Order Fuzzy Initial Value Problem

Consider the following linear second order FIVP

$$y'' + a(x)y' + b(x)y = c(x), \qquad 0 \leqslant x \leqslant 1, \tag{6.1}$$

$$y(0) = \hat{\beta}, \tag{6.2}$$

$$y'(0) = \hat{\gamma}, \tag{6.3}$$

where $\hat{\beta}$ and $\hat{\gamma}$ are fuzzy numbers $a(x)$, and $b(x)$ are continues functions on $[0,1]$, and $c(x)$ is fuzzy function. Let the α-levels of $y'(x), y(x)$, $c(x)$, $\hat{\beta}$, and $\hat{\gamma}$ be given by

$$\begin{aligned}
y'_\alpha(x) &= \left[y'_{1\alpha}(x), y'_{2\alpha}(x)\right], \\
y_\alpha(x) &= \left[y_{1\alpha}(x), y_{2\alpha}(x)\right], \\
\hat{\beta} &= \left[\beta_1, \beta_2\right], \\
\hat{\gamma} &= \left[\gamma_1, \gamma_2\right],
\end{aligned}$$

and

$$c_\alpha(x) = \left[c_{1\alpha}(x), c_{2\alpha}(x)\right].$$

To solve Problem $(6.1) - (6.3)$, four cases should be implemented.

Case l: Let $a(x) \geqslant 0, b(x) \geqslant 0$ for all $x \in [0,1]$. Then

$$\left[y''_{1\alpha}(x), y''_{2\alpha}(x)\right] + a(x) \odot \left[y'_{1\alpha}(x), y'_{2\alpha}(x)\right] + b(x) \odot \left[y_{1\alpha}(x), y_{2\alpha}(x)\right] = \left[c_{1\alpha}(x), c_{2\alpha}(x)\right]$$

which produce the following system

$$y''_{1\alpha}(x)+a(x)y'_{1\alpha}(x)+b(x)y_{1\alpha}(x)=c_{1\alpha}(x), \quad y_1(0)=\beta_1,\ y'_1(0)=\gamma_1 \tag{6.4}$$

and

$$y''_{2\alpha}(x)+a(x)y'_{2\alpha}(x)+b(x)y_{2\alpha}(x)=c_{2\alpha}(x), \quad y_2(0)=\beta_2,\ y'_2(0)=\gamma_2. \tag{6.5}$$

Then, the method described in chapter 4 will be implemented to Problems (6.4) and (6.5), separately.

Case 2: Let $a(x)\geqslant 0, b(x)<0$ for all $x\in[0,1]$. Then

$$\left[y''_{1\alpha}(x),y''_{2\alpha}(x)\right]+a(x)\odot\left[y'_{1\alpha}(x),y'_{2\alpha}(x)\right]+b(x)\odot[y_{1\alpha}(x),y_{2\alpha}(x)]=[c_{1\alpha}(x),c_{2\alpha}(x)]$$

which implies that

$$y''_{1\alpha}(x)+a(x)y'_{1\alpha}(x)+b(x)y_{2\alpha}(x)=c_{1\alpha}(x), \quad y_{1\alpha}(0)=\beta_1,\ y'_{1\alpha}=\gamma_1 \tag{6.6}$$

and

$$y''_{2\alpha}(x)+a(x)y'_{2\alpha}(x)+b(x)y_{1\alpha}(x)=c_{2\alpha}(x), \quad y_{2\alpha}(0)=\beta_2,\ y'_{2\alpha}=\gamma_2. \tag{6.7}$$

Let

$$Y_\alpha(x)=\begin{pmatrix} y_{1\alpha}(x) \\ y_{2\alpha}(x) \end{pmatrix}, \quad C_\alpha(x)=\begin{pmatrix} c_{1\alpha}(x) \\ c_{2\alpha}(x) \end{pmatrix},$$

$$\lambda=\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \quad \lambda'=\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \quad B(x)=\begin{pmatrix} 0 & b(x) \\ b(x) & 0 \end{pmatrix}.$$

Then, Equation (6.6) and (6.7) can be written in the matrix form as

$$Y''_{\alpha}(x)+a(x)Y'_{\alpha}(x)+B(x)Y_{\alpha}(x)=C_{\alpha}(x), \quad Y_{\alpha}(0)=\lambda,\ Y'_{\alpha}(0)=\lambda'. \tag{6.8}$$

Implement the proposed method which discussed in Chapter 4 to solve Problem (6.8). Then, $y_{1\alpha}(x)$ and $y_{2\alpha}(x)$ can be found.

Case 3: Let $a(x)<0, b(x)\geqslant 0$ for all $x\in[0,1]$. Then,

$$\left[y''_{1\alpha}(x),y''_{2\alpha}(x)\right]+a(x)\odot\left[y'_{1\alpha}(x),y'_{2\alpha}(x)\right]+b(x)\odot\left[y_{1\alpha}(x),y_{2\alpha}(x)\right]=\left[c_{1\alpha}(x),c_{2\alpha}(x)\right]$$

which implies that

$$y''_{1\alpha}(x)+a(x)y'_{2\alpha}(x)+b(x)y_{1\alpha}(x)=c_{1\alpha}(x),\ y_{1\alpha}(0)=\beta_1,\ y'_{1\alpha}(0)=\gamma_1, \tag{6.9}$$

$$y''_{2\alpha}(x)+a(x)y'_{1\alpha}(x)+b(x)y_{2\alpha}(x)=c_{2\alpha}(x),\ y_{2\alpha}(0)=\beta_2,\ y'_{2\alpha}(0)=\gamma_2. \tag{6.10}$$

Let

$$Y_{\alpha}(x)=\begin{pmatrix} y_1\alpha(x) \\ y_{2\alpha}(x)\end{pmatrix}, \quad C_{\alpha}(x)=\begin{pmatrix} c_{1\alpha}(x) \\ c_{2\alpha}(x)\end{pmatrix},$$

$$\lambda=\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}, \quad \lambda'=\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}, \quad A(x)=\begin{pmatrix}0 & a(x)\\ a(x) & 0\end{pmatrix}.$$

Then, Equation (6.9) and (6.10) can be written in the matrix form as

$$Y''_{\alpha}(x)+A(x)Y'_{\alpha}(x)+b(x)Y_{\alpha}(x)=C_{\alpha}(x), \quad Y_{\alpha}(0)=\lambda,\ Y'_{\alpha}(0)=\lambda'. \tag{6.11}$$

Implement the proposed method which discussed in Chapter 4 to solve Problem (6.11). Then $y'_{1\alpha}(x)$ and $y'_{2\alpha}(x)$ can be found.

Case 4: Let $a(x) < 0, b(x) < 0$ for all $x \in [0,1]$.

$$\left[y''_{1\alpha}(x), y''_{2\alpha}(x)\right] + a(x) \odot \left[y'_{1\alpha}(x), y'_{2\alpha}(x)\right] + b(x) \odot \left[y_{1\alpha}(x), y_{2\alpha}(x)\right] = \left[c_{1\alpha}(x), c_{2\alpha}(x)\right].$$

which implies that

$$y''_{1\alpha}(x) + a(x)y'_{2\alpha}(x) + b(x)y_{2\alpha}(x) = c_{1\alpha}(x), \;\; y_{1\alpha}(0) = \beta_1, \;\; y'_{1\alpha}(0) = \gamma_1 \qquad (6.12)$$

$$y''_{2\alpha}(x) + a(x)y'_{1\alpha}(x) + b(x)y_{1\alpha}(x) = c_{2\alpha}(x), \;\; y_{2\alpha}(0) = \beta_2, \;\; y'_{2\alpha}(0) = \gamma_2. \qquad (6.13)$$

Let

$$Y_\alpha(x) = \begin{pmatrix} y_1\alpha(x) \\ y_{2\alpha}(x) \end{pmatrix}, \quad C_\alpha(x) = \begin{pmatrix} c_{1\alpha}(x) \\ c_{2\alpha}(x) \end{pmatrix},$$

$$\lambda = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \quad \lambda' = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \quad A(x) = \begin{pmatrix} 0 & a(x) \\ a(x) & 0 \end{pmatrix} \quad B(x) = \begin{pmatrix} 0 & b(x) \\ b(x) & 0 \end{pmatrix}.$$

Then, Equation (6.12) and (6.13) can be written in the matrix form as

$$Y''_\alpha(x) + A(x)Y'_\alpha(x) + B(x)Y_\alpha(x) = C_\alpha(x), \quad Y_\alpha(0) = \lambda, \;\; Y'_\alpha(0) = \lambda'. \qquad (6.14)$$

Implement the proposed method which discussed in Chapter 4 to solve problem (6.14). Then $y_{1\alpha}(x)$ and $y_{2\alpha}(x)$ can be found.

Example 6.1.1. Consider the following problem

$$y^{''} + y^{'} = x^2, \qquad 0 \leq x \leq 1$$

subject to

$$y(0) = \hat{\beta}$$
$$y^{'}(0) = \hat{\gamma}$$

where $\hat{\beta} = (0,1,2)$ and $\hat{\gamma} = (1,2,3)$. Then, the $\alpha - cut$ of $y, y', y'', \hat{\beta}$, and $\hat{\gamma}$ are

$$y_\alpha = [y_{1\alpha}, y_{2\alpha}], \quad y^{'}_\alpha = [y^{'}_{1\alpha}, y^{'}_{2\alpha}], \quad y^{''}_\alpha = [y^{''}_{1\alpha}, y^{''}_{2\alpha}],$$
$$\hat{\beta} = [\alpha, 2-\alpha], \quad \hat{\gamma} = [\alpha+1, 3-\alpha].$$

Then,

$$y^{''}_{1\alpha} + y^{'}_{1\alpha} = x^2, \quad y_\alpha(0) = \alpha, \quad y'_\alpha(0) = \alpha + 1,$$
$$y^{''}_{2\alpha} + y'_{2\alpha} = x^2, \quad y_2\alpha(0) = 2-\alpha, \quad y'_{2\alpha}(0) = 3-\alpha.$$

Then, the exact solution is

$$y_\alpha(x) = \left[2\alpha + 1 - (\alpha+1)e^{-x} - \frac{1}{3}x^3 - x^2, 5 - 2\alpha - (3-\alpha)e^{-x} - \frac{1}{3}x^3 - x^2\right].$$

Using $n = 8$, the absolute error in $y_{1\alpha}$ and $y_{2\alpha}$ are given in Table (6.1).

Table 6.1: The absolute errors in Example 6.1.1

x_k	$Abs.\ error\ of\ y_{1\alpha}$	$Abs.\ error\ of\ y_{2\alpha}$
0	0	0
0.1	$2.3*10^{-14}$	$2.4*10^{-14}$
0.2	$2.4*10^{-14}$	$2.5*10^{-14}$
0.3	$2.5*10^{-14}$	$2.6*10^{-14}$
0.4	$2.6*10^{-14}$	$2.7*10^{-14}$
0.5	$2.7*10^{-14}$	$2.8*10^{-14}$
0.6	$2.8*10^{-14}$	$3.1*10^{-14}$
0.7	$3.0*10^{-14}$	$3.2*10^{-14}$
0.8	$3.1*10^{-14}$	$3.4*10^{-14}$
0.9	$3.3*10^{-14}$	$3.6*10^{-14}$
1	$3.5*10^{-14}$	$3.8*10^{-14}$

Example 6.1.2. Consider the following problem

$$y'' + (-1) \odot y' + y = 1$$

subject to

$$y(0) = \hat{\beta}$$

$$y'(0) = \hat{\gamma}$$

where $\hat{\beta} = (-1, 0, 1)$ and $\hat{\gamma} = (0, 1, 2)$. Then, the α-cut of $y, g', y'', \hat{\beta}$, and $\hat{\gamma}$ are

$$y_\alpha = [y_{1\alpha}, y_{2\alpha}], \quad y'_\alpha = \left[y'_{1\alpha}, y'_{2\alpha}\right], \quad y''_\alpha \left[y''_{1\alpha}, y''_{2\alpha}\right],$$

$$\hat{\beta} = [\alpha - 1, 1 + \alpha], \quad \hat{\gamma} = [\alpha, 2 - \alpha].$$

Then,

$$y''_{1\alpha} - y'_{2\alpha} + y_{1\alpha} = 1, \qquad y_{1\alpha}(0) = \alpha - 1, \quad y'_{1\alpha}(0) = \alpha$$

$$y'_{2\alpha} - y'_{1\alpha} + y_{2\alpha} = 1, \qquad y_{2\alpha}(0) = 1 + \alpha, \quad y'_{2}(0) = 2 - \alpha.$$

Then, Using Mathematica, the exact solution is

$$\begin{aligned} y_\alpha(x) = \Bigg[-\frac{1}{3} e^{-x/2} \Bigg(& 3\cos\left[\frac{\sqrt{3}x}{2}\right] + 3e^{x}\cos\left[\frac{\sqrt{3}x}{2}\right] - 3\alpha e^{x}\cos\left[\frac{\sqrt{3}x}{2}\right] \\ & -3e^{x/2}\cos\left[\frac{\sqrt{3}x}{2}\right]^2 + 3\sqrt{3}\sin\left[\frac{\sqrt{3}x}{2}\right] - 2\sqrt{3}\alpha\sin\left[\frac{\sqrt{3}x}{2}\right] - 3\sqrt{3}e^{x}\sin\left[\frac{\sqrt{3}x}{2}\right] \\ & +\sqrt{3}\alpha e^{t}\sin\left[\frac{\sqrt{3}t}{2}\right] - 3e^{x/2}\sin\left[\frac{\sqrt{3}x}{2}\right]^2 \Bigg), -\frac{1}{3} e^{-x/2} \Bigg(-3\cos\left[\frac{\sqrt{3}x}{2}\right] \\ & +3e^{x}\cos\left[\frac{\sqrt{3}x}{2}\right] - 3\alpha e^{t}\cos\left[\frac{\sqrt{3}x}{2}\right] - 3e^{x/2}\cos\left[\frac{\sqrt{3}x}{2}\right]^2 - 3\sqrt{3}\sin\left[\frac{\sqrt{3}x}{2}\right] \\ & +2\sqrt{3}\alpha\sin\left[\frac{\sqrt{3}x}{2}\right] - 3\sqrt{3}e^{x}\sin\left[\frac{\sqrt{3}x}{2}\right] + \sqrt{3}\alpha e^{x}\sin\left[\frac{\sqrt{3}x}{2}\right] \\ & -3e^{x/2}\sin\left[\frac{\sqrt{3}x}{2}\right]^2 \Bigg) \Bigg]. \end{aligned}$$

Using $n = 8$, the absolute error in $y_{1\alpha}$ and $y_{2\alpha}$ are given in Table (6.2).

Table 6.2: The absolute errors in Example 6.1.2

x_k	$Abs.\ error\ of\ y_{1\alpha}$	$Abs.\ error\ of\ y_{2\alpha}$
0	0	0
0.1	$3.4*10^{-13}$	$3.6*10^{-13}$
0.2	$3.6*10^{-13}$	$3.8*10^{-13}$
0.3	$3.9*10^{-13}$	$4.1*10^{-13}$
0.4	$4.2*10^{-13}$	$4.4*10^{-13}$
0.5	$4.5*10^{-13}$	$4.6*10^{-13}$
0.6	$4.8*10^{-13}$	$4.9*10^{-13}$
0.7	$5.2*10^{-13}$	$5.3*10^{-13}$
0.8	$5.6*10^{-13}$	$5.7*10^{-13}$
0.9	$5.9*10^{-13}$	$6.0*10^{-13}$
1	$6.2*10^{-13}$	$6.3*10^{-13}$

6.2 Nonlinear Second Order Fuzzy Initial Value Problem

Consider the following problem

$$y'' = f\left(x, y, y'\right), \quad 0 \leqslant x \leq 1,$$

subject to

$$y(0) = \hat{\beta}, \quad y'(0) = \hat{\gamma}.$$

Let the α-cut of $y(x), y'(x), y''(x), \hat{\beta}, \hat{\gamma}$, and $f(x, y, y')$ be given by

$$y_\alpha(x) = [y_{1\alpha}(x), y_{2\alpha}(x)], \qquad y'_\alpha(x) = \left[y'_{1\alpha}(x), y'_{2\alpha}(x)\right],$$

$$y''_\alpha(x) = \left[y''_{1\alpha}(x), y''_{2\alpha}(x)\right], \quad \hat{\beta} = [\beta_{1\alpha}, \beta_{2\alpha}], \quad \hat{\gamma} = [\gamma_{1\alpha}, \gamma_{2\alpha}],$$

$$f_\alpha\left(x,y,y'\right) = \left[f_{1\alpha}\left(x,y,y'\right), f_{2\alpha}\left(x,y,y'\right)\right]$$

where

$$f_{1\alpha}\left(x,y,y'\right) = \min\left\{f(x,u,v) : u \in [y_{1\alpha}, y_{2\alpha}], v \in \left[y'_{1\alpha}, y'_{2\alpha}\right]\right\}$$

and

$$f_{2\alpha}\left(x,y,y'\right) = \max\left\{f(x,u,v) : u \in [y_{1\alpha}, y_{2\alpha}], v \in \left[y'_{1\alpha}, y'_{2\alpha}\right]\right\}.$$

Then,

$$y''_{1\alpha} = f_{1\alpha}\left(x,y,y'\right), \quad y_{1\alpha}(0) = \beta_{1\alpha}, \quad y'_{1\alpha}(0) = \gamma_{1\alpha},$$

and

$$y''_{2\alpha} = f_{2\alpha}\left(x,y,y'\right), \quad y_{2\alpha}(0) = \beta_{2\alpha}, \quad y'_{2\alpha}(0) = \gamma_{2\alpha}.$$

Then,

$$y_{1\alpha n}(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik}\hat{\psi}_i(x) f_{1\alpha}\left(x_k, y_{\alpha k}, y'_{\alpha k}\right)$$

and

$$y_{2\alpha n}(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik}\hat{\psi}_i(x) f_{2\alpha}\left(x_k, y_{\alpha k}, y'_{\alpha k}\right).$$

Example 6.2.1. Consider the following problem

$$y'' = -\left(y'(x)\right)^2$$

subject to

$$y(0) = \hat{\beta}, \quad y'(0) = \hat{\gamma}$$

where

$$\hat{\beta} = [\alpha, 2-\alpha], \qquad \hat{\gamma} = [1+\alpha, 3-\alpha].$$

Then,

$$y''_{1\alpha}(x) = f_{1\alpha}\left(x, y, y'\right), \qquad y_{1\alpha}(0) = \alpha, \quad y'_{1\alpha}(0) = 1 + \alpha,$$

and

$$y''_{2a}(x) = f_{2\alpha}\left(x_1 y, y'\right), \qquad y_{2\alpha}(0) = 2 - \alpha, \quad y'_{2\alpha}(0) = 3 - \alpha,$$

where

$$f\left(x, y, y'\right) = -\left(y'(x)\right)^2.$$

Then, the exact solution is

$$y_\alpha(x) = [\ln\left((\alpha e^\alpha + e^\alpha)x + e^x\right), \ln\left((3e^{2-\alpha} - \alpha e^{2-\alpha}x) + e^{2\alpha}\right)]$$

Then, using the method proposed in Chapter 4, one gets

$$y_{1\alpha n}(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} \hat{\psi}_i(x) f_{1\alpha}\left(x_k, y_{\alpha k}, y'_{\alpha k}\right)$$

$$y_{2\alpha n}(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} \hat{\psi}_i(x) f_{2\alpha}\left(x_k, y_{\alpha k}, y'_{\alpha k}\right)$$

where

$$f_{1\alpha}\left(x_k, y_{\alpha_k}, y'_{\alpha_k}\right) = \min\left\{-v^2 : v \in [y'_{1\alpha}\left(x_k\right), y'_{2\alpha}\left(x_k\right)]\right\}$$

and

$$f_{2\alpha}\left(x_k, y_{\alpha_k}, y'_{\alpha_k}\right) = \max\left\{-v^2 : v \in [y'_{1\alpha}\left(x_k\right), y'_{2\alpha}\left(x_k\right)]\right\}$$

Let $n = 8$. Let $E_1(x_k)$ and $E_2(x_k)$ be the absolute error in $y_{1\alpha}$ and $y_{2\alpha}$ respectively. The results are reported in Table (6.3).

Table 6.3: The absolute error of Example 6.2.1

x_k	$E_1(x_k)$	$E_2(x_k)$
0	0	0
0.1	$3.1*10^{-12}$	$2.9*10^{-12}$
0.2	$3.3*10^{-12}$	$3.2*10^{-12}$
0.3	$3.7*10^{-12}$	$3.5*10^{-12}$
0.4	$4.1*10^{-12}$	$3.9*10^{-12}$
0.5	$4.5*10^{-12}$	$4.3*10^{-12}$
0.6	$4.8*10^{-12}$	$4.7*10^{-12}$
0.7	$5.2*10^{-12}$	$5.1*10^{-12}$
0.8	$5.5*10^{-12}$	$5.4*10^{-12}$
0.9	$5.8*10^{-12}$	$5.7*10^{-12}$
1	$6.2*10^{-12}$	$6.0*10^{-12}$

Example 6.2.2. Consider the following problem

$$y'' = x^2 \odot y'(x) \oplus 2x \odot y(x) \oplus x \odot \hat{\beta}$$

subject to

$$y(0) = \hat{\beta} \quad , \quad y'(0) = \hat{\gamma}$$

where

$$\hat{\beta} = [1+\alpha, 3-\alpha], \quad \hat{\gamma} = [0,0].$$

Then,

$$y_1''(x) = f_{1\alpha}\left(x, y, y'\right), \qquad y_{1\alpha}(0) = 1+\alpha, \quad y_{1\alpha}'(0) = 0$$

and

$$y''_{2\alpha}(x) = f_{2\alpha}\left(x, y, y'\right), \qquad y_{2x}(0) = 3 - \alpha, \qquad y_{2\alpha}(0) = 0$$

where

$$f\left(x, y, y'\right) = x^2 \odot y'(x) \oplus 2x \odot y(x) \oplus x \odot \hat{\beta}.$$

Then, the exact solution is given by

$$\left[\left(e^{(x^3/3)} - 1\right)(1-\alpha), \left(2e^{(x^3/3)} - 1\right)(3-\alpha)\right]$$

Then using the method which proposed in Chapter 4, one gets

$$y_{1\alpha n}(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} \hat{\psi}_i(x) f_1 \alpha \left(x_k, y_{\alpha k}, y'_{\alpha k}\right),$$

$$y_{2an}(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} \hat{\psi}_i(x) f_{2\alpha} \left(x_k, y_{\alpha k}, y'_{\alpha k}\right),$$

where

$$f_{1\alpha}\left(x_k, y_{\alpha k}, y'_{\alpha k}\right) = x^2 y'_{2\alpha} + 2xy_{1\alpha} + x\beta_1,$$

$$f_{2\alpha}\left(x_k, y_{\alpha k}, y'_{\alpha k}\right) = x^2 y'_{2\alpha} + 2xy_{1\alpha} + x\beta_2.$$

Let $n = 8$. Let $E_1(x_k)$ and $E_2(x_k)$ be the absolute error in $y_{1\alpha}$ and $y_{2\alpha}$, respectively. The results reported in Table (6.4).

Table 6.4: The Absolute Error of Example 6.2.2

x_k	$E_1(x_k)$	$E_2(x_k)$
0	0	0
0.1	$2.7 * 10^{-14}$	$2.5 * 10^{-14}$
0.2	$2.9 * 10^{-14}$	$2.7 * 10^{-14}$
0.3	$3.1 * 10^{-14}$	$2.8 * 10^{-14}$
0.4	$3.4 * 10^{-14}$	$3.0 * 10^{-14}$
0.5	$3.5 * 10^{-14}$	$3.2 * 10^{-14}$
0.6	$3.7 * 10^{-14}$	$3.4 * 10^{-14}$
0.7	$3.9 * 10^{-14}$	$3.7 * 10^{-14}$
0.8	$4.2 * 10^{-14}$	$3.9 * 10^{-14}$
0.9	$4.4 * 10^{-14}$	$4.1 * 10^{-14}$
1	$4.7 * 10^{-14}$	$4.4 * 10^{-14}$

Chapter 7: Conclusion

In this thesis, the analysis of the reproducing kernel method has been presented for first and second order fuzzy initial value problems. It Started by preliminaries about fuzzy number and differentiation, and then highlighted the direct method for solving linear and nonlinear fuzzy problems. When the direct method was used to solve the problem, complicated optimization problems that is difficult to solve appeared. The proposed method was based on RKM and Gram Schmidt process. The structure of the RKM was explained and supported by several examples. The numerical results showed the efficiency of the proposed method. The absolute errors were computed using Mathematica. This thesis was divided into 7 chapters. Chapter 1 presented the literature review. Later the preliminaries of fuzzy number and fuzzy function were illustrated. Then direct method for solving fuzzy initial and value problems was discussed . In addition, the RKM for solving ordinary initial value problems was presented and analyzed. FIV problems of first and second order were discussed and investigated in the chapter 5 and 6 respectively. Finally some conclusion were drawn in chapter 7. For the future work, the fuzzy boundary value problems should be investigated using RKM by implementing the shooting method. Moreover, several applications for this method should be investigated, such as Fuzzy Strum- Liouville problems and the delay fuzzy initial value problems.

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