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ON THE GENERALIZED HARDY-LITTLEWOOD MAXIMAL OPERATOR

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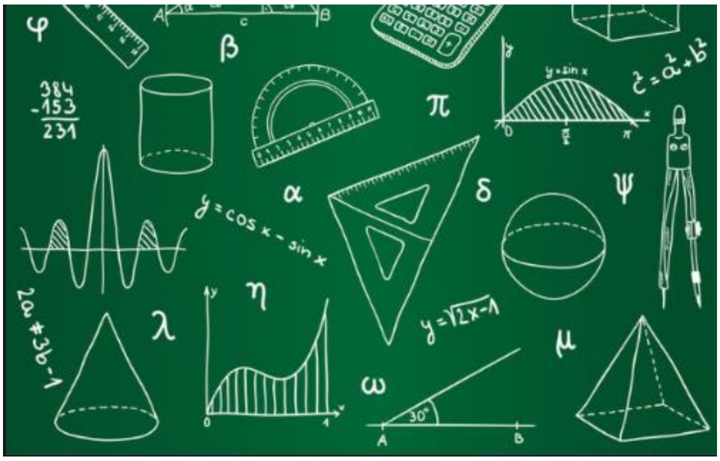

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Namarig Hashim Hassan

This thesis is submitted in partial fulfilment of the requirements for the degree of Master of Science in Mathematics

Under the Supervision of Dr. Salem Ben Said

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Declaration of Original Work

I, Namarig Hashim Hassan, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis, entitled *"On the Generalized Hardy-Littlewood Maximal Operator"*, hereby, solemnly declare that this thesis is my own original research work that has been done and prepared by me under the supervision of Dr. Salem Ben Said, in the College of Science at the UAEU. This work has not previously been presented or published, or formed the basis for the award of any academic degree, diploma or a similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my thesis have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this thesis.

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Abstract

In this report we introduce and then study a maximal operator $\mathcal{M}_{k,n}$ that generalizes the classical one introduced by Hardy and Littlewood in the rank one case. More precisely, for $k \geq 0$ and an integer $n \geqslant 1$,

$$\mathcal{M}_{k,n}f(x) = \sup_{r>0} \frac{1}{\mu_{k,n}(] - r.r[)} \Big| \int_{\mathbb{R}} f(y)\tau_x^{k,n}(\chi_r; y)d\mu_{k,n}(y)\Big|,$$

where the measure $\mu_{k,n}$ is given by $d\mu_{k,n}(y) = |y|^{2k+\frac{2}{n}-2}dy$, and $\tau_x^{k,n}$ is a certain translation operator.

The main result is to prove the weak $(1,1)$ inequality and the strong (p,p) inequality for $\mathcal{M}_{k,n}$, with $1 < p \leq \infty$. The approach uses geometric and analytic tools. One of the major technical obstacles is the lack of known properties of the translation operator $\tau_x^{k,n}$. The strategy is to introduce an uncentered maximal operator associated to intervals of type $I(x,r) =]\max\{0, |x|^{\frac{1}{n}} - r^{\frac{1}{n}}\}^n, (|x|^{\frac{1}{n}} + r^{\frac{1}{n}})^n[$ which controls the maximal operator $\mathcal{M}_{k,n}$. To do so, one needs to prove a Vitaly type covering lemma for the intervals $\big\{I(x_j, r_j)\big\}_j$ together with a sharp estimate for $\mu_{k,n}(I(x_j, r_j))$. The main result generalizes the case $n = 1$ proved by Deleaval, and the case $n = 2$ proved by Ben Said and Deleaval.

Keywords: Hardy-Littlewood maximal operator, Generalized Fourier transform, Vitali type lemma, Strong and Weak type inequalities, Convolution structure, Translation operator.

Title and Abstract (in Arabic)

عن المعامل الأقصى لهاردي و ليتلوود

الملخص

في هذا التقرير، نقدم ثم ندرس المعامل الأقصى $(M_{k,n})$ الذي يعمم المعامل الكلاسيكي الذي قدمه هاردي وليتلوود في الحالة الأولى. بتعبير أدق، ل (N) أي عدد صحيح و (k) أي عدد أكبر أو يساوي الصفر،

$$M_{k,n}f(x) = \sup_{r>0} \frac{1}{\mu_{k,n}(]-r,r[)} \left| \int_{\mathbb{R}} \tau_x^{kn}(\chi_r; y)\, d\mu_{k,n}(y) \right|,$$

حيث المقياس $(\mu_{k,n})$ يعطى بشكل $(d\mu_{k,n}(y) = |y|^{2k+\frac{2}{n}-2}\, dy)$ و (τ_x^{kn}) هو معامل تحويل معين. النتيجة الرئيسية هي إثبات المتباينة الضعيفة $(1,1)$ والمتباينة القوية (P,P) لـ $(1 < P \leq \infty)$. يستخدم النهج أدوات هندسية وتحليلية. تتمثل إحدى العقبات الرئيسية في عدم معرفة خصائص معامل التحويل (τ_x^{kn}). الاستراتيجية هي إدخال معامل أقصى غير مركزي مرتبط بفترات من النوع

$$I(x,r) = \left] \max\left\{0, \left(|x|^{\frac{1}{n}} - r^{\frac{1}{n}}\right)^n\right\}, \left(|x|^{\frac{1}{n}} + r^{\frac{1}{n}}\right)^n \right[$$

والذي يتحكم في المعامل الأقصى. للقيام بذلك، نحتاج إلى إثبات نظرية من نوع "Vitaly" للفترات $(\{I(x_j,y_j)\}_j)$ للوصول لتقدير دقيق لمقياس الفترات $(\mu_{k,n}(I(x_j,y_j)))$. النتيجة الرئيسية تعمم الحالة $(n=1)$ أثبتتها روسلر، والحالة $(n=2)$ أثبتها بن سعيد وديليفال.

مفاهيم البحث الرئيسية: المعامل الأقصى لهاردي و ليتلوود، تحويل فورييه المعمم، نظرية فيتالي، معامل التحويل، المتباينات الضعيفة والقوية.

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I take this opportunity to express my sincere love and warmest gratitude to my parents and my family. Special thanks go to my sisters. I would have never accomplished this work without their continuous support and encouragements.

Dedication

To my beloved family.

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Chapter 1: Introduction

1.1 Hardy-Littlewood Maximal Operators

In 1930, G. H. Hardy and J. E. Littlewood introduced a maximal operator $\mathcal{M}$, defined on the space of locally integrable functions f on $\mathbb{R}$ [1]. Several years later, the maximal operator $\mathcal{M}$ was generalized by Wiener to functions defined on $\mathbb{R}^N$ [2]. This continued the revolutionary change in analysis that started around the late 1800's. Harold Bohr, a danish mathematician once said "Nowadays, there are only three really great English Mathematicians: Hardy, Littlewood, and Hardy-Littlewood".

The Hardy-Littlewood maximal operator is a very important tool in the theory of differentiation of functions, Fourier analysis (especially in the theory of singular integrals), in studying Sobolev functions, and also in complex and Harmonic analysis. Generally speaking, this maximal operator can be thought of as follows. Considering a certain collection of sets $\mathcal{C}$ in $\mathbb{R}^N$, and then taking any function f that is locally integrable, at each x, the maximal average value of f is measured with respect to the collection $\mathcal{C}$, translated by x. More precisely,

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy,$$

where $B(x,r)$ is the ball of radius r centred at the point x. The simplest example of such maximal operator is the one defined on $\mathbb{R}$ by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y)|dy = \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |\tau_x f(y)|dy,$$

where τ_x denotes the Euclidean translation operator $\tau_x f(y) = f(x+y)$. Then, it is of fundamental importance to obtain certain regularity properties of the operator. Such as, weak type inequalities and L^p-boundness. More precisely, it's well known by now that

$$\|\mathcal{M}f\|_{L^p} \leq c_1 \|f\|_{L^p} \tag{1.1.1}$$

for all $1 < p \leq \infty$ and any $f \in L^p(\mathbb{R}^N)$. Also,

$$\|\mathcal{M}f\|_{L^{1,\infty}} \leq c_2 \|f\|_{L^1} \tag{1.1.2}$$

for any $f \in L^1(\mathbb{R}^N)$. Here $L^{1,\infty}$ stands for the Lorentz space. It is worth mentioning that $\mathcal{M}f$ is not in $L^1(\mathbb{R}^N)$ whenever $f \in L^1(\mathbb{R}^N)$.

There are several proofs for the above two fundamental inequalities. The most well-known uses Fourier analysis associated to the Euclidean Fourier transform

$$\mathcal{F}f(x) = c \int_{\mathbb{R}^N} f(y) e^{i\langle x,y \rangle} dy.$$

Some of the applications of these inequalities are in the proof of the Lebesgue differentiation theorem [3], the Rademacher differentiation theorem, Fatou's theorem on nontangential convergence, and fractional integration theorem. See for instance [4], and [5].

It is interesting to mention that Antonios D. Melas [6] was able to find the exact value of the best possible constant c_2 for the weak-type $(1,1)$ inequality for the one-dimensional centered Hardy-Littlewood maximal operator. It was the first time anyone ever precisely evaluated the best constant for one of the fundamental inequalities satisfied by a centered maximal operator.

Along the years, Mathematicians have been working to expand the Hardy-Littlewood maximal operator to different frame works (more precisely for different integral transforms). For instance, singular integral operators [7] and [8], fractional integral operators [9] and Poisson-Szegö integrals [10]. See also [11], [12], [13], [14], [15] and [16].

1.2 Motivation

The Dunkl theory is a significant generalization of the classical Fourier analysis and the theory of special functions in several variables. In the late 70's, it became progressively clear that radial Fourier analysis on flat symmetric spaces and the theory of special functions in one variable, are closely related. Generally speaking, it turned out that spherical functions on one-dimensional flat symmetric spaces can be written in terms of the classical Bessel function.

In the 80's, several attempts were made to generalize the above connection to higher ranks. The motivation for this subject comes to some extent from the harmonic analysis on flat symmetric spaces and the growing interest in the theory of special functions of several variables. The major breakthrough came with the discovery of the so-called rational Dunkl operators introduced by Dunkl in [17]. These operators are commuting differential-reflection operators associated to finite reflection groups on some finite-dimensional Euclidean space. This discovery have led to a very rich Dunkl theory. The early contributions to this theory go back to Koornwinder [18], Heckman [19], Opdam [20], and Dunkl [17]. In a series of papers, Dunkl built up the framework for a theory of an integral transform in several variables related to reflection groups, called the Dunkl transform. Since then, this theory has attracted considerable attention as it embraces in a unified way harmonic analysis on flat symmetric spaces and the corresponding theory of spherical functions in several variables. See for instance, [21], [22], [23], [24] and [25].

Beside Fourier analysis and multivariable special functions, the Dunkl theory also has deep and fruitful interactions with algebra (degenerate Hecke algebras) and probability (Feller processes with jumps). An equally important motivation to study Dunkl operators originates in their relevance for the study of quantum many body systems of Calogero-Moser type. Recently, such models have gained considerable interest in mathematical physics. A good bibliography is contained in [26].

In 2012, the seminal paper [27] by Ben Saïd, Kobayashi and Ørsted gave a far reaching generalization of the Dunkl theory (and, in fact, of the entire Hermite semigroup of operators, of which the Dunkl transform was a part) by introducing a positive real parameter a as a deformation parameter of the Dunkl theory. See also, [28]. In particular, a (k, a)-generalized Fourier transform $\mathcal{F}_{k,a}$ has been constructed and acting on a concrete Hilbert function space deforming $L^2(\mathbb{R}^N)$. The parameter k is a multiplicity function coming from the Dunkl theory. The case $a = 2$ gives the known Dunkl Fourier analysis [17], while the case $a = 1$ gives a new framework and it is of particular interest, as it is related to the so-called Laguerre semigroup and the minimal unitary representations of $O(n+1; 2)$ in the work of Kobayashi and Mano [29] and [30]. This new setting built up in [27] and [28] by Ben Said, Kobayashi and Ørsted has attracted an increasing interest from international researchers, the literature bears witness, e.g., [31], [32], [33], [34], [35], [36], [37] and [38]. Several questions were addressed at length in several papers but many additional problems were left unsolved.

In this thesis, we present a challenging problem which fit into the above described line of research associated to $\mathcal{F}_{k,a}$ when the dimension $N = 1$ and the parameter $a = \frac{2}{n}$ where $n \in \mathbb{N}_{>0}$. More precisely we will introduce and then study a generalized Hardy-Littlewood maximal operator $\mathcal{M}_{k,a}$ in the rank one case and with $a = \frac{2}{n}$.

1.3 Framework and Results

In this thesis we are concerned with the case $N = 1$, $a > 0$, and $k \geqslant 0$. Assume that $2k > 1 - a$. The (k, a)-generalized Fourier transform $\mathcal{F}_{k,a}$ takes the form

$$\mathcal{F}_{k,a}f(y) = \int_{\mathbb{R}} f(x)B_{k,a}(x, y)d\mu_{k,a}$$

for $f \in L^1(\mathbb{R}, d\mu_{k,a})$, where

$$d\mu_{k,a} = 2^{-1}a^{-((2k-1)/a)}|x|^{2k+a-2}dx$$

and

$$B_{k,a}(x, y) = \left(\widetilde{J}_{(2k-1)/a}\left(\frac{2}{a}|xy|^{a/2}\right) + \frac{xy}{(ai)^{2/a}}\widetilde{J}_{(2k+1)/a}\left(\frac{2}{a}|xy|^{a/2}\right)\right).$$

Here $\widetilde{J}_\nu$ denotes the normalized Bessel function

$$\widetilde{J}_\nu(\omega) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \omega^{2\ell}}{2^{2\ell}\ell!\gamma(\nu + \ell + 1)}.$$

It is worth mentioning that $\mathcal{F}_{k,a}$ includes:

- the classical Fourier transform [39] ($k = 0$ and $a = 2$),
- the Dunkl transform [17] (k arbitrary and $a = 2$),
- the Hankel transform [29] and [30] ($k = 0$ and $a = 1$),
- the k-Hankel transform [35] (k arbitrary and $a = 1$).

As the Euclidean translation $\tau_x : f \mapsto f(\cdot + x)$ plays a crucial role in Fourier analysis, it is natural to define a generalized translation operator by means of the transform $\mathcal{F}_{k,a}$.

In 2020, Boubatra, Negzaoui and Sifi [36] were able to prove the following product formula for the kernel $B_{k,a}(x,\xi)$ when $a = \frac{2}{n}$, with $n \in \mathbb{N}$

$$B_{k,a}(x,\xi)\, B_{k,a}(y,\xi) = \int_{\mathbb{R}} B_{k,a}(z,\xi)\, \mathcal{K}_{k,a}(x,y,z)\, d\mu_{k,a}(z), \tag{1.3.1}$$

where $\mathcal{K}_{k,a}(x,y,\,.)$ is a compactly supported kernel. The above product formula had been previously proved for $n = 1$ by Rösler [40] and for $n = 2$ by Ben Saïd [33]. In view of the given product formula, the appropriate translation operator for the transform $\mathcal{F}_{k,\frac{2}{n}}$ will be

$$\tau_x^{k,n} f(y) = \int_{\mathbb{R}} f(z)\, \mathcal{K}_{k,n}(x,y,z)\, d\mu_{k,n}(z).$$

In particular,

$$\mathcal{F}_{k,a}\left(\tau_x^{k,a} f\right)(\xi) = B_{k,a}(x,\xi)\, \mathcal{F}_{k,a}(f)(\xi).$$

By the Plancherel theorem for $\mathcal{F}_{k,a}$ together with the fact $|B_{k,a}(x,y)| \leq C$, we immediately deduce that $\tau_x^{k,a}$ is bounded on $L^2(\mathbb{R}, d\mu_{k,a})$. The $L^p(\mathbb{R}, d\mu_{k,a})$-boundedness of the generalized translation operator for $p \geq 1$ and $p \neq 2$ was proved in [36]. We pin down that for arbitrary dimension (i.e. for $\mathbb{R}^N$) and for $a = 1$, the generalized translation operator was recently investigated in [34].

The main goal of this thesis is to introduce and study the generalized Hardy-Littlewood maximal operator $\mathcal{M}_{k,n}$ defined by

$$\mathcal{M}_{k,n} f(x) = \sup_{r>0} \frac{1}{\mu_{k,n}(]-r,r[)} \left| \int_{\mathbb{R}} f(y) \tau_x^{k,n} \chi_r(y) d\mu_{k,n}(y) \right|, \qquad x \in \mathbb{R},$$

which reduces to the classical maximal operator $\mathcal{M}$ when the parameter $k = 0$ and $n = 1$ (up to an absolute value). The study of $\mathcal{M}_{k,n}$ not only contains intrinsic interest but it opens potentially interesting studies such as singular integral operators associated with $\mathcal{F}_{k,n}$, for instance.

The main result is to establish the analogue of the inequalities (1.1.1) and (1.1.2) for $\mathcal{M}_{k,n}$. It is worth mentioning that the obscure structure of the translation operator $\tau_x^{k,n}$, mainly the kernel $\mathcal{K}_{k,n}(x,y,z)$ in (1.3.1), makes the study of the maximal operators $\mathcal{M}_{k,n}$ difficult. However, it was proved that for $k > \frac{n-1}{2n}$ where $n \in \mathbb{N}$,

$$\mu_{k,n}\Big(\Big\{x \in \mathbb{R} : \mathcal{M}_{k,n}f(x) > \lambda\Big\}\Big) \lesssim \|f\|_{k,1} \tag{1.3.2}$$

for every $f \in L^1(\mathbb{R}, d\mu_{k,n})$ and for every $\lambda > 0$. This is the so-called weak $(1,1)$ type inequality. Further, for every $f \in L^p(\mathbb{R}, d\mu_{k,n})$ with $1 < p \leq \infty$,

$$\|\mathcal{M}_{k,n}f\|_{k,p} \lesssim \|f\|_{k,p}, \tag{1.3.3}$$

the so-called strong type (p,p) inequality.

Even though we have some information about the translation operator $\tau_x^{k,n}$, it was impossible to prove the above main inequalities directly. One way to do it is to construct a more handy maximal operator $\mathbb{M}_{k,n}$ which will control the Hardy-Littlewood maximal operator $\mathcal{M}_{k,n}$ in the sense that

$$\mathcal{M}_{k,n}f(x) \lesssim \mathbb{M}_{k,n}f(x). \tag{1.3.4}$$

As we shall see, our strategy of constructing the more convenient maximal operator $\mathbb{M}_{k,n}$ follows from the fact that we have to bypass some problems occurring with the structure of the translation operators and preventing us from proceeding directly by standard techniques.

In order to construct the operator $\mathbb{M}_{k,n}$, the main idea is to eliminate finely the translation operator. For the construction, the main idea is to introduce for $x \in \mathbb{R}$ and $r > 0$, the intervals

$$I(x,r) = \Big]\Big(max\{0, |x|^{\frac{1}{n}} - r^{\frac{1}{n}}\}\Big)^n, \Big(|x|^{\frac{1}{n}} + r^{\frac{1}{n}}\Big)^n\Big[,$$

and then to prove the following sharp estimate

$$|\tau_x^{k,n}(\chi_r)(y)| \lesssim \frac{\mu_{k,n}(]-r,r[)}{\mu_{k,n}(I(x,r))}. \tag{1.3.5}$$

This result generalizes an estimate proved by Bloom and Xu [41] in the framework of one-dimensional Bessel-Kingman hypergroups. The above sharp inequality plays a crucial role since it allowed to construct the maximal operator $\mathbb{M}_{k,n}$. More precisely, we are naturally brought to consider the following operator

$$\mathbb{M}_{k,n}f(x) = \sup_{r>0} \frac{1}{\mu_{k,n}(I(x,r))} \int_{\{y\in\mathbb{R}:|y|\in I(x,r)\}} |f(y)| d\mu_{k,n}(y).$$

Therefore the inequality (1.3.4) becomes obvious. Hence, the weak-type $(1,1)$ and the strong (p,p) estimates for $\mathcal{M}_{k,n}$ follow from the one for $\mathbb{M}_{k,n}$.

The proof of the weak-type $(1,1)$ estimate for $\mathbb{M}_{k,n}$ depends heavily on a covering lemma of Vitali-type for the intervals $I(x,r)$. We provide a proof of the lemma to highlight the non-obvious doubling property of the measure $\mu_{k,n}$ and the engulfing property of the intervals $I(x,r)$. More precisely, it is proved that

$$\mu_{k,n}(I(x,2r)) \lesssim \mu_{k,n}(I(x,r)) \qquad \forall x \in \mathbb{R},\ r>0,$$

and that if E is a $\mu_{k,n}$-measurable subset in $\mathbb{R}_+^*$ and covered by a finite collection of intervals $\{I(x_i,r_i)\}_{1\le i\le L}$ covering E, then there exists a disjoint subcollection $I(x_{m_1},r_{m_1}),\ldots,I(x_{m_\ell},r_{m_\ell})$ that satisfies

$$\mu_{k,n}(E) \lesssim \sum_{i=1}^{\ell} \mu_{k,n}(I(x_{m_i},r_{m_i})). \tag{1.3.6}$$

Finally, by means of Marcinkiewicz interpolation, we get the assertion that $\mathbb{M}_{k,n}$, and therefore $\mathcal{M}_{k,n}$, is strong-type (p,p) for all $1<p\le\infty$.

In Chapter 1 we will recall some basic properties of the L^p spaces. For instance, we will give the proofs of some important inequalities such as Hölder's inequality and Minkowski's inequality. Furthermore, we give the proof of the Marcinkiewicz interpolation theorem since it will be a greatly important tool that will help us in the process of achieving the main result of our thesis.

Chapter 2 gives a brief introduction to the theory of Dunkl operators. We build this theory from the foundation of it, that is root systems, finite reflection groups or (Coxeter groups) and multiplicity functions. Building on these concepts, we define Dunkl operators. Further, we shall introduce the so-called Dunkl intertwining operator which is used to define the kernel $B_{k,a}(x, y)$ appearing in the integral transform $\mathcal{F}_{k,a}$.

Chapter 3 starts to shed a light on our framework as it introduces the kernel $B_{k,a}(x, y)$ and some of its main properties. For instance, the boundedness of $B_{k,a}(x, y)$, which will be of a particular importance to define the (k, a)-generalized Fourier transform $\mathcal{F}_{k,a}$. Due to the significant role of $\mathcal{F}_{k,a}$ in the proof of the main result of this thesis, we list some of its properties including the inversion formula and the Plancherel theorem for $\mathcal{F}_{k,a}$.

Our main result relies heavily on what we discuss in chapter 4, where we will assume that $a = \frac{2}{n}$. We introduce the generalized translation operator $\tau_x^{k,n}$. In particular the boundedness of $\tau_x^{k,n}$ from $L^p(\mathbb{R}, d\mu_{k,n})$ into itself. By means of the generalized translation operator, a convolution structure was defined such that

$$\mathcal{F}_{k,n}(f \star_{k,n} g)(\lambda) = \mathcal{F}_{k,n}(f)(\lambda)\mathcal{F}_{k,n}(g)(\lambda).$$

In the last chapter, we will be defining and studying the generalized Hardy–Littlewood maximal operator $\mathcal{M}_{k,n}$ defined above. One of the main results is the Vitali type covering lemma 1.3.6. Further, a detailed construction of $\mathbb{M}_{k,n}$ and the sharp inequality 1.3.5 are given. We close this chapter by the weak type $(1, 1)$ and the strong (p, p) inequalities for the uncentered maximal operator $\mathbb{M}_{k,n}$, which lead to the main goal of this thesis (inequalities 1.3.2 and 1.3.3). The main result of this thesis was proved by Deleaval for $n = 1$ [42] and by Ben Said and Deleaval for $n = 2$ [35].

Chapter 2: The L^p Spaces and the Marcinkiewicz Interpolation Theorem

2.1 Introduction

In this chapter, we will recall some basic facts about L^p spaces as well as some important inequalities like Hölder's, and Minkowski's inequalities as well as their proofs. Further we will give he Marcinkiewicz interpolation theorem and its proof. This interpolation theorem will play a crucial role in the main result of this thesis. The main references are [43], [44], and [45].

2.2 L^p Spaces

Definition 2.2.1. *Consider a measure space* $(X, \mathcal{M}, \mu)$ *and assume that* $0 < p < \infty$. *Let* $f : X \to \mathbb{R}$ *be a measurable function, then we define*

$$\|f\|_{L^p} := \left(\int_X |f(x)|^p dx\right)^{\frac{1}{p}}$$

and

$$\|f\|_{L^\infty} := \operatorname*{ess\,sup}_{x\in X} |f(x)|.$$

Definition 2.2.2. *The space* $L^p(X)$ *is the set*

$$L^p(X) = \left\{ f : X \to \mathbb{R} \,\middle|\, \|f\|_{L^p} < \infty \right\}.$$

The space $L^p(X)$ satisfies the following vector space properties:

Properties 2.2.3 ([44])**.** *(1) For each* $\alpha \in \mathbb{R}$, *if* $f \in L^p(X)$ *then* $\alpha f \in L^p(X)$.

(2) If $f, g \in L^p(X)$, *then* $f + g \in L^p(X)$ *since,*

$$|f(x) + g(x)|^p \leqslant 2^{p-1} \left(|f(x)|^p + |g(x)|^p\right).$$

(3) If $p \geqslant 1$, *then the triangle inequality*

$$\|f + g\|_{L^p}^p \leqslant \|f\|_{L^p}^p + \|g\|_{L^p}^p$$

holds.

Definition 2.2.4. *Let $p \geqslant 1$ and define q such that*

$$\frac{1}{q} + \frac{1}{p} = 1.$$

Then, p and q are called conjugate exponents.

Theorem 2.2.5 (Hölder's inequality [44])**.** *Assume that $1 \leqslant p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$, and that p and q are conjugate exponents. If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$. In fact we have,*

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Note that in the case where $p = q = 2$, Hölder's inequality becomes exactly the Cauchy-Schwarz inequality as follows

$$\int |f(x)g(x)|dx \leqslant \left(\int |f(x)|^2 dx \right)^{\frac{1}{2}} \left(\int |g(x)|^2 dx \right)^{\frac{1}{2}}.$$

To prove this theorem we will need the following two lemmas.

Lemma 2.2.6. *For $\lambda \in (0, 1)$, the following inequality is true,*

$$x^\lambda \leq (1 - \lambda) + \lambda x.$$

Proof. Consider $f(x) = (1 - \lambda) + \lambda x - x^\lambda$, therefore,

$$f'(x) = \lambda - \lambda x^{\lambda - 1} = \lambda \left(1 - x^{\lambda - 1}\right)$$

takes the value zero only in the case that either $\lambda \in \{0, 1\}$, which is not possible since $\lambda \in (0, 1)$, or $\left(1 - x^{\lambda - 1}\right) = 0$. Which means that $x = 1$ is the critical point of f. Specifically, the minimum occurs at $x = 1$ with the value

$$f(1) = 0 \leq (1 - \lambda) + \lambda x - x^\lambda.$$

□

Lemma 2.2.7. *Let $\lambda \in (0,1)$, and let us assume that $a, b \geqslant 0$. Then the following inequality holds*

$$a^{\lambda} b^{1-\lambda} \leqslant \lambda a + (1-\lambda) b.$$

Proof. It is clear in the case that either $a = 0$ or $b = 0$, therefore let us consider the case that $a, b > 0$. Let $x = \frac{a}{b}$, then, using Lemma 2.2.6 we get

$$\left(\frac{a}{b}\right)^{\lambda} \leqslant (1-\lambda) + \lambda \left(\frac{a}{b}\right)$$
$$a^{\lambda} b^{1-\lambda} \leqslant (1-\lambda) b + \lambda a,$$

which is the desiered inequality. □

Now let us prove the Hölder's inequality in Theorem (2.2.5).

Proof. Let

$$\lambda = \frac{1}{p}, \quad a = \frac{|f(x)|^p}{\|f\|_{L^p}^p}, \quad \text{and} \quad b = \frac{|g(x)|^q}{\|g\|_{L^p}^q}$$

for all $x \in X$. Then,

$$a^{\lambda} b^{1-\lambda} = a^{1/p} b^{1-1/p} = a^{1/p} b^{1/q}.$$

Using Lemma 2.2.7, we get,

$$\frac{|f(x)| \cdot |g(x)|}{\|f\|_{L^p} \|g\|_{L^q}} \leqslant \frac{1}{p} \frac{|f(x)|^p}{\|f\|_{L^p}^p} + \frac{1}{q} \frac{|g(x)|^q}{\|g\|_{L^q}^q}.$$

Integrating both sides gives us

$$\int_X \frac{|f(x)| \cdot |g(x)|}{\|f\|_{L^p} \|g\|_{L^q}} dx \leqslant \int_X \left(\frac{1}{p} \frac{|f(x)|^p}{\|f\|_{L^p}^p} + \frac{1}{q} \frac{|g(x)|^q}{\|g\|_{L^q}^q} \right) dx = \frac{1}{p} + \frac{1}{q} = 1.$$

□

Theorem 2.2.8 (Minkowski's inequality [44])**.** *Let $1 \leqslant p \leqslant \infty$ and assume that $f, g \in L^p$, then,*

$$\|f + g\|_{L^p} \leqslant \|f\|_{L^p} + \|g\|_{L^p}.$$

Proof. If we assume that $f+g=0$ a.e., then the statement is trivial. So, let us assume that $f+g\neq 0$ a.e. Then, we consider the following

$$\begin{aligned}|f(x)+g(x)|^p &= |f(x)+g(x)||f(x)+g(x)|^{p-1}\\ &\le (|f(x)|+|g(x)|)|f(x)+g(x)|^{p-1}.\end{aligned}$$

Now, we integrate both sides over X.

$$\begin{aligned}\int_X |f(x)+g(x)|^p dx &\leqslant \int_X \left[(|f(x)|+|g(x)|)|f(x)+g(x)|^{p-1}\right] dx\\ &\leqslant (\| f \|_{L^p}+\| g\|_{L^p}) \left\| |f+g|^{p-1}\right\|_{L^q}.\end{aligned}$$

Since $q=\frac{p}{p-1}$,

$$\left\| |f+g|^{p-1}\right\|_{L^q} = \left(\int_X |f(x)+g(x)|^p dx\right)^{\frac{1}{q}},$$

consequently,

$$\begin{aligned}\left(\int_X |f(x)+g(x)|^p dx\right)^{1-\frac{1}{q}} &\leqslant \|f\|_{L^p}+\|g\|_{L^q},\\ \left(\int_X |f(x)+g(x)|^p dx\right)^{\frac{1}{p}} &\leqslant \|f\|_{L^p}+\|g\|_{L^q},\\ \|f+g\|_{L^p} &\leqslant \|f\|_{L^p}+\|g\|_{L^q},\end{aligned}$$

which concludes the proof. □

Corollary 2.2.9. *For $1\leqslant p\leqslant\infty$, $L^p(X)$ is a normed vector space.*

2.3 The Marcinkiewicz Interpolation Theorem

Definition 2.3.1. *Recall that $(X,\mathcal{M},\mu)$ is a measurable space, and f is a measurable function on X. Let us define the distribution function $\lambda_f:(0,\infty)\longrightarrow[0,\infty]$ by*

$$\lambda_f(\alpha)=\mu\Big(\{x:|f(x)|>\alpha\}\Big).$$

Definition 2.3.2. *Let $0<p\leqslant\infty$ and assume that f is a measurable function on X. We define $[f]_p$ as*

$$[f]_p=\left(\sup_{\alpha>0}\alpha^p\lambda_f(\alpha)\right)^{\frac{1}{p}}$$

for $0 < p < \infty$, *and*

$$[f]_\infty = \|f\|_{L_\infty}.$$

Definition 2.3.3. *Consider the set of all* $f \in (X, \mathcal{M}, \mu)$ *where* $[f(x)]_p < \infty$. *We denote this set by weak* $\mathbf{L}^p$.

Notice that:

(1) $[f(x)]_p \leqslant \|f\|_{L^p}$, which implies $L^p \subseteq$ weak $\mathbf{L}^p$.

(2) For $0 < p < \infty$, $[\,\cdot\,]_p$ does not satisfy the triangle inequality, therefore is not a norm.

Definition 2.3.4. *Assume that* U *and* V *are vector spaces. A map* $T : U \longrightarrow V$ *is called sublinear if it satisfies*

$$|T(cf)(x)| = |c||T(f)(x)|$$

for all $c \in \mathbb{R}$, *and*

$$|T(f+g)(x)| \leq |T(f)(x)| + |T(g)(x)|$$

Definition 2.3.5. *Let* T *be a map from some vector space* V *of measurable functions on* $(X, \mathcal{M}, \mu)$ *to the space of all measurable functions on* $(Y, \mathcal{N}, \nu)$. *A sublinear map* T *is said to be strong type* (p, q), *for* $1 \leqslant p \leqslant \infty$ *and* $1 \leqslant q \leqslant \infty$, *if*

(1) $L^p(X, d\mu) \subseteq V$.

(2) T *maps* $L^p(X, d\mu)$ *into* $L^q(Y, d\nu)$.

(3) There exists a constant $c > 0$ *so that*

$$\|T(f)\|_q \leqslant c\|f\|_{L^p}$$

for all $f \in L^p(X, d\mu)$.

Definition 2.3.6. *A sublinear map T is said to be weak type (p,q), for $1 \leqslant p \leqslant \infty, 1 \leqslant q < \infty$, if*

(1) $L^p(X, d\mu) \subseteq V$.

(2) *T maps $L^p(X, d\mu)$ into weak $\mathbf{L}^q$.*

(3) *There exists a constant $c > 0$ such that*

$$[T(f)]_q \leqslant c\|f\|_{L^p}$$

for all $f \in L^p(X, d\mu)$.

Now we will state the Marcinkiewicz interpolation theorem and its proof. For this purpose, let us denote the set of all μ-measurable functions on a vector space U by $\mathcal{M}(U, \mu)$.

Theorem 2.3.7 ([45]). *Let $1 \leqslant p_0 < p_1 \leqslant \infty$ and assume that $T : \mathcal{M}(U, \mu) \longrightarrow \mathcal{M}(V, \nu)$ is a sublinear map of weak type (p_0, p_0) and (p_1, p_1). Then, for every $p_0 < p < p_1$, the operator T is of strong type (p, p).*

In order to prove this theorem, we will need the following two lemmas.

Lemma 2.3.8. *If $f \in L^p$, where $1 \leqslant p < \infty$, we have*

(1) *For every $\alpha > 0$, λ_f satisfies*

$$\lambda_f(\alpha) \leqslant \frac{\|f\|_{L^p}^p}{\alpha^p}.$$

(2) *We may rewrite $\|f\|_{L^p}^p$ as*

$$\|f\|_{L^p}^p = p \int_0^\infty t^{p-1} \lambda_f(t) dt.$$

Proof. (1) From the definition in (2.2.1) we have

$$\begin{aligned}
\|f\|_{L^p}^p &= \int_X |f(x)|^p d\mu(x) \\
&\geqslant \int_{\{x\in X:|f(x)|>\alpha\}} |f(x)|^p d\mu(x) \\
&\geqslant \int_{\{x\in X:|f(x)|>\alpha\}} \alpha^p d\mu(x) \\
&= \alpha^p \mu\Big(\Big\{x \in X : |f(x)| > \alpha\Big\}\Big) \\
&= \alpha^p \lambda_f(\alpha).
\end{aligned}$$

(2) Let $A = \{(x,s) : |f(x)|^p > s\}$. Then,

$$\begin{aligned}
\int_X |f(x)|^p d\mu(x) &= \int_X \int_0^{|f(x)|^p} ds d\mu(x) \\
&= \int_{X\times[0,\infty)} 1_A(x,s) ds d\mu(x) \\
&= \int_0^\infty \int_X 1_A(x,s) d\mu(x) ds \\
&= \int_0^\infty \mu\Big(\{x \in X : |f(x)|^p > s\} \Big) ds \\
&= \int_0^\infty \mu\Big(\{x \in X : |f(x)|^p > t^p\} \Big) pt^{p-1} dt \\
&= p\int_0^\infty t^{p-1} \mu\Big(\{x \in X : |f(x)| > t\}\Big) dt \\
&= p\int_0^\infty t^{p-1}\lambda_f(t) dt.
\end{aligned}$$

□

Lemma 2.3.9. *If* $f \in L^p$*, where* $1 \leqslant p < \infty$*, then*

$$\int_{\{x:|f(x)|>\alpha\}} |f(x)|^p d\mu(x) = \lambda_f(\alpha)\alpha^p + p\int_\alpha^\infty t^{p-1}\lambda_f(t) dt,$$

and

$$\int_{\{x:|f(x)|\leqslant\alpha\}} |f(x)|^p d\mu(x) = -\lambda_f(\alpha)\alpha^p + p\int_0^\alpha t^{p-1}\lambda_f(t) dt.$$

Proof. (1) Let $A = \{(x,s) : |f(x)|^p > s\}$, then

$$\begin{aligned}
&\int_{\{x:|f(x)|>\alpha\}} |f(x)|^p d\mu(x) \\
&\qquad = \int_{\{x:|f(x)|>\alpha\}} \int_0^{|f(x)|^p} ds d\mu(x) \\
&\qquad = \int_{\{x:|f(x)|>\alpha\}} \int_0^{\alpha^p} ds d\mu(x) + \int_{\{x:|f(x)|>\alpha\}} \int_{\alpha^p}^{|f(x)|^p} ds d\mu(x) \\
&\qquad = \alpha^p \int_{\{x:|f(x)|>\alpha\}} d\mu(x) + \int_{X\times[\alpha^p,\infty)} 1_A(x,s) ds d\mu(x) \\
&\qquad = \alpha^p \mu\Big(\{x : |f(x)| > \alpha\}\Big) + \int_{X\times[\alpha^p,\infty)} 1_A(x,s) d\mu(x) ds \\
&\qquad = \alpha^p \lambda_f(\alpha) + \int_{\alpha^p}^{\infty} \mu\left(\{x : |f(x)|^p > s\}\right) ds \\
&\qquad = \alpha^p \lambda_f(\alpha) + \int_{\alpha}^{\infty} \mu\left(\{x : |f(x)|^p > t^p\}\right) p t^{p-1} dt \\
&\qquad = \alpha^p \lambda_f(\alpha) + p \int_{\alpha}^{\infty} t^{p-1} \lambda_f(t) dt.
\end{aligned}$$

$$\begin{aligned}
(2)\ &\int_{\{x:|f(x)|\leqslant\alpha\}} |f(x)|^p d\mu(x) \\
&\qquad = \int_X |f(x)|^p d\mu(x) - \int_{\{x:|f(x)|>\alpha\}} |f(x)|^p d\mu(x) \\
&\qquad = p \int_0^{\infty} t^{p-1} \lambda_f(t) dt - \alpha^p \lambda_f(\alpha) - p \int_{\alpha}^{\infty} t^{p-1} \lambda_f(t) dt.
\end{aligned}$$

Then,

$$\int_{\{x:|f(x)|\leqslant\alpha\}} |f(x)|^p d\mu(x) = -\lambda_f(\alpha)\alpha^p + p \int_0^{\alpha} t^{p-1} \lambda_f(t) dt.$$

□

Now, let us prove the Marcenkiewics interpolation Theorem (2.3.7).

Proof. Let $f \in L^p$ and define

$$g_\alpha(x) = \begin{cases} f(x) & \text{if} \quad |f(x)| \leqslant \alpha \\ 0 & \text{if} \quad |f(x)| > \alpha \end{cases}$$

and

$$h_\alpha(x) = \begin{cases} 0 & \text{if} \quad |f(x)| \leqslant \alpha \\ f(x) & \text{if} \quad |f(x)| > \alpha \end{cases}$$

To complete the proof we have to prove the following claim.

Claim: $g_\alpha \in L^{p_1}$ and $h_\alpha \in L^{p_0}$.

$$
\begin{aligned}
\|g_\alpha(x)\|_{L^{p_1}}^{p_1} = \int_X |g_\alpha(x)|^{p_1}\, d\mu(x) &= \alpha^{p_1} \int_X \left| \frac{g_\alpha(x)}{\alpha} \right|^{p_1} d\mu(x) \\
&\leqslant \alpha^{p_1} \int_X \left| \frac{g_\alpha(x)}{\alpha} \right|^{p} d\mu(x) \\
&\leqslant \alpha^{p_1 - p} \int_X |f(x)|^p d\mu(x) \\
&< \infty.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\|h_\alpha\|_{L^{p_0}}^{p_0} = \int_X |h_\alpha(x)|^{p_0}\, d\mu(x) &= \alpha^{p_0} \int_X \left| \frac{h_\alpha(x)}{\alpha} \right|^{p_0} d\mu(x) \\
&\leqslant \alpha^{p_0} \int_X \left| \frac{h_\alpha(x)}{\alpha} \right|^{p} d\mu(x) \\
&\leqslant \alpha^{p_0 - p} \int_X |f(x)|^p d\mu(x) \\
&< \infty.
\end{aligned}
$$

Therefore, $f(x) = g_\alpha(x) + h_\alpha(x)$ and, since T is sublinear, we have

$$|T\left(f(x)\right)| \leq |T\left(g_\alpha(x)\right)| + |T\left(h_\alpha(x)\right)|.$$

Suppose $|T(f(x))| > \alpha$. Then

$$|T\left(g_\alpha(x)\right)| + |T\left(h_\alpha(x)\right)| > \alpha,$$

which implies that either $|T\left(g_\alpha(x)\right)| > \frac{\alpha}{2}$, or $|T\left(h_\alpha(x)\right)| > \frac{\alpha}{2}$. Therefore,

$$\{x : |T(f(x))| > \alpha\} \subseteq \left\{x : |Tg_\alpha(x)| > \frac{\alpha}{2}\right\} \cup \left\{x : |Th_\alpha(x)| > \frac{\alpha}{2}\right\}.$$

Hence,

$$\lambda_{Tf}(\alpha) \leqslant \lambda_{Tg_\alpha}\left(\frac{\alpha}{2}\right) + \lambda_{Th_\alpha}\left(\frac{\alpha}{2}\right).$$

Since T is weak (p_1, p_1), there exists a constant $c > 0$ so that $[T(g_\alpha)]_{p_1} \leqslant c \|g_\alpha\|_{L^{p_1}}$, in particular,

$$\left(\sup_{\alpha'>0} \alpha'^{p_1} \lambda_{Tg_\alpha}(\alpha')\right)^{1/p_1} \leqslant c \left(\int_X |g(x)|^{p_1} d\mu(x)\right)^{1/p_1},$$

therefore we get

$$\left(\frac{\alpha}{2}\right)^{p_1} \lambda_{Tg_\alpha}\left(\frac{\alpha}{2}\right) \leqslant c_1 \int_X |g(x)|^{p_1} d\mu(x).$$

Consequently,

$$\begin{aligned}\lambda_{Tg_\alpha}\left(\frac{\alpha}{2}\right) &\leqslant c_1 \left(\frac{2}{\alpha}\right)^{p_1} \int_X |g(x)|^{p_1} d\mu(x)\\ &= c_1 \left(\frac{2}{\alpha}\right)^{p_1} \int_{\{x|f(x)|\leqslant\alpha\}} |f(x)|^{p_1} d\mu(x).\end{aligned}$$

Similarly, since T is weak (p_0, p_0), we obtain

$$\lambda_{Th_\alpha}\left(\frac{\alpha}{2}\right) \leqslant c_2 \left(\frac{2}{\alpha}\right)^{p_1} \int_{\{x:|f(x)|>\alpha\}} |f(x)|^{p_0} d\mu(x).$$

Therefore we have the following

$$\begin{aligned}\|T(f)\|_{L^p}^p &= p\int_0^\infty \alpha^{p-1}\lambda_{Tf}(\alpha)d\alpha\\ &\leqslant p\int_0^\infty \alpha^{p-1}\lambda_{Tg_\alpha}\left(\frac{\alpha}{2}\right)d\alpha + p\int_0^\infty \alpha^{p-1}\lambda_{Th_\alpha}\left(\frac{\alpha}{2}\right)d\alpha\\ &\leqslant c_1 p 2^{p_1}\int_0^\infty \frac{\alpha^{p-1}}{\alpha^{p_1}}\int_{\{x=|f(x)|\leqslant\alpha\}} |f(x)|^{p_1} d\mu(x)d\alpha\\ &\quad + c_2 p 2^{p_0}\int_0^\infty \frac{\alpha^{p-1}}{\alpha^{p_0}}\int_{\{x:|f(x)|>\alpha\}} |f(x)|^{p_0} d\mu(x)d\alpha\end{aligned}$$

and by Lemma 2.3.9, we get

$$\begin{aligned}
\|T(f)\|_{L^p}^p &\leqslant c_1 p 2^{p_1} \int_0^\infty \alpha^{p-p_1-1} \left(-\lambda_f(\alpha)\alpha^{p_1} + p_1 \int_0^\alpha t^{p_1-1}\lambda_f(t)dt \right) d\alpha \\
&\quad + c_2 p 2^{p_0} \int_0^\infty \alpha^{p-p_0-1} \left(\lambda_f(\alpha)\alpha^{p_0} + p_0 \int_\alpha^\infty t^{p_0-1}\lambda_f(t)dt \right) d\alpha \\
&\leqslant c_1 p_1 p 2^{p_1} \int_0^\infty \alpha^{p-p_1-1} \int_0^\alpha t^{p_1-1}\lambda_f(t)dtd\alpha \\
&\quad + pc_2 2^{p_0} \int_0^\infty \alpha^{p-1}\lambda_f(\alpha)d\alpha \\
&\quad + pp_0 c_2 2^{p_0} \int_0^\infty \alpha^{p-p_0-1} \int_\alpha^\infty t^{p_0-1}\lambda_f(t)dtd\alpha.
\end{aligned}$$

That is,

$$\begin{aligned}
\|T(f)\|_{L^p}^p &= c_1 p_1 p 2^{p_1} \int_0^\infty \int_t^\infty \alpha^{p-p_1-1} t^{p_1-1}\lambda_f(t)d\alpha dt + c_2 2^{p_0}\|f\|_{L^p}^p \\
&\quad + pp_0 c_2 2^{p_0} \int_0^\infty \int_0^t \alpha^{p-p_0-1} t^{p_0-1}\lambda_f(t)d\alpha dt \\
&= c_2 2^{p_0}\|f\|_{L^p}^p + c_1 p_1 p 2^{p_1} \int_0^\infty t^{p_1-1}\lambda_f(t) \int_t^\infty \alpha^{p-p_1-1} d\alpha dt \\
&\quad + pp_0 c_2 2^{p_0} \int_0^\infty t^{p_0-1}\lambda_f(t) \int_0^t \alpha^{p-p_0-1} d\alpha dt \\
&= c_2 2^{p_0}\|f(t)\|_{L^p}^p + c_1 p_1 p 2^{p_1} \int_0^\infty \frac{1}{p_1 - p} t^{p_1-1}\lambda_f(t)dt \\
&\quad + pp_0 c_2 2^{p_0} \int_0^\infty \frac{1}{p - p_0} t^{p_0-1}\lambda_f(t)dt \\
&= c_3\|f\|_{L^p}^p + c_4\|f\|_{L^p}^p + c_5\|f\|_{L^p}^p \\
&= c_6\|f\|_{L^p}^p.
\end{aligned}$$

The case when $p_1 = \infty$ is similar by taking

$$g_\alpha(x) = \begin{cases} f(x) & \text{if} \quad |f(x)| \leqslant \frac{\alpha}{2c} \\ \frac{\alpha}{2c}\,\mathrm{sign}(f(x)) & \text{if} \quad |f(x)| > \frac{\alpha}{2c} \end{cases}$$

and

$$h_\alpha(x) = f(x) - g_\alpha(x),$$

for some positive constant c. □

Chapter 3: Dunkl Operators

3.1 Introduction

The main goal of this chapter is to provide an introduction to the theory of Dunkl operators, and to give some of their properties. The main reference of the chapter is [40]. General references are [46], [47], [48] and [49]. An introduction to reflection groups and root systems can be found in [50] and [51]. We do not intend to give a complete survey, but rather focus on those aspects which will be important in the context of this thesis.

3.2 Root Systems and Coxeter Groups

Consider $(\mathbb{R}^N, \langle\cdot,\cdot\rangle)$, with the scalar product $\langle x, y\rangle = \sum_{j=1}^{N} x_j y_j$. We define the reflection σ_α for $\alpha \in \mathbb{R}^N \backslash \{0\}$ by

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{|\alpha|^2}\alpha,$$

where $|\alpha|^2 = \langle \alpha, \alpha\rangle$.

Definition 3.2.1 (Root systems)**.** *Let $R \subset \mathbb{R}^N \backslash \{0\}$ be a finite set. Then R is considered to be a root system, if it satisfies*

(1) $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in R$.

(2) $\sigma_\alpha(R) = R$ for all $\alpha \in R$.

From now on we assume that R is normalized in the sense that $\langle \alpha, \alpha\rangle = 2$ for all $\alpha \in R$; this simplifies formulas, but is no loss of generality for our purposes.

Definition 3.2.2 (The Coxeter group)**.** *The reflection group or Coxeter group G is the finite group generated by all the reflections $\{\sigma_\alpha, \alpha \in R\}$.*

Example 3.2.3 ([40])**.** *Root system of type A_{N-1}. Let the symmetric group in N elements be denoted by S_N. It acts on $\mathbb{R}^N$ by permuting the standard basis vectors $e_1, \ldots, e_N$. Each transposition (ij) acts as a reflection σ_{ij} sending $e_i - e_j$ to its negative. S_N is a finite reflection group since it is generated by transpositions. A* *root system of S_N is given by*

$$R = \{\pm (e_i - e_j), 1 \leqslant i < j \leqslant N\}.$$

Example 3.2.4 ([40])**.** *Root system of type B_N. Here G is the reflection group in $\mathbb{R}^N$ generated by the transpositions σ_{ij} just like in type A_{N-1} in the previous example, together with the sign changes $\sigma_i : e_i \mapsto -e_i$, for $i \in \{1, \ldots, N\}$. The group of sign changes is isomorphic to $\mathbb{Z}_2^N$, intersects S_N trivially. Thus, G is isomorphic with the semidirect product $S_N \ltimes \mathbb{Z}_2^N$. The corresponding root system is given by*

$$R = \{\pm e_i, 1 \leqslant i \leqslant N, \pm (e_i \pm e_j), 1 \leqslant i < j \leqslant N\}.$$

3.3 Dunkl Operators

Let us introduce the multiplicity function where every root is paired with a certain parameter k.

Definition 3.3.1 (Multiplicity functions)**.** *A function $k : R \to \mathbb{R}^+$ on the root system R such that $k(g\alpha) = k(\alpha)$, $g \in G$ (we say invariant with respect to the Coxeter group) is called a multiplicity function on R. The vector space containing all multiplicity functions on R is denoted by K.*

Definition 3.3.2. *Let $k \in K$. Then for $\xi \in \mathbb{R}^N$, the Dunkl operator $T_\xi := T_\xi(k)$ is defined by*

$$T_\xi f(x) := \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}.$$

Here ∂_ξ denotes the directional derivative corresponding to ξ, and R_+ is a fixed positive subsystem of R. For the i-th standard basis vector $\xi = e_i \in \mathbb{R}^N$ we use the abbreviation $T_i = T_{e_i}$.

Notice that if f is G-invariant then

$$T_\xi f = \partial_\xi f.$$

Example 3.3.3 ([40])**.** *In the case $N = 1$, the root system becomes $R = \{\pm 1\}$ and $\sigma(x) = -x$. The Dunkl operator $T := T_1$ associated with the parameter $k \geq 0$ is given by*

$$Tf(x) = f'(x) + \frac{k}{x}(f(x) - f(-x)).$$

Example 3.3.4 ([40])**.** *Dunkl operators of type A_{N-1}. Suppose $G = S_N$ with root system of type A_{N-1}. As all transpositions σ_{ij} are conjugate in S_N, the vector space of multiplicity functions is one-dimensional. The Dunkl operators associated with the parameter $k \geq 0$ are given by*

$$T_i^S = \partial_i + k \cdot \sum_{j \neq i} \frac{1 - \sigma_{ij}}{x_i - x_j} \quad (i = 1, \ldots, N).$$

Example 3.3.5 ([40])**.** *Dunkl operators of type B_N. Suppose R is a root system of type B_N. There are two conjugacy classes of reflections in G, leading to multiplicity functions of the form $k = (k_0, k_1)$. The associated Dunkl operators are given by*

$$T_i^B = \partial_i + k_1 \frac{1 - \sigma_i}{x_i} + k_0 \cdot \sum_{j \neq i} \left[\frac{1 - \sigma_{ij}}{x_i - x_j} + \frac{1 - \tau_{ij}}{x_i + x_j} \right] \quad (i = 1, \ldots, N),$$

where $\tau_{ij} := \sigma_{ij}\sigma_i\sigma_j$.

Notation 3.3.6. *(1) C^m is the space consisting of all continuous functions that are also m differentiable.*

(2) $\mathscr{S}\left(\mathbb{R}^N\right)$ denotes the Schwartz space of rapidly decreasing functions on $\mathbb{R}^N$,

$$\mathscr{S}\left(\mathbb{R}^N\right) := \left\{ f \in C^\infty\left(\mathbb{R}^N\right) : \left\| x^\beta \partial^\alpha f \right\|_{L^\infty} < \infty \;\; \textit{for all} \;\; \alpha, \beta \in \mathbb{Z}_+^N \right\}.$$

The Dunkl operators T_ξ have the following properties

Properties 3.3.7 ([40])**.** *(1) If $f \in C^m\left(\mathbb{R}^N\right)$ with $m \geqslant 1$, then $T_\xi f \in C^{m-1}\left(\mathbb{R}^N\right)$.*

(2) T_ξ leaves $C_c^\infty\left(\mathbb{R}^N\right)$ and $\mathscr{S}\left(\mathbb{R}^N\right)$ invariant.

(3) If $f, g \in C^1\left(\mathbb{R}^N\right)$ and at least one of them is G-invariant, then

$$T_\xi(fg) = T_\xi(f) \cdot g + f \cdot T_\xi(g).$$

We state here another important property of the Dunkl operator T_ξ.

Theorem 3.3.8 ([40]). *Let w_k denote the weight function on $\mathbb{R}^N$ defined by*

$$w_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)}.$$

Then,

$$\int_{\mathbb{R}^N} T_\xi f(x) g(x) w_k(x) dx = - \int_{\mathbb{R}^N} f(x) T_\xi g(x) w_k(x) dx.$$

The most interesting property of the Dunkl operators, which is the foundation for rich analytic structures related with them, is the following

Theorem 3.3.9 ([40]). *For fixed k, the Dunkl operators $T_{\xi_i} = T_{\xi_i}(k)$, $\xi_i \in \mathbb{R}^N$, commute.*

We now state the Dunkl Laplacian Δ_k, which is defined by

$$\Delta_k := \sum_{j=1}^{N} T_i^2. \tag{3.3.1}$$

Theorem 3.3.10 ([40]).

$$\Delta_k = \Delta + 2 \sum_{\alpha \in R_+} k(\alpha) \delta_\alpha \quad \textit{with} \quad \delta_\alpha f(x) = \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle^2}$$

here Δ and ∇ denote the usual Laplacian and gradient respectively.

Example 3.3.11. *For $N = 1$, the Dunkl Laplacian is T^2 and is given by*

$$T^2 f(x) = f''(x) + \frac{2k}{x} f'(x) + \frac{k}{x^2}(f(-x) - f(x)),$$

where $k \geq 0$.

Proof.

$$\begin{aligned}
T^2 f(x) &= \left(f'(x) + \frac{k}{x}(f(x) - f(-x))\right)' + \frac{k}{x}\left(f'(x) + \frac{k}{x}(f(x) - f(-x))\right) \\
&\quad - \frac{k}{x}\left(f'(-x) + \frac{k}{-x}(f(-x) - f(x))\right) \\
&= f''(x) + \frac{k}{x}\left(f'(x) - f'(-x)\right) - \frac{k}{x^2}\left(f(x) - f(-x)\right) + \frac{k}{x}f'(x) \\
&\quad + \frac{k^2}{x^2}\left(f(x) - f(-x)\right) - \frac{k}{x}f'(-x) + \frac{k^2}{x^2}\left(f(-x) - f(x)\right) \\
&= f''(x) + \frac{k}{x}f'(x) - \frac{k}{x}f'(-x) - \frac{k}{x^2}f(x) + \frac{k}{x^2}f(-x) + \frac{k}{x}f'(x) \\
&\quad + \frac{k^2}{x^2}f(x) - \frac{k^2}{x^2}f(-x) - \frac{k}{x}f'(-x) + \frac{k^2}{x^2}f(-x) - \frac{k^2}{x^2}f(x) \\
&= f''(x) + \frac{2k}{x}f'(x) - \frac{k}{x^2}f(x) + \frac{k}{x^2}f(-x) \\
&= f''(x) + \frac{2k}{x}f'(x) + \frac{k}{x^2}\left(f(-x) - f(x)\right)
\end{aligned}$$

□

Example 3.3.12 ([40])**.** *The Dunkl Laplacian for the type A_{N-1} is given by*

$$\Delta_k^S = \Delta + 2k \sum_{1 \le i < j \le N} \frac{1}{x_i - x_j}\left[(\partial_i - \partial_j) - \frac{1 - \sigma_{ij}}{x_i - x_j}\right],$$

where σ_{ij} stands for the transposition (ij).

3.4 The Dunkl Intertwining Operator

It first appeared that for multiplicity functions $k \geqslant 0$, the associated commutative algebra of Dunkl operators is intertwined with the algebra of usual partial differential operators by a unique linear and homogeneous isomorphism on polynomials on $\mathbb{R}^N$ [46].

Theorem 3.4.1 ([40])**.** *There exists a unique linear isomorphism (intertwining operator) V_k such that*

$$T_\xi V_k = V_k \partial_\xi.$$

The following very important theorem is due to Rösler [40].

Theorem 3.4.2 ([40]). *For each $x \in \mathbb{R}^N$ there exists a unique probability measure μ_x^k on the Borel- σ-algebra of $\mathbb{R}^N$ such that*

$$V_k f(x) = \int_{\mathbb{R}^N} f(\xi) d\mu_x^k(\xi) \tag{3.4.1}$$

for all $f \in \mathscr{S}\left(\mathbb{R}^N\right)$. The representing measures μ_x^k are compactly supported and they satisfy

$$\mu_{rx}^k(B) = \mu_x^k\left(r^{-1}B\right), \quad \mu_{gx}^k(B) = \mu_x^k\left(g^{-1}(B)\right)$$

for each $r > 0, g \in G$ and each Borel set $B \subseteq \mathbb{R}^N$.

Example 3.4.3 (The rank-one case [40]). *For $k > 0$, this amounts to the following integral representation*

$$V_k f(x) = \frac{\Gamma(k+1/2)}{\Gamma(1/2)\Gamma(k)} \int_{-1}^{1} f(xt)(1-t)^{k-1}(1+t)^k dt.$$

For instance,

$$V_k\left(x^{2n}\right) = \frac{\left(\frac{1}{2}\right)_n}{\left(k+\frac{1}{2}\right)_n} x^{2n}; \quad V_k\left(x^{2n+1}\right) = \frac{\left(\frac{1}{2}\right)_{n+1}}{\left(k+\frac{1}{2}\right)_{n+1}} x^{2n+1},$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer-symbol.

Chapter 4: The (k, a)-Generalized Fourier Transform

4.1 Introduction

In the following chapter, we will start to shed a light on the framework of the main result of the thesis as we define the (k, a)-generalized Fourier transform and state some of its crucial properties. The (k, a)-generalized Fourier transform was introduced by Ben Saïd, Kobayashi Ørsted in [33] and [27].

4.2 The Kernel $B_{k,a}(x, y)$

First we shall introduce the normalized Bessel function $\widetilde{I}_\nu(w)$.

Definition 4.2.1. *Define $\widetilde{I}_\nu(w)$ as*

$$\begin{aligned}\widetilde{I}_\lambda(w) &:= \left(\frac{w}{2}\right)^{-\lambda} I_\lambda(w) = \sum_{\ell=0}^{\infty} \frac{w^{2\ell}}{2^{2\ell}\ell!\Gamma(\lambda+\ell+1)} \\ &= \frac{1}{\sqrt{\pi}\Gamma\left(\lambda+\frac{1}{2}\right)} \int_{-1}^{1} e^{wt}\left(1-t^2\right)^{\lambda-\frac{1}{2}} dt.\end{aligned}$$

Let us also state the re-normalized Bessel function of the first kind J_ν.

Definition 4.2.2.

$$\widetilde{J}_\nu(\omega) := \left(\frac{\omega}{2}\right)^{-\nu} J_\nu(\omega) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \omega^{2\ell}}{2^{2\ell}\ell!\gamma(\nu+\ell+1)}.$$

Comparing this definition with the normalized I-Bessel function defined in (4.2.1) we get

$$\widetilde{J}_\nu(\omega) = \widetilde{I}_\nu(-i\omega) = \widetilde{I}_\nu(i\omega).$$

Definition 4.2.3. *The Gegenbauer polynomials $C_m^\alpha(t)$ are explicitly defined for $\alpha > 0$ and $m \in \mathbb{N}$ by*

$$C_m^\alpha(t) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{[m/2]} (-1)^k \frac{\Gamma(m-k+\alpha)}{k!(m-2k)!}(2t)^{m-2k}, \tag{4.2.1}$$

Let us introduce the function $\mathscr{I}(b, \nu; w; t)$.

Definition 4.2.4. *Let $\widetilde{I}_\lambda(w)$ be the normalized I-Bessel function defined in (4.2.1), and $C_m^\nu(t)$ the Gegenbauer polynomial found in (4.2.1). Consider the following infinite sum:*

$$\mathscr{I}(b,\nu;w;t) = \frac{\Gamma(b\nu+1)}{\nu}\sum_{m=0}^{\infty}(m+\nu)\left(\frac{w}{2}\right)^{bm}\widetilde{I}_{b(m+\nu)}(w)C_m^\nu(t). \tag{4.2.2}$$

Lemma 4.2.5 ([27])**.** *The summation in the left hand side of* (4.2.2) *is absolutely and uniformly convergent on any compact subset of*

$$U := \{(b,\nu,w,t) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{C} \times [-1,1] : 1 + b\nu > 0\}.$$

Specifically, $\mathscr{I}(b,\nu;w;t)$ is continuous on U.

Let R be a root system and $k : R \longrightarrow \mathbb{R}^+, \alpha \mapsto k_\alpha$, be a multiplicity function. Henceforth $a > 0$.

Definition 4.2.6. *For r, s positive real numbers, $z \in \mathbb{C}^+ \backslash i\pi\mathbb{Z}$, and t a continuous function on the interval $[-1,1]$. $h_{k,a}(r,s;z;t)$ is given by*

$$\begin{aligned} h_{k,a}(r,s;z;t) = &\frac{\exp\left(-(1/a)\left(r^a+s^a\right)\coth(z)\right)}{\sinh(z)^{(2\langle k\rangle+N+a-2)/a}} \\ &\times \mathscr{I}\left(\frac{2}{a}, \frac{2\langle k\rangle+N-2}{2}; \frac{2(rs)^{a/2}}{a\sinh(z)}; t\right), \end{aligned}$$

where $\langle k\rangle$ is given by

$$\langle k\rangle = \sum_{\alpha\in R^+} k_\alpha. \tag{4.2.3}$$

Here, we give examples for the function $h_{k,a}$.

Example 4.2.7 ([27])**.** *For $(a = 1)$ the series expantion of $h_{k,a}$ can be expressed as the following*

$$\begin{aligned} h_{k,1}(r,s;z;t) = &\frac{\exp\left(-1\left(r+s\right)\coth(z)\right)}{\sinh(z)^{(2\langle k\rangle+N-1)}} \\ &\times \Gamma\left(\langle k\rangle + \frac{N-1}{2}\right)\widetilde{I}_{\langle k\rangle+(N-3)/2}\left(\frac{\sqrt{2}(rs)^{\frac{1}{2}}}{\sinh z}(1+t)^{\frac{1}{2}}\right), \end{aligned} \tag{4.2.4}$$

and for the case $(a = 2)$ *we have*

$$h_{k,2}(r,s;z;t) = \frac{\exp\left(-(1/2)\left(r^2+s^2\right)\coth(z)\right)}{\sinh(z)^{(\langle k\rangle+\frac{N}{2})}} \exp\left(\frac{rst}{\sinh z}\right). \tag{4.2.5}$$

Define the kernel function $\Lambda_{k,a}\left(x,y;z\right)$ as

$$\Lambda_{k,a}\left(r\omega, s\eta; z\right) = \widetilde{V}_k(h_{k,a}(r,s;z;\cdot))(\omega,\eta)$$

and $\widetilde{V}_k$ is given in terms of the Dunkl intertwining operator by

$$(\widetilde{V}_k h)(x,y) := (V_k h_y)(x) = \int_{\mathbb{R}^N} h(\langle \xi, y\rangle) d\mu_x^k(\xi),$$

where μ_x^k is the measure that appears in (3.4.1).

It's time now to state the kernel $B_{k,a}$. For a multiplicity function $k \geqslant 0$, $a > 0$ and $2\langle k\rangle + N > max(1, 2-a)$ we introduce the kernel $B_{k,a}(\xi, x)$ which is given by

$$B_{k,a}(x,y) = e^{i\pi((2\langle k\rangle+N+a-2)/2a)}\Lambda_{k,a}\left(x,y;i\frac{\pi}{2}\right). \tag{4.2.6}$$

Example 4.2.8 ([27])**.** *For* $N = 1, a > 0, k \geqslant 0$ *and* $2k > 1 - a$*, the kernel* $B_{k,a}$ *can be written in an explicit way as follows*

$$\begin{aligned} B_{k,a}(x,y) =& \Gamma\left(\frac{2k+a-1}{a}\right)\left(\widetilde{J}_{(2k-1)/a}\left(\frac{2}{a}|xy|^{a/2}\right)\right. \\ &\left. + \frac{xy}{(ai)^{2/a}}\widetilde{J}_{(2k+1)/a}\left(\frac{2}{a}|xy|^{a/2}\right)\right). \end{aligned}$$

Example 4.2.9 ([27])**.** *By substituting* $z = \pi i/2$ *into* (4.2.4) *and* (4.2.5)*, we get the following formulas:*

$$\begin{aligned} h_{k,a}\left(r,s;\frac{\pi i}{2};t\right) =& \Gamma\left(\langle k\rangle + \frac{N-1}{2}\right)e^{-(\pi i/2)(2\langle k\rangle+N-1)} \\ & \widetilde{J}_{\langle k\rangle+(N-3)/2}\left(\sqrt{2}(rs)^{1/2}(1+t)^{1/2}\right) \end{aligned} \tag{4.2.7}$$

for a=1, and

$$h_{k,a}\left(r, s; \frac{\pi i}{2}; t\right) = e^{-(\pi i/2)(\langle k\rangle + N/2)} e^{-irst} \tag{4.2.8}$$

for a=2. Using (4.2.7) and (4.2.8) together with the definition of the kernel $B_{k,a}$ in (4.2.6), we get the following : For $a = 1$ the kernel $B_{k,a}(x, y)$ is given by

$$B_{k,a}(r\omega, s\eta) = \Gamma\left(\langle k\rangle + \frac{N-1}{2}\right) \widetilde{V}_k\left(\widetilde{J}_{\langle k\rangle+(N-3)/2}(\sqrt{2rs(1+\cdot)})\right)(\omega, \eta)$$

and for $a = 2$ it becomes the following

$$B_{k,a}(r\omega, s\eta) = \widetilde{V}_k\left(e^{-irs\cdot}\right)(\omega, \eta).$$

The kernel $B_{k,a}(\xi, x)$ satisfies the following differential-difference equations:

$$E^x B_{k,a}(\xi, x) = E^\xi B_{k,a}(\xi, x)$$
$$\|\xi\|^{2-a} \Delta_k B_{k,a}(\xi, x) = -\|x\|^a B_{k,a}(\xi, x),$$
$$\|x\|^{2-a} \Delta_k B_{k,a}(\xi, x) = -\|\xi\|^a B_{k,a}(\xi, x),$$

where Δ_k denotes the Dunkl Laplacian operator given by 3.3.1 and E is the Euler's operator given by

$$E^x f = \sum_{j=0}^{\infty} x_j \partial_{x_j}$$

We shall now shed a light some basic properties of $B_{k,a}$ the kernel defining the generalized Fourier transform.

Properties 4.2.10 ([27]). *Let $B_{k,a}(\cdot,\cdot)$ be the kernel defined in* (4.2.6). *Then $B_{k,a}(\cdot,\cdot)$ satisfies:*

(1) $B_{k,a}(\lambda x, \xi) = B_{k,a}(x, \lambda\xi)$ *for* $\lambda > 0$.

(2) $B_{k,a}(gx, g\xi) = B_{k,a}(x, \xi)$ *for* $g \in G$.

(3) $B_{k,a}(\xi, x) = B_{k,a}(x, \xi)$.

(4) $B_{k,a}(0, x) = 1$.

Furthermore, it was shown by [52] that the kernel $B_{k,a}$ can be bounded as follows

Theorem 4.2.11 ([27]). *Let $k \geqslant 0$ and $x, y \in \mathbb{R}^N$. Then*

- *For $a = \{1, 2\}$, $|B_{k,a}(x, y)| \leqslant 1$.*
- *For $N = 1$ and $a = \frac{2}{n}$, we have $|B_{k,a}(x, y)| \leqslant C$ for some positive constant C.*

4.3 The (k, a)-Generalized Fourier Transform

Let us begin by introducing the normalization constant $c_{k,n}$ that is given for $a > 0$ and a multiplicity function k by

$$c_{k,n} := \left(\int_{\mathbb{R}^N} exp\left(-\frac{1}{a}\|x\|^a\right) \vartheta_{k,a}(x)dx \right)^{-1},$$

where the density function $\vartheta_{k,a}(x)$ on $\mathbb{R}^N$ is given by

$$\vartheta_{k,a}(x) := \|x\|^{a-2} \prod_{\alpha \in R} |\langle \alpha, x \rangle|^{k_\alpha}.$$

We define the following integral transform using the kernel $B_{k,n}$ that was given in (4.2.6), as well as the normalizing constant $c_{k,a}$. The generalized transform $\mathcal{F}_{k,a}$ can be expressed on $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)dx\right)$ as

$$\mathcal{F}_{k,a}f(\xi) = c_{k,a} \int_{\mathbb{R}^N} B_{k,a}(\xi, x) f(x) \vartheta_{k,a}(x) dx,$$

Example 4.3.1 ([27]). *The (k, a)-generalized Fourier transform reduces to the Euclidean Fourier transform for $(a = 2, k \equiv 0)$,[39], to the Dunkl transform for $(a = 2, k > 0)$[17], to the generalized Hankel transform by Mano and Kobayashi [29] for $(a = 1, k \equiv 0)$, and to the k-Hankel transform [33] for $(a = 1, k > 0)$.*

Example 4.3.2 ([27]). *For $N = 1, a > 0, k \geqslant 0$ and $2k > 1 - a$, the integral transform $\mathcal{F}_{k,n}$ takes the following form*

$$\begin{aligned}\mathcal{F}_{k,a}f(y) =& 2^{-1}a^{-((2k-1)/a)} \int_{\mathbb{R}} f(x) \left(\widetilde{J}_{(2k-1)/a}\left(\frac{2}{a}|xy|^{a/2}\right) \right. \\ & \left. + \frac{xy}{(ai)^{2/a}} \widetilde{J}_{(2k+1)/a}\left(\frac{2}{a}|xy|^{a/2}\right)\right) |x|^{2k+a-2} dx.\end{aligned}$$

Let us now state a collection of formulas and properties of the (k,a)-generalized Fourier transform.

Theorem 4.3.3 (Plancherel's formula [27])**.** *Let k be a multiplicity function on the root system R that is non-negative, $a > 0$ and let a and k satisfy the inequality $a + 2\langle k\rangle + N > 2$.Then, $\mathcal{F}_{k,a}$ is a unitary operator on $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)dx\right)$. That is, the (k,a)-generalized Fourier transform is a bijective linear operator satisfying*

$$\left\|\mathcal{F}_{k,a}(f)\right\|_{L^2} = \|f\|_{L^2} \quad \textit{for any } f \in L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)dx\right).$$

In addition to the Plancherel's formula, there are some interesting properties of the (k,a)-generalized Fourier transform that are worth mentioning and that will later prove to be beneficial for the purpose of this thesis. Let us state here

Theorem 4.3.4 (Inversion formula [27])**.** *Let k be a multiplicity function on the root system R with $k \geqslant 0$.*

(1) Let $r > 0$ and $r \in \mathbb{N}$. Then, suppose the inequality $2\langle k\rangle + N > 2 - a$ with $a = 2/r$ is satisfied.Then $\mathcal{F}_{k,2/r}$ is a unitary operator on $L^2\left(\mathbb{R}^N, \vartheta_{k,1/r}(x)dx\right)$. specifically, the inversion formula is given by

$$\left(\mathcal{F}_{k,2/r}^{-1} f\right)(x) = \left(\mathcal{F}_{k,2/r} f\right)(x)$$

(2) Now, let $r \in \mathbb{N}$, $r \geqslant 0$ and $a = \frac{2}{2r+1}$, and suppose the following inequality holds $2\langle k\rangle + N > 2 - a$. Then $\mathcal{F}_{k,2/(2r+1)}$ is a unitary operator of order 4 on $L^2\left(\mathbb{R}^N, \vartheta_{k,2/(2r+1)}(x)dx\right)$. In fact, the inversion formula is given as

$$\left(\mathcal{F}_{k,2/(2r+1)}^{-1} f\right)(x) = \left(\mathcal{F}_{k,2/(2r+1)} f\right)(-x).$$

We now state the following important properties of $\mathcal{F}_{k,a}$

Theorem 4.3.5 ([27])**.** *The unitary operator $\mathcal{F}_{k,a}$ satisfies the following intertwining relations on a dense subspace of $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)dx\right)$.*

(1) $\mathcal{F}_{k,a} \circ E = -(E + N + 2\langle k\rangle + a - 2) \circ \mathcal{F}_{k,a}$.

(2) $\mathcal{F}_{k,a} \circ \|x\|^a = -\|x\|^{2-a}\Delta_k \circ \mathcal{F}_{k,a}$.

(3) $\mathcal{F}_{k,a} \circ \|x\|^{2-a}\Delta_k = -\|x\|^a \circ \mathcal{F}_{k,a}$.

Here E *denotes the Euler operator*

$$Ef = \sum_{j=0}^{\infty} x_j \partial_{x_j}$$

and Δ_k *denotes the Dunkl Laplacian operator* (3.3.1).

Chapter 5: A Convolution Structure in the Rank-One Case

5.1 Introduction

In this chapter we will restrict our self to the rank-one case. We define and study a translation operator $\tau_{k,a}$ and a convolution structure associated to the (k,a)-generalized Fourier transform $\mathcal{F}_{k,a}$ for $a = \frac{2}{n}$. The translation operator and the convolution structure share many important properties with their analogous in the classical Fourier theory. The main references are [36], [35]

5.2 The Translation Operator

From now on we will assume that $a = \frac{2}{n}$ where n is an integer, and the dimension $N = 1$. Here the parameter $k > 0$. Recall the (k,a)-generalized Fourier transform denoted by $\mathcal{F}_{k,a} = \mathcal{F}_{k,\frac{2}{n}}$, and defined for $f \in L^1(\mathbb{R}, d\mu_{k,n})$, by

$$\mathcal{F}_{k,n}f(\lambda) = \int_{\mathbb{R}} f(x)B_{k,n}(x,\lambda)d\mu_{k,n}(x), \lambda \in \mathbb{R}$$

where $B_{k,n}$ is the generalized Hankel function, defined by

$$\begin{aligned} B_{k,n}(\lambda, x) = j_{kn-\frac{n}{2}}&\left(n|\lambda x|^{\frac{1}{n}}\right) \\ &+ (-i)^n \left(\frac{n}{2}\right)^n \frac{\Gamma(kn - \frac{n}{2}+1)}{\Gamma(kn+\frac{n}{2}+1)} \lambda x j_{kn+\frac{n}{2}}\left(n|\lambda x|^{\frac{1}{n}}\right). \end{aligned} \tag{5.2.1}$$

The function j_α is the normalized Bessel function of first kind and order $\alpha > \frac{-1}{2}$, and is given by

$$j_\alpha(z) = \Gamma(\alpha+1)\left(\frac{z}{2}\right)^{-\alpha} J_\alpha(z) = \Gamma(\alpha+1)\sum_{m=0}^{+\infty} \frac{(-1)^m}{m!\Gamma(\alpha+m+1)}\left(\frac{z}{2}\right)^{2m}.$$

The measure $d\mu_{k,n}(x)$ is defined by

$$d\mu_{k,n}(x) = (M_{k,n})^{-1}|x|^{2k+\frac{2}{n}-2}dx, \tag{5.2.2}$$

where

$$M_{k,n} = 2\left(\frac{2}{n}\right)^{kn-\frac{n}{2}} \Gamma\left(kn+1-\frac{n}{2}\right).$$

Recall from 4.2.11 that the kernel $B_{k,n}$ is bounded by a constant C.

Notation 5.2.1. *For $x, y \in \mathbb{R}^\star, z \in \mathbb{R}, n \in \mathbb{N}^\star$ and $k > \frac{n-1}{2n}$, let*

$$\sigma_{x,y,z}^n = \frac{|x|^{\frac{2}{n}} + |y|^{\frac{2}{n}} - |z|^{\frac{2}{n}}}{2|xy|^{\frac{1}{n}}},$$

and

$$\xi_{k,n}(x,y,z) = \frac{n! \operatorname{sgn}(xy)}{(2kn-n)_n} C_n^{kn-\frac{n}{2}} \left(\sigma_{x,y,z}^n\right).$$

Furthermore let us define

$$\begin{aligned}\mathcal{K}_{k,n}(x,y,z) = \frac{M_{k,n}}{2n} K_B^{kn-\frac{n}{2}} &\left(|x|^{\frac{1}{n}}, |y|^{\frac{1}{n}}, |z|^{\frac{1}{n}}\right) \\ &\times \left\{1 + (-1)^n \xi_{k,n}(x,y,z) + \xi_{k,n}(z,x,y) + \xi_{k,n}(y,z,x)\right\}.\end{aligned} \tag{5.2.3}$$

Here K_B^α, and C_n^α are defined as

$$K_B^\alpha(u,v,w) = 2^{-2\alpha+1} \frac{\left\{\left[(u+v)^2 - w^2\right]\left[w^2 - (u-v)^2\right]\right\}^{\alpha-\frac{1}{2}}}{(uvw)^{2\alpha}},$$

$$C_m^\alpha(t) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{[m/2]} (-1)^k \frac{\Gamma(m-k+\alpha)}{k!(m-2k)!} (2t)^{m-2k},$$

Let us state some important properties that will prove to be crucial later in this section:

Properties 5.2.2 ([36])**.** *Let $n \in \mathbb{N}^\star$ and $k > \frac{n-1}{2n}$, then*

(i) The mapping $(x,y,z) \to \sigma_{x,y,z}^n$ is homogeneous of degree 0.

(ii) We have

$$|\xi_{k,n}(x,y,z)| \leqslant 1, \quad x,y,z \in \mathbb{R}^\star \tag{5.2.4}$$

(iii) The kernel $\mathcal{K}_{k,n}$ satisfies

$$\begin{cases} \mathcal{K}_{k,n}(x,y,z) = \mathcal{K}_{k,n}(y,x,z) \\ \mathcal{K}_{k,n}(x,y,z) = \mathcal{K}_{k,n}\left((-1)^n x, z, y\right) \\ \mathcal{K}_{k,n}(x,y,z) = \mathcal{K}_{k,n}\left(z, (-1)^n y, x\right) \end{cases}$$

Theorem 5.2.3 ([36]). *For $\lambda, x, y \in \mathbb{R}$, we have*

$$B_{k,n}(\lambda, x)B_{k,n}(\lambda, y) = \int_{\mathbb{R}} B_{k,n}(\lambda, z)d\nu_{k,n}^{x,y}(z),$$

where

$$d\nu_{k,n}^{x,y}(z) = \begin{cases} \mathcal{K}_{k,n}(x,y,z)d\mu_{k,n}(z) & \text{if} \quad xy \neq 0 \\ d\delta_x(z) & \text{if} \quad y = 0 \\ d\delta_y(z) & \text{if} \quad x = 0 \end{cases}, \tag{5.2.5}$$

where δ is the dirac measure and $d\mu_{k,n}$ is as in 5.2.2.

Here we have some properties of the measure $\nu_{k,n}^{x,y}$ that was given in 5.2.5.

Properties 5.2.4 ([36]). *For $n \in \mathbb{N}^\star, k > \frac{n-1}{2n}$ and $x, y \in \mathbb{R}$, we have*
(i) $\operatorname{supp}\left(\nu_{k,n}^{x,y}\right)(\mathbb{R}) \subset I_{x,y} = \left\{z \in \mathbb{R}/ \left||x|^{\frac{1}{n}} - |y|^{\frac{1}{n}}\right| < |z|^{\frac{1}{n}} < |x|^{\frac{1}{n}} + |y|^{\frac{1}{n}}\right\}$. *(ii)* $\nu_{k,n}^{x,y}(\mathbb{R}) = 1$. *(iii)* $\left\|\nu_{k,n}^{x,y}\right\| \leqslant 4$.

5.3 The Convolution Structure

Let the space of bounded continuous functions on $\mathbb{R}$ and the space of continuous functions on $\mathbb{R}$ with compact support be denoted by $\mathcal{C}_b(\mathbb{R})$ and $\mathcal{C}_c(\mathbb{R})$ respectively.

Recall, the generalized Fourier transform $\mathcal{F}_{k,n}$ is defined by 5.2 for $n \in \mathbb{N}^\star$ and $k > \frac{n-1}{2n}$. $\mathcal{F}_{k,n}^{-1}$ can be defined as

$$\mathcal{F}_{k,n}^{-1}(g)(x) = \mathcal{F}_{k,n}(g)\left((-1)^n x\right), \quad x \in \mathbb{R}.$$

Let us denote the even part of a function f by f_e, and the odd part by f_o.

Proposition 5.3.1 ([36]). *Let $\lambda > 0,\ n \in \mathbb{N}^\star,\ k > \frac{n-1}{n}$, and $f \in \mathcal{C}_c(\mathbb{R})$. Then*

$$\mathcal{F}_{k,n}(f)(\lambda) = \frac{1}{2n^{kn-\frac{n}{2}+1}}\mathcal{H}_{kn-\frac{n}{2}}\left(G_n\left(f_e\right)\right)\left(|\lambda|^{\frac{1}{n}}\right) + \frac{(-i)^n\lambda}{n!2^n n^{kn-\frac{n}{2}+1}}\mathcal{H}_{kn-\frac{n}{2}}\left(J_n\left(f_o\right)\right)\left(|\lambda|^{\frac{1}{n}}\right),$$

where $\mathcal{H}_\alpha$ is the Hankel transform defined by

$$\mathcal{H}_\alpha(f)(\lambda) = \frac{1}{2^{\alpha-1}\gamma(\alpha+1)} \int_0^{+\infty} f(t) j_\alpha(t\lambda) t^{2\alpha+1} dt,$$

and g_n and J_n are the functions defined on $\mathbb{R}_+$by

$$G_n(f_e)(t) = f_e\left(\left(\frac{t}{n}\right)^n\right),$$

$$J_n\left(f_o\right)(t) = \int_s^\infty f_o\left(\left(\frac{t}{n}\right)^n\right)\left(t^2 - s^2\right)^{n-1} t^{-n+1} dt.$$

Now, we define an important operator that will be appearing in many theorems and proofs in this thesis.

Definition 5.3.2 ([36])**.** *Let $x \in \mathbb{R}$ and $f \in \mathcal{C}_b(\mathbb{R})$. For $n \in \mathbb{N}^\star$ and $k > \frac{n-1}{2n}$, we define the translation operator $\tau_y^{k,n}$ by*

$$\tau_x^{k,n} f(y) = \int_{\mathbb{R}} f(z) d\nu_{k,n}^{x,y}(z), \quad y \in \mathbb{R}, \tag{5.3.1}$$

where $d\nu_{k,n}^{x,y}$ is given by 5.2.5.

We state here the properties of this translation operator.

Properties 5.3.3 ([36])**.** *Let $n \in \mathbb{N}^\star, k > \frac{n-1}{n}, x \in \mathbb{R}$ and $f \in \mathcal{C}_b(\mathbb{R})$. Then*

(i) $\tau_x^{k,n} f(y) = \tau_y^{k,n} f(x)$.

(ii) $\tau_0^{k,n} f = f$.

(iii) $\tau_x^{k,n} \tau_y^{k,n} = \tau_y^{k,n} \tau_x^{k,n}$.

If we suppose also that $f \in \mathcal{C}_c(\mathbb{R})$, then

(iv) $\mathcal{F}_{k,n}\left(\tau_x^{k,n} f\right)(\lambda) = B_{k,n}\left(\lambda, (-1)^n x\right) \mathcal{F}_{k,n}(f)(\lambda)$.

Proof. i) is a consequence of the property $\mathcal{K}_{k,n}(x,y,z) = \mathcal{K}_{k,n}(y,x,z)$.
ii) follows from the fact that $B_{k,n}(\lambda, 0) = 1$.
iii) follows from i).
iv) Assume $f \in \mathcal{C}_c(\mathbb{R})$, then from the definition of the translation operator $\tau_x^{k,n}$ in 5.3.1

together with Fubini's theorem, we get

$$\begin{aligned}\mathcal{F}_{k,n}\left(\tau_x^{k,n} f\right)(\lambda) &= \int_{\mathbb{R}} \tau_x^{k,n} f(y) B_{k,n}(\lambda, y) d\mu_{k,n}(y) \\ &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(z)\mathcal{K}_{k,n}(x,y,z) d\mu_{k,n}(z)\right] B_{k,n}(\lambda, y) d\mu_{k,n}(y) \\ &= \int_{\mathbb{R}} f(z) \left[\int_{\mathbb{R}} \mathcal{K}_{k,n}(x,y,z) B_{k,n}(\lambda, y) d\mu_{k,n}(y)\right] d\mu_{k,n}(z).\end{aligned}$$

The property $\mathcal{K}_{k,n}(x,y,z) = \mathcal{K}_{k,n}\left((-1)^n x, z, y\right)$, gives

$$\begin{aligned}&\mathcal{F}_{k,n}\left(\tau_x^{k,n} f\right)(\lambda) \\ &\quad= \int_{\mathbb{R}} f(z) \left[\int_{\mathbb{R}} \mathcal{K}_{k,n}\left((-1)^n x, z, y\right) B_{k,n}(\lambda, y) d\mu_{k,n}(y)\right] d\mu_{k,n}(z).\end{aligned}$$

Using Theorem 5.2.3, we obtain

$$\mathcal{F}_{k,n}\left(\tau_x^{k,n} f\right)(\lambda) = B_{k,n}\left(\lambda, (-1)^n x\right) \mathcal{F}_{k,n}(f)(\lambda).$$

□

Definition 5.3.4. *We define $L^p\left(\mathbb{R}, d\mu_{k,n}\right)$ as the space of real valued functions f that are $\mu_{k,n}$-measurable such that*

$$\left(\int_{\mathbb{R}} |f(x)|^p d\mu_{k,n}(x)\right)^{\frac{1}{p}} < \infty$$

and

$$\|f\|_{k,\infty} := \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)|.$$

Lemma 5.3.5 ([36])**.** *Let $n \in \mathbb{N}^\star$ and $k > \frac{n-1}{2n}, 1 \leqslant p \leqslant \infty, f \in L^p\left(\mathbb{R}, d\mu_{k,n}\right)$ and $x \in \mathbb{R}$. Then*

$$\left\|\tau_x^{k,n}(f)\right\|_{k,p} \leqslant 4\|f\|_{k,p}, \quad x \in \mathbb{R}.$$

Proof. Let us consider the following cases:

Case 1: $p = \infty$ is obvious.

Case 2: If $p = 1$, the assertion follows from Fubini-Tonelli's theorem, the property $\mathcal{K}_{k,n}(x,y,z) = \mathcal{K}_{k,n}\left((-1)^n x, z, y\right)$ and iii) of Properties 5.3.3

Case 3: Let $1 < p < +\infty$ and q the conjugate exponent of p. Then by Hölder's inequality, we have

$$\left|\tau_x^{k,n} f(y)\right|^p \leq \int_{\mathbb{R}} |f(z)|^p \left|\mathcal{K}_{k,n}(x,y,z)\right| d\mu_{k,n}(z) \left(\int_{\mathbb{R}} \left|\mathcal{K}_{k,n}(x,y,z)\right| d\mu_{k,n}(z)\right)^{\frac{p}{q}}.$$

Therefore

$$\left\|\tau_x^{k,n} f\right\|_{k,p}^p \leqslant 4^{\frac{p}{q}} \int_{\mathbb{R}} \int_{\mathbb{R}} |f(z)|^p \left|\mathcal{K}_{k,n}(x,y,z)\right| d\mu_{k,n}(z) d\mu_{k,n}(y).$$

Invoking the property $\mathcal{K}_{k,n}(x,y,z) = \mathcal{K}_{k,n}\left((-1)^n x, z, y\right)$ and Fubini's theorem, we get

$$\begin{aligned}\left\|\tau_y^{k,n} f\right\|_{k,p}^p &\leqslant 4^{\frac{p}{q}} \int_{\mathbb{R}} |f(z)|^p \int_{\mathbb{R}} \left|\mathcal{K}_{k,n}\left((-1)^n x, z, y\right)\right| d\mu_{k,n}(y) d\mu_{k,n}(z) \\ &= 4^p \|f\|_{k,p}^p.\end{aligned}$$

Thus,

$$\left\|\tau_y^{k,n} f\right\|_{k,p} \leqslant 4\|f\|_{k,p}$$

which is the desired inequality. □

Definition 5.3.6. *The convolution product of two functions f and g on $\mathbb{R}$ is defined by*

$$f \star_{k,n} g(x) = \int_{\mathbb{R}} f(y) \tau_x^{k,n} g\left((-1)^n y\right) d\mu_{k,n}(y)$$

for suitable functions f and g.

The following are properties of the convolution product $\star_{k,n}$.

Properties 5.3.7 ([36])**.** *(i)* $f \star_{k,n} g = g \star_{k,n} f$.

(ii) $(f \star_{k,n} g) \star_{k,n} h = f \star_{k,n} (g \star_{k,n} h)$.

Proof. (i) By using Fubini's theorem and the property $\mathcal{K}_{k,n}(x,y,z) = \mathcal{K}_{k,n}\left((-1)^n x, z, y\right)$, we obtain

$$\begin{aligned} f \star_{k,n} g(x) &= \int_{\mathbb{R}} f(y) \left[\int_{\mathbb{R}} g(z) \mathcal{K}_{k,n}\left(x, (-1)^n y, z\right) d\mu_{k,n}(z) \right] d\mu_{k,n}(y) \\ &= \int_{\mathbb{R}} g(z) \left[\int_{\mathbb{R}} f(y) \mathcal{K}_{k,n}\left((-1)^n x, z, (-1)^n y\right) d\mu_{k,n}(y) \right] d\mu_{k,n}(z). \end{aligned}$$

So, using the property $\mathcal{K}_{k,n}\left((-1)^n x, z, (-1)^n y\right) = \mathcal{K}_{k,n}\left(x, (-1)^n z, y\right)$, we get

$$\begin{aligned} f \star_{k,n} g(x) &= \int_{\mathbb{R}} g(z) \left[\int_{\mathbb{R}} f(y) \mathcal{K}_{k,n}\left(x, (-1)^n z, y\right) d\mu_{k,n}(y) \right] d\mu_{k,n}(z) \\ &= \int_{\mathbb{R}} g(z) \tau_x^{k,n} f\left((-1)^n z\right) d\mu_{k,n}(z) \\ &= g \star_{k,n} f(x). \end{aligned}$$

□

Proposition 5.3.8 (Young inequality [36] and [35])**.** *For* p, q, r *such that* $1 \leqslant p, q, r \leqslant \infty$ *and* $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$, *and for* $f \in L^p\left(\mathbb{R}, d\mu_{k,n}\right)$ *and* $g \in L^q\left(\mathbb{R}, d\mu_{k,n}\right)$, *the convolution product* $f \star_k g$ *is a well defined element in* $L^r\left(\mathbb{R}, d\mu_{k,n}\right)$ *and*

$$\|f \star_{k,n} g\|_{k,r} \leqslant 4 \|f\|_{k,p} \|g\|_{k,q}.$$

For every $R > 0$, let us denote by $\mathcal{C}_R^\infty(\mathbb{R})$ the space of smooth functions on $\mathbb{R}$ which are supported in $[-R, R]$. Then

Proposition 5.3.9 ([36])**.** *For* $f \in \mathcal{C}_{R_1}^\infty(\mathbb{R})$ *and* $g \in \mathcal{C}_{R_2}^\infty(\mathbb{R})$, *then* $f \star_{k,n} g \in \mathcal{C}_{R_1+R_2}^\infty(\mathbb{R})$ *and we have*

$$\mathcal{F}_{k,n}\left(f \star_{k,n} g\right) = \mathcal{F}_{k,n}(f) \mathcal{F}_{k,n}(g).$$

Proof. Using Fubini's theorem, we have

$$\begin{aligned} &\mathcal{F}_{k,n}\left(f \star_{k,n} g\right)(\lambda) \\ &= \int_{\mathbb{R}} B_{k,n}(\lambda, x) f \star_{k,n} g(x) d\mu_{k,n}(x) \\ &= \int_{\mathbb{R}} f(z) \int_{\mathbb{R}} g(y) \int_{\mathbb{R}} B_{k,n}(\lambda, x) \mathcal{K}_{k,n}\left(x, (-1)^n y, z\right)\ d\mu_{k,n}(x)\ d\mu_{k,n}(y)\ d\mu_{k,n}(z). \end{aligned}$$

Using the property $\mathcal{K}_{k,n}\left(x,(-1)^n y,z\right) = \mathcal{K}_{k,n}(y,z,x)$ along with Theorem 5.2.3, leads to

$$\begin{aligned}
&\mathcal{F}_{k,n}\left(f \star_{k,n} g\right)(\lambda) \\
&\qquad = \left(\int_{\mathbb{R}} f(z) B_{k,n}(\lambda,z) d\mu_{k,n}(z)\right)\left(\int_{\mathbb{R}} g(y) B_{k,n}(\lambda,y) d\mu_{k,n}(y)\right) \\
&\qquad = \mathcal{F}_{k,n}(f)(\lambda)\mathcal{F}_{k,n}(g)(\lambda).
\end{aligned}$$

□

Chapter 6: The Generalized Hardy-Littlewood Maximal Operator

6.1 Introduction

In this chapter, we will define and study the generalized Hardy–Littlewood maximal operator $\mathcal{M}_{k,a}$ associated with the one-dimensional generalized Fourier transform $\mathcal{F}_{k,a}$. For this operator to which covering methods do not apply, we will construct a geometric maximal operator $\mathbb{M}_{k,a}$, which controls pointwise the maximal operator $\mathcal{M}_{k,a}$, and for which we can use the machinery of real analysis to obtain a maximal theorem. Therefore, proving the above mentioned conjecture reduces to proving the same conjecture for the geometric maximal operator that will control $\mathcal{M}_{k,a}$.

6.2 A Covering Lemma of Vitali-Type

In this section we are going to be stating and showing a covering lemma of Vitali-type. This lemma is essential for us in order to be able to reach the proof of the main result of this thesis. From now on, we will be using the symbol ($\lesssim$) which can be read as "less up to a constant", meaning the left hand side is bounded by some scalar multiple of the right hand side of the inequality.

Definition 6.2.1. *For $x \in \mathbb{R}$ and $r > 0$, let*

$$I(x,r) := \left] \left(max\{0, |x|^{\frac{1}{n}} - r^{\frac{1}{n}}\}\right)^n, \left(|x|^{\frac{1}{n}} + r^{\frac{1}{n}}\right)^n \right[,$$

where $I(x,r)$ comes from the previously-stated support of the measure $\nu_{k,n}^{x,y}$ 5.2.4.

Lemma 6.2.2. *The measure $\mu_{k,n}$ is doubling for the intervals $I(x,r)$ such that:*
(1) $0 < \mu_{k,n}(I(x,r)) < \infty$ for all $I(x,r)$.
(2) $\mu_{k,n}(I(x,2r)) \lesssim \mu_{k,n}(I(x,r))$ for all $x \in \mathbb{R}$ and $r > 0$.

Proof. Part (1). The measure $\mu_{k,n}$ is a positive measure and therefore $\mu_{k,n}(I(x,r))$ is positive. Any continuous function on a closed interval is Riemann-integrable and thus finite.

The proof of part (2) will consider three cases.

(i) assume $|x|^{\frac{1}{n}} \leq r^{\frac{1}{n}} \leq (2r)^{\frac{1}{n}}$ then

$$\begin{aligned}
\mu_{k,n}(I(x,2r)) &= \int_0^{\left(|x|^{\frac{1}{n}}+(2r)^{\frac{1}{n}}\right)^n} d\mu_{k,n}(z) \\
&= (M_{k,n})^{-1}\frac{(|x|^{\frac{1}{n}}+(2r)^{\frac{1}{n}})^{2nk+2-n}}{(2k+\frac{2}{n}-1)} \\
&\lesssim (|x|^{\frac{1}{n}}+(2)^{\frac{1}{n}}r^{\frac{1}{n}})^{2nk+2-n} \\
&= ((2)^{\frac{1}{n}}(\frac{|x|^{\frac{1}{n}}}{(2)^{\frac{1}{n}}}+r^{\frac{1}{n}}))^{2nk+2-n} \\
&\leq ((2)^{\frac{1}{n}}(|x|^{\frac{1}{n}}+r^{\frac{1}{n}}))^{2nk+2-n} \\
&\lesssim (|x|^{\frac{1}{n}}+r^{\frac{1}{n}})^{2nk+2-n}
\end{aligned}$$

therefore we obtain the following,

$$\mu_{k,n}(I(x,2r)) \lesssim \mu(I(x,r)).$$

(ii) Now assume that $r^{\frac{1}{n}} \leq |x|^{\frac{1}{n}} \leq (2r)^{\frac{1}{n}}$.

Then, in those circumstances the measure of the interval becomes as follows,

$$\begin{aligned}
\mu_{k,n}(I(x,2r)) &= \int_0^{(|x|^{\frac{1}{n}}+(2r)^{\frac{1}{n}})^n} d\mu_{k,n}(z) \\
&= (M_{k,n})^{-1}\frac{(|x|^{\frac{1}{n}}+(2r)^{\frac{1}{n}})^{2nk+2-n}}{(2k+\frac{2}{n}-1)} \\
&\lesssim (2(2)^{\frac{1}{n}}r^{\frac{1}{n}})^{2nk+2-n} \\
&\lesssim (r^{\frac{1}{n}})^{2nk+2-n} \\
&= r^{2k+\frac{2}{n}-1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\mu_{k,n}(I(x,r)) &= \int_{(|x|^{\frac{1}{n}}-r^{\frac{1}{n}})^n}^{(|x|^{\frac{1}{n}}+r^{\frac{1}{n}})^n} d\mu_{k,n}(z)\\ &\geq \int_{((2r)^{\frac{1}{n}}-r^{\frac{1}{n}})^n}^{(2r^{\frac{1}{n}})^n} d\mu_{k,n}(z)\\ &= (M_{k,n})^{-1}\frac{(2^n r)^{2k+\frac{2}{n}-1}-(r((2)^{\frac{1}{n}}-1)^n)^{2k+\frac{2}{n}-1}}{(2k+\frac{2}{n}-1)}\\ &= (M_{k,n})^{-1}\frac{2^{2nk+2-n}r^{2k+\frac{2}{n}-1}-r^{2k+\frac{2}{n}-1}((2)^{\frac{1}{n}}-1)^{2nk+2-n}}{(2k+\frac{2}{n}-1)}\\ &= (M_{k,n})^{-1}\frac{2^{2nk+2-n}-((2)^{\frac{1}{n}}-1)^{2nk+2-n}}{(2k+\frac{2}{n}-1)}\,r^{2k+\frac{2}{n}-1}.\end{aligned}$$

So we will arrive again at the result,

$$\mu_{k,n}(I(x,2r)) \lesssim \mu(I(x,r)).$$

(iii) For the last case let $|x|^{\frac{1}{n}} \geqslant (2r)^{\frac{1}{n}}$, then,

$$\begin{aligned}\mu_{k,n}(I(x,2r)) &= \int_{(|x|^{\frac{1}{n}}-(2r)^{\frac{1}{n}})^n}^{(|x|^{\frac{1}{n}}+(2r)^{\frac{1}{n}})^n} d\mu_{k,n}(z)\\ &\leqslant \int_{(|x|^{\frac{1}{n}}-r^{\frac{1}{n}})^n}^{(2|x|^{\frac{1}{n}})^n} d\mu_{k,n}(z)\\ &= (M_{k,n})^{-1}\ \frac{(2|x|^{\frac{1}{n}})^{2nk+2-n}-(|x|^{\frac{1}{n}}-r^{\frac{1}{n}})^{2nk+2-n}}{(2k+\frac{2}{n}-1)},\end{aligned}$$

and,

$$\begin{aligned}\mu_{k,n}(I(x,r)) &= \int_{(|x|^{\frac{1}{n}}-r^{\frac{1}{n}})^n}^{(|x|^{\frac{1}{n}}+r^{\frac{1}{n}})^n} d\mu_{k,n}(z)\\ &= (M_{k,n})^{-1}\ \frac{(|x|^{\frac{1}{n}}+r^{\frac{1}{n}})^{2nk+2-n}-(|x|^{\frac{1}{n}}-r^{\frac{1}{n}})^{2nk+2-n}}{(2k+\frac{2}{n}-1)}.\end{aligned}$$

Futhermore, since the following statement is true,

$$\begin{aligned}|x|^{\frac{1}{n}}+r^{\frac{1}{n}} &\geqslant |x|^{\frac{1}{n}}\\ 2(|x|^{\frac{1}{n}}+r^{\frac{1}{n}}) &\geqslant 2|x|^{\frac{1}{n}}\\ |x|^{\frac{1}{n}}+r^{\frac{1}{n}} &\gtrsim 2|x|^{\frac{1}{n}}\end{aligned}$$

then,

$$\mu_{k,n}(I(x,2r)) \lesssim \mu(I(x,r)).$$

□

Here, we state the Vitali-Type covering lemma that will shed a light on the engulfing property of the intrvals $I(x,r)$. For the purpose of proving this lemma we will denote the diameter of an interval $I(x,y)$ by $diam(I(x,r))$.

Lemma 6.2.3. *Assume that E is a $\mu_{k,n}$-measurable subset of $\mathbb{R}_+^*$, and that there exists a finite collection of intervals $\{I(x_j,r_j)\}_{1\leqslant j\leqslant L}$ covering E. Then, there exists a disjoint subcollection $I(x_{m_1},r_{m_1}),\ldots,I(x_{m_\ell},r_{m_\ell})$ of the intervals $I(x_j,r_j)$ that satisfies the following.*

$$\mu_k(E) \lesssim \sum_{i=1}^{\ell} \mu_{k,n}\left(I\left(x_{m_i},r_{m_i}\right)\right) \tag{6.2.1}$$

Proof. We follow a regular selection method. The interval with the biggest diameter in the collection would be selected first, let it be denoted as $I(x_{m_1},r_{m_1})$, then all other intervals that have an intersection with this interval should be removed. Repeat this procedure until all intervals are either selected or removed. After this greedy algorithm, we end up with a subcollection of disjoint intervals $I(x_{m_1},r_{m_1}),\ldots,I(x_{m_\ell},r_{m_\ell})$.

To get the desired inequality, we will invoke the doubling property that was proved previously in Lemma 6.2.2. In addition, we must prove that every removed interval $I(x_j,r_j)$ is included in some dilated version of a selected interval, precisely, $I(x_{m_i},cr_{m_i}), 1\leqslant i\leqslant \ell$, for some positive constant $c\geqslant 1$.

To do so, let us assume that one of the removed intervals is given by $I(x_j,r_j)$. Thus, due to our selection method, there exists a smallest $i, 1\leqslant i\leqslant \ell$, such that the intersection $I(x_j,r_j)\cap I(x_{m_i},r_{m_i})$ is nonempty, and $diam\left(I\left(x_j,r_j\right)\right) \leqslant diam\left(I\left(x_{n_l},r_{n_l}\right)\right)$.

Then, there exists some constant $c\geqslant 1$ such that $I(x_j,r_j)\subset I(x_{m_i},c\,r_{m_i})$. We divide the proof of this statement into two cases.

(i) For the first case, let us assume that $I(x_j,r_j) = \left]0, \left(|x_j|^{\frac{1}{n}} + r_j^{\frac{1}{n}}\right)^n\right[$. Here, we have two possible subcases.

(a) First consider that $I(x_{m_i}, r_{m_i}) = \left]0, \left(|x_{m_i}|^{\frac{1}{n}} + r_{m_i}^{\frac{1}{n}}\right)^n\right[$.
That is, $|x_{m_i}|^{\frac{1}{n}} \leqslant r_{m_i}^{\frac{1}{n}}$. Since

$$diam(I(x_j, r_j)) \leqslant diam(I(x_{m_i}, r_{m_i}))$$

then, it is clear to see that

$$(I(x_j, r_j)) \subset (I(x_{m_i}, r_{m_i})).$$

(b) Now, let us consider the case where $r_{m_i}^{\frac{1}{n}} \leqslant |x_{m_i}|^{\frac{1}{n}}$. Consequently, the interval $I(x_{m_i}, r_{m_i})$ takes the form

$$I(x_{m_i}, r_{m_i}) = \left]\left(|x_{m_i}|^{\frac{1}{n}} - r_{m_i}^{\frac{1}{n}}\right)^n, \left(|x_{m_i}|^{\frac{1}{n}} + r_{m_i}^{\frac{1}{n}}\right)^n\right[.$$

then, by the diameter property we obtain the following

$$\left(|x_j|^{\frac{1}{n}} + r_j^{\frac{1}{n}}\right)^n \leqslant \left(|x_{m_i}|^{\frac{1}{n}} + r_{m_i}^{\frac{1}{n}}\right)^n - \left(|x_{m_i}|^{\frac{1}{n}} - r_{m_i}^{\frac{1}{n}}\right)^n.$$

Moreover, since $I(x_{m_i}, r_{m_i}) \cap I(x_j, r_j) \neq \phi$, then,

$$\left(|x_{m_i}|^{\frac{1}{n}} - r_{m_i}^{\frac{1}{n}}\right)^n \leqslant \left(|x_j|^{\frac{1}{n}} + r_j^{\frac{1}{n}}\right)^n$$

and from the above inequalities we get

$$\begin{aligned}
\left(|x_{m_i}|^{\frac{1}{n}} - r_{m_i}^{\frac{1}{n}}\right)^n &\leqslant \left(|x_{m_i}|^{\frac{1}{n}} + r_{m_i}^{\frac{1}{n}}\right)^n - \left(|x_{m_i}|^{\frac{1}{n}} - r_{m_i}^{\frac{1}{n}}\right)^n \\
2\left(|x_{m_i}|^{\frac{1}{n}} - r_{m_i}^{\frac{1}{n}}\right)^n &\leqslant \left(|x_{m_i}|^{\frac{1}{n}} + r_{m_i}^{\frac{1}{n}}\right)^n \\
(2\left(|x_{m_i}|^{\frac{1}{n}} - r_{m_i}^{\frac{1}{n}}\right)^n)^{\frac{1}{n}} &\leqslant (\left(|x_{m_i}|^{\frac{1}{n}} + r_{m_i}^{\frac{1}{n}}\right)^n)^{\frac{1}{n}} \\
(2)^{\frac{1}{n}}|x_{m_i}|^{\frac{1}{n}} - (2)^{\frac{1}{n}} r_{m_i}^{\frac{1}{n}} &\leqslant |x_{m_i}|^{\frac{1}{n}} + r_{m_i}^{\frac{1}{n}} \\
(2)^{\frac{1}{n}}|x_{m_i}|^{\frac{1}{n}} - |x_{m_i}|^{\frac{1}{n}} &\leqslant r_{m_i}^{\frac{1}{n}} + (2)^{\frac{1}{n}} r_{m_i}^{\frac{1}{n}} \\
\left((2)^{\frac{1}{n}} - 1\right)|x_{m_i}|^{\frac{1}{n}} &\leqslant \left((2)^{\frac{1}{n}} + 1\right) r_{m_i}^{\frac{1}{n}} \\
|x_{m_i}|^{\frac{1}{n}} &\leqslant \frac{\left((2)^{\frac{1}{n}} + 1\right)}{\left((2)^{\frac{1}{n}} - 1\right)} r_{m_i}^{\frac{1}{n}} \qquad (6.2.2)
\end{aligned}$$

which implies that the form of the interval $I\left(x_{m_i},\left(\frac{(2)^{\frac{1}{n}}+1}{(2)^{\frac{1}{n}}-1}\right)^n r_{m_i}\right)$ will be given by

$$I\left(x_{m_i},\left(\frac{(2)^{\frac{1}{n}}+1}{(2)^{\frac{1}{n}}-1}\right)^n r_{m_i}\right)=\left]0,\left(|x_{m_i}|^{\frac{1}{n}}+\frac{(2)^{\frac{1}{n}}+1}{(2)^{\frac{1}{n}}-1}|r_{m_i}|^{\frac{1}{n}}\right)^n\right[.$$

Due to the form of $I(x_{m_i},r_{m_i})$ together with the inequality (6.2.2),

$$r_{m_i}^{\frac{1}{n}}\leqslant|x_{m_i}|^{\frac{1}{n}}\leqslant\frac{(2)^{\frac{1}{n}}+1}{(2)^{\frac{1}{n}}-1}r_{m_i}^{\frac{1}{n}}$$

and from the inequality obtained by the diameter property

$$\left(|x_j|^{\frac{1}{n}}+r_j^{\frac{1}{n}}\right)^n\leqslant\left(|x_{m_i}|^{\frac{1}{n}}+r_{m_i}^{\frac{1}{n}}\right)^n,$$

then,

$$\left(|x_j|^{\frac{1}{n}}+r_j^{\frac{1}{n}}\right)^n\leqslant\left(|x_{m_i}|^{\frac{1}{n}}+\frac{(2)^{\frac{1}{n}}+1}{(2)^{\frac{1}{n}}-1}r_{m_i}^{\frac{1}{n}}\right)^n.$$

It is clear from the above inequality and the forms of the two intervals that

$$(I(x_j,r_j))\subset I\left(x_{m_i},\left(\frac{2^{\frac{1}{n}}+1}{2^{\frac{1}{n}}-1}\right)^n r_{m_i}\right).$$

(ii) For the second case, let us now assume that $|x_j|^{\frac{1}{n}}\geqslant(r_j)^{\frac{1}{n}}$. Then, the interval is given by

$$I(x_j,r_j)=\left]\left(|x_j|^{\frac{1}{n}}-r_j^{\frac{1}{n}}\right)^n,\left(|x_j|^{\frac{1}{n}}+r_j^{\frac{1}{n}}\right)^n\right[.$$

Now, consider the case where $|x_{m_i}|^{\frac{1}{n}}\leqslant r_{m_i}^{\frac{1}{n}}$. That is,

$$I(x_{m_i},r_{m_i})=\left]0,\left(|x_{m_i}|^{\frac{1}{n}}+r_{m_i}^{\frac{1}{n}}\right)^n\right[.$$

If we assume that

$$\left(|x_{m_i}|^{\frac{1}{n}}+r_{m_i}^{\frac{1}{n}}\right)^n\geqslant\left(|x_j|^{\frac{1}{n}}+r_j^{\frac{1}{n}}\right)^n.$$

Then, it is clear from the forms of the two intervals that $I(x_j, r_j) \subset I(x_{m_i}, r_{m_i})$.
If not, then from the fact that $I(x_j, r_j) \cap I(x_{m_i}, r_{m_i}) \neq \emptyset$ we get

$$\left(|x_j|^{\frac{1}{n}} - r_j^{\frac{1}{n}}\right)^n \leqslant \left(|x_{m_i}|^{\frac{1}{n}} + r_{m_i}^{\frac{1}{n}}\right)^n \leqslant \left(|x_j|^{\frac{1}{n}} + r_j^{\frac{1}{n}}\right)^n, \tag{6.2.3}$$

and from the diameter property

$$\begin{aligned}\left(|x_j|^{\frac{1}{n}} + r_j^{\frac{1}{n}}\right)^n &= \left(|x_j|^{\frac{1}{n}} - r_j^{\frac{1}{n}}\right)^n + diam\left(I\left(x_j, r_j\right)\right) \\ &\leqslant \left(|x_j|^{\frac{1}{n}} - r_j^{\frac{1}{n}}\right)^n + diam\left(I\left(x_{m_i}, r_{m_i}\right)\right) \\ &\leqslant \left(|x_j|^{\frac{1}{n}} - r_j^{\frac{1}{n}}\right)^n + \left(|x_{m_i}|^{\frac{1}{n}} + r_{m_i}^{\frac{1}{n}}\right)^n.\end{aligned}$$

By the inequality 6.2.3 and from the form of $I(x_{m_i}, r_{m_i})$ we get the following

$$\begin{aligned}\left(|x_j|^{\frac{1}{n}} + r_j^{\frac{1}{n}}\right)^n &\leqslant 2\left(|x_{m_i}|^{\frac{1}{n}} + r_{m_i}^{\frac{1}{n}}\right)^n \\ &\leqslant \left((2)^{\frac{1}{n}}|x_{m_i}|^{\frac{1}{n}} + (2)^{\frac{1}{n}} r_{m_i}^{\frac{1}{n}}\right)^n \\ &\leqslant \left((2^{n+1} r_{m_i})^{\frac{1}{n}}\right)^n \\ &\leqslant \left(|x_{m_i}|^{\frac{1}{n}} + (2^{n+1} r_{m_i})^{\frac{1}{n}}\right)^n,\end{aligned}$$

and since

$$|x_{m_i}|^{\frac{1}{n}} \leqslant r_{m_i}^{\frac{1}{n}} \leqslant (2^{n+1} r_{m_i})^{\frac{1}{n}}.$$

Then, the form of the interval $I\left(x_{m_i}, 2^{n+1} r_{m_i}\right)$ would be given by

$$I\left(x_{m_i}, 2^{n+1} r_{m_i}\right) = \left(0, \left(|x_{m_i}|^{\frac{1}{n}} + (2^{n+1} r_{m_i})^{\frac{1}{n}}\right)^n\right).$$

Therefore, it is clear from the forms of the two intervals and the diameter property that

$$I(x_j, r_j) \subset I(x_{m_i}, 2^{n+1} r_{m_i}).$$

Now, consider the case where $|x_{m_i}|^{\frac{1}{n}} \geqslant r_{m_i}^{\frac{1}{n}}$.
Which leads to the Interval $I(x_{m_i}, r_{m_i})$ taking the form

$$I(x_{m_i}, r_{m_i}) = \left](|x_{m_i}|^{\frac{1}{n}} - r_{m_i}^{\frac{1}{n}})^n, (|x_{m_i}|^{\frac{1}{n}} + r_{m_i}^{\frac{1}{n}})^n\right[.$$

Here, we have three possible cases, the first being the case where

$$\begin{aligned}\left(|x_{m_i}|^{\frac{1}{n}} - (r_{m_i})^{\frac{1}{n}}\right)^n &\leqslant \left(|x_j|^{\frac{1}{n}} - (r_j)^{\frac{1}{n}}\right)^n \\ &\leqslant \left(|x_j|^{\frac{1}{n}} + (r_j)^{\frac{1}{n}}\right)^n \\ &\leqslant \left(|x_{m_i}|^{\frac{1}{n}} + (r_{m_i})^{\frac{1}{n}}\right)^n.\end{aligned}$$

Then, clearly,

$$I(x_j, r_j) \subset I(x_{m_i}, r_{m_i}).$$

In the second case consider

$$\begin{aligned}\left(|x_{m_i}|^{\frac{1}{n}} - (r_{m_i})^{\frac{1}{n}}\right)^n &\leqslant \left(|x_j|^{\frac{1}{n}} - (r_j)^{\frac{1}{n}}\right)^n \\ &\leqslant \left(|x_{m_i}|^{\frac{1}{n}} + (r_{m_i})^{\frac{1}{n}}\right)^n \\ &\leqslant \left(|x_j|^{\frac{1}{n}} + (r_j)^{\frac{1}{n}}\right)^n,\end{aligned}$$

then, by the diameter rule,

$$\begin{aligned}&\left(|x_j|^{\frac{1}{n}} + (r_j)^{\frac{1}{n}}\right)^n \\ &\quad\leqslant \left(|x_{m_i}|^{\frac{1}{n}} + (r_{m_i})^{\frac{1}{n}}\right)^n - \left(|x_{m_i}|^{\frac{1}{n}} - (r_{m_i})^{\frac{1}{n}}\right)^n + \left(|x_j|^{\frac{1}{n}} - (r_j)^{\frac{1}{n}}\right)^n \\ &\quad= 2\left(|x_{m_i}|^{\frac{1}{n}} + (r_{m_i})^{\frac{1}{n}}\right)^n - \left(|x_{m_i}|^{\frac{1}{n}} - (r_{m_i})^{\frac{1}{n}}\right)^n \\ &\quad= 2\sum \binom{m}{k} |x_{m_i}|^{\frac{n-k}{n}} |r_{m_i}|^{\frac{k}{n}} - \sum \binom{n}{k} |x_{m_i}|^{\frac{n-k}{n}} |r_{m_i}|^{\frac{k}{n}} (-1)^k \\ &\quad= \sum \binom{m}{k} |x_{m_i}|^{\frac{n-k}{n}} |r_{m_i}|^{\frac{k}{n}} (2 - (-1)^k) \\ &\quad\leqslant \sum \binom{m}{k} |x_{m_i}|^{\frac{n-k}{n}} (3|r_{m_i}|^{\frac{1}{n}})^k \\ &\quad= \left(|x_{m_i}|^{\frac{1}{n}} + 3r_{m_i}^{\frac{1}{n}}\right)^n.\end{aligned}$$

Note that we have two possible sub-cases here, one being that $|x_{m_i}|^{\frac{1}{n}} \leqslant 3(r_{m_i})^{\frac{1}{n}}$. Then,

$$I(x_{m_i}, 3^n r_{m_i}) = (0, (|x_{m_i}|^{\frac{1}{n}} + 3r_{m_i}^{\frac{1}{n}})^n),$$

hence,

$$I(x_j, r_j) \subset I(x_{m_i}, 3^n r_{m_i}).$$

Conversely, if $|x_{m_i}|^{\frac{1}{n}} \geqslant 3(r_{m_i})^{\frac{1}{n}}$, then

$$I(x_{m_i}, 3^n r_{m_i}) = ((|x_{m_i}|^{\frac{1}{n}} - 3r_{m_i}^{\frac{1}{n}})^n, (|x_{m_i}|^{\frac{1}{n}} + 3r_{m_i}^{\frac{1}{n}})^n),$$

and

$$I(x_j, r_j) \subset I(x_{m_i}, 3^n r_{m_i})$$

follows consequently.

The last possible case is

$$\begin{aligned}\left(|x_j|^{\frac{1}{n}} - (r_j)^{\frac{1}{n}}\right)^n &\leqslant \left(|x_{m_i}|^{\frac{1}{n}} - (r_{m_i})^{\frac{1}{n}}\right)^n \\ &\leqslant \left(|x_j|^{\frac{1}{n}} + (r_j)^{\frac{1}{n}}\right)^n \\ &\leqslant \left(|x_{m_i}|^{\frac{1}{n}} + (r_{m_i})^{\frac{1}{n}}\right)^n\end{aligned}$$

which is to say that $|x_{m_i}|^{\frac{1}{n}} \geqslant r_{m_i}^{\frac{1}{n}} \geqslant 0$. Then there must be a constant C big enough to get $|x_{m_i}|^{\frac{1}{n}} \leqslant C(r_{m_i})^{\frac{1}{n}}$ and thus the interval would be $I(x_{m_i}, C^n r_{m_i}) = \left]0, \left(|x_{m_i}|^{\frac{1}{n}} + (r_{m_i})^{\frac{1}{n}}\right)^n\right[$. consequently, $I(x_j, r_j) \subset I(x_{m_i}, C^n r_{m_i})$.

Now, we have accomplished the proof of the claim which is that every interval we haven't selected ($I(x_j, r_j)$) is contained in some inflation of a selected interval $I(x_{m_i}, r_{m_i})$. This, as well as the doubling property produce the following

$$\begin{aligned}\mu_{k,n}(E) &\leqslant \mu_{k,n}(\cup_{j=1}^{m} I(x_j, r_j)) \\ &\leqslant \mu_{k,n}(\cup_{i=1}^{\ell} I(x_{m_i}, Cr_{m_i})) \\ &\lesssim \mu_{k,n}(\cup_{i=1}^{\ell} I(x_{m_i}, Cr_{m_i})) \\ &= \sum_{i=1}^{\ell} \mu_{k,n}(I(x_{m_i}, r_{m_i}))\end{aligned}$$

which concludes the proof of Lemma 6.2.3. $\square$

6.3 A Sharp Estimate for the Generalized Translation Operator

In the following section, a control of the translate of the characteristic function χ_r of the interval $]-r, r[$, where $r > 0$ will be proved. This estimate will have a critical role in the proof of the main result of this thesis.

The following theorem is in fact the main result of this section.

Theorem 6.3.1. *For every $x \in \mathbb{R}^*$ and for almost every $y \in \mathbb{R}^*$,we have*

$$|\tau_x^{k,n}\chi_r(y)| \lesssim \frac{\mu_{k,n}(]-r,r[)}{\mu_{k,n}(I(x,r))}.$$

The proof of this theorem depends mostly on the proposition below.

Proposition 6.3.2. *For every $x \in \mathbb{R}^*$ and almost every $y \in \mathbb{R}^*$, we have*

$$|\tau_x^{k,n}\chi_r(y)| \lesssim \left(\frac{r}{|x|}\right)^{2k-1+\frac{1}{n}}.$$

In order to prove Proposition 6.3.2 we need the following three lemmas.

Lemma 6.3.3. *for all $x, y \in \mathbb{R}^*$,*

$$|B_{k,n}(x,y)| \lesssim |xy|^{-k-\frac{1}{2n}+\frac{1}{2}}$$

Proof. By the Definition 5.2.1 of $B_{k,n}(x,y)$ we have

$$\begin{aligned}
&B_{k,n}(x,y)\\
&\quad= \Gamma(kn-\frac{n}{2}+1)\left(\frac{n|xy|^{\frac{1}{n}}}{2}\right)^{-kn+\frac{n}{2}} J_{kn-\frac{n}{2}}\left(n|yx|^{\frac{1}{n}}\right)\\
&\quad+ (-i)^n\left(\frac{n}{2}\right)^n\Gamma(kn-\frac{n}{2}+1)xy\left(\frac{n|xy|^{\frac{1}{n}}}{2}\right)^{-kn-\frac{n}{2}} J_{kn+\frac{n}{2}}\left(n|xy|^{\frac{1}{n}}\right)\\
&= \Gamma(kn-\frac{n}{2}+1)2^{kn-\frac{n}{2}}\frac{J_{kn-\frac{n}{2}}(n|xy|^{\frac{1}{n}})}{(n|xy|^{\frac{1}{n}})^{kn-\frac{n}{2}}}\\
&\quad+ (-i)^n\left(\frac{n}{2}\right)^n\Gamma(kn-\frac{n}{2}+1)2^{kn+\frac{n}{2}}xy\frac{J_{kn+\frac{n}{2}}(n|xy|^{\frac{1}{n}})}{(n|xy|^{\frac{1}{n}})^{kn+\frac{n}{2}}},
\end{aligned}$$

and by using $\sup\limits_{u\geq 0} u^{1/2}|J_\nu(u)| < +\infty$, we get,

$$\begin{aligned}|B_{k,n}(x,y)| &\lesssim \frac{(|xy|^{\frac{1}{n}})^{\frac{-1}{2}}}{(|xy|^{\frac{1}{n}})^{kn-\frac{n}{2}}} + |xy|\frac{(|xy|^{\frac{1}{n}})^{\frac{-1}{2}}}{(|xy|^{\frac{1}{n}})^{kn+\frac{n}{2}}} \\ &= |xy|^{-\frac{1}{2n}}|xy|^{-k+\frac{1}{2}} + |xy||xy|^{-\frac{1}{2n}}|xy|^{-k-\frac{1}{2}} \\ &= |xy|^{-\frac{1}{2n}-k+\frac{1}{2}} + |xy|^{-\frac{1}{2n}-k+\frac{1}{2}} \\ &= 2|xy|^{-k+\frac{1}{2}-\frac{1}{2n}} \\ &\lesssim |xy|^{-k+\frac{1}{2}-\frac{1}{2n}}.\end{aligned}$$

□

Lemma 6.3.4. *The generalized Fourier transform of the characteristic function* χ_r *satisfies the following*

$$|\mathcal{F}_{k,n}(\chi_r)(x)| \lesssim r^{2k+\frac{2}{n}-1}, \quad \forall x \in \mathbb{R} \tag{6.3.1}$$

and

$$|\mathcal{F}_{k,n}(\chi_r)(x)| \lesssim \frac{r^{k-\frac{1}{2}+\frac{1}{2n}}}{x^{k-\frac{1}{2}+\frac{3}{2n}}}, \quad \forall x \in \mathbb{R}^* \tag{6.3.2}$$

Proof. let us recall that

$$\mathcal{F}_{k,n}f(y) = \int_{\mathbb{R}} f(x)B_{k,n}(x,y)d\mu_{k,n}(x), \quad y \in \mathbb{R}.$$

Since $\chi_r(x) \leqslant 1$ and $|B_{k,n}(x,\lambda)| \leqslant C$ then,

$$\begin{aligned}\big|\mathcal{F}_{k,n}(\chi_r)(y)\big| &= \left|\int_{\mathbb{R}} \chi_r(x)B_{k,n}(x,y)d\mu_{k,n}(x)\right| \\ &\leqslant C\int_{-r}^{r} d\mu_{k,n}(x) \\ &= 2C\int_0^r d\mu_{k,n}(x) \\ &= 2Cr^{2k+\frac{2}{n}-1}\end{aligned}$$

Now, from Proposition 5.3.1 we have,

$$\mathcal{F}_{k,n}(f)(x) = \mathcal{F}_{k,n}G_n(\widetilde{f_e})(x) + \mathcal{F}_{k,n}J_n(\widetilde{f_o})(x),$$

and since $\chi_r(y)$ is an even function, thus,

$$
\begin{aligned}
&\mathcal{F}_{k,n}(\chi_r)(\lambda)\\
&\quad= \mathcal{H}_{kn-\frac{n}{2}}(G_n(\widetilde{\chi_r}))(y)\\
&\quad= \int_0^\infty \chi_r\left(\left(\frac{t}{n}\right)^n\right) j_{kn-\frac{n}{2}}(t\lambda^{\frac{1}{n}})t^{2kn-n+1}dt\\
&\quad= \int_0^\infty \chi_r(T^n) j_{kn-\frac{n}{2}}(nT\lambda^{\frac{1}{n}})nT^{2kn-n+1}dT\\
&\quad\leqslant \int_0^{r^{\frac{1}{n}}} j_{kn-\frac{n}{2}}(nT\lambda^{\frac{1}{n}})nT^{2kn-n+1}dT\\
&\quad\leqslant \int_0^{r^{\frac{1}{n}}} \frac{J_{kn-\frac{n}{2}}(nT\lambda^{\frac{1}{n}})}{(T\lambda^{\frac{1}{n}})^{kn-\frac{n}{2}}} T^{2kn-n+1}dT\\
&\quad= r^{\frac{1}{n}} \int_0^1 \frac{J_{kn-\frac{n}{2}}(n\lambda^{\frac{1}{n}}r^{\frac{1}{n}}U)}{\lambda^{k-\frac{1}{2}}r^{\frac{1}{n}(kn-\frac{n}{2})}U^{kn-\frac{n}{2}}} r^{\frac{2kn-n+1}{n}}U^{2kn-n+1}dU\\
&\quad= r^{\frac{1}{n}} \int_0^1 J_{kn-\frac{n}{2}}(n\lambda^{\frac{1}{n}}r^{\frac{1}{n}}U)\lambda^{-k+\frac{1}{2}}r^{-k+\frac{1}{2}}r^{2k-1+\frac{1}{n}}U^{2kn-n+1}U^{-kn+\frac{n}{2}}dU\\
&\quad= r^{\frac{2}{n}+k-\frac{1}{2}}\lambda^{-k+\frac{1}{2}} \int_0^1 J_{kn-\frac{n}{2}}(n\lambda^{\frac{1}{n}}r^{\frac{1}{n}}U)U^{kn-\frac{n}{2}+1}dU\\
&\quad\leqslant \frac{r^{k-\frac{1}{2}+\frac{1}{2n}}}{\lambda^{k-\frac{1}{2}+\frac{3}{2n}}}
\end{aligned}
$$

Here we have used

$$\int_0^1 J_\nu(xy)y^{\nu+1}dy = x^{-1}J_{\nu+1}(x)$$

and $\sup\limits_{u\geq 0} u^{1/2}|J_\nu(u)| < +\infty$. □

Definition 6.3.5. *For* $t > 0$, *and* $x \in \mathbb{R}$, *let*

$$q_t(x) = \left(\frac{2}{n}\right)^{kn-\frac{n}{2}+1} t^{-kn+\frac{n}{2}-1}e^{\frac{-|x|^{\frac{2}{n}}}{t}}.$$

The third lemma needed to prove Proposition 6.3.2 is the following:

Lemma 6.3.6. *for* $t > 0$, q_t *satisfies*

$$\|q_t\|_{k,1} = 1$$

and

$$\mathcal{F}_{k,n}(q_t)(\lambda) = e^{\frac{-t|\lambda|^{\frac{2}{n}}n^2}{4}}$$

for all $x \in \mathbb{R}$.

Proof. Since

$$\mathcal{F}_{k,n}(f_e)(\lambda) = c\int_0^{+\infty} f_e(x) j_{kn-\frac{n}{2}}(n|\lambda x|^{\frac{1}{n}})|x|^{2k+\frac{2}{n}-2}dx.$$

Then,

$$\mathcal{F}_{k,n}\left(\left(\frac{2}{n}\right)^{kn-\frac{n}{2}+1} t^{-kn+\frac{n}{2}-1} e^{\frac{-|x|^{\frac{2}{n}}}{t}}\right)(\lambda)$$
$$= c\int_0^{+\infty}\left(\left(\frac{2}{n}\right)^{kn-\frac{n}{2}+1} t^{-kn+\frac{n}{2}-1} e^{\frac{-|x|^{\frac{2}{n}}}{t}}\right) j_{kn-\frac{n}{2}}(n|\lambda x|^{\frac{1}{n}})|x|^{2k+\frac{2}{n}-2}dx.$$

Let $y = x^{\frac{1}{n}}$, then,

$$\mathcal{F}_{k,n}\left(\left(\frac{2}{n}\right)^{kn-\frac{n}{2}+1} t^{-kn+\frac{n}{2}-1} e^{\frac{-|x|^{\frac{2}{n}}}{t}}\right)(\lambda)$$
$$= c\int_0^{+\infty}\left(\left(\frac{2}{n}\right)^{kn-\frac{n}{2}+1} t^{-kn+\frac{n}{2}-1} e^{\frac{-y^2}{t}}\right) j_{kn-\frac{n}{2}}(n|\lambda|^{\frac{1}{n}}y)y^{2nk+2-2n}\, n\, y^{n-1}dy$$
$$= c\int_0^{+\infty}\left(\left(\frac{2}{n}\right)^{kn-\frac{n}{2}+1} t^{-kn+\frac{n}{2}-1} e^{\frac{-y^2}{t}}\right) \frac{J_{kn-\frac{n}{2}}(n|\lambda|^{\frac{1}{n}}y)}{(|\lambda|^{\frac{1}{n}}y)^{kn-\frac{n}{2}}}\, n\, y^{2nk-n+1}dy$$
$$= |\lambda|^{-k+\frac{1}{2}}\left(\frac{2}{n}\right)^{kn-\frac{n}{2}+1}\int_0^{+\infty}\left(t^{-kn+\frac{n}{2}-1} e^{\frac{-y^2}{t}}\right) J_{kn-\frac{n}{2}}(n|\lambda|^{\frac{1}{n}}y)\, n\, y^{nk-\frac{n}{2}+1}dy,$$

let $b = n|\lambda|^{\frac{1}{n}}, \nu = nk - \frac{n}{2}$, and $a^2 = \frac{1}{t}$. Then,

$$\mathcal{F}_{k,n}\left(\left(\frac{2}{n}\right)^{kn-\frac{n}{2}+1} t^{-kn+\frac{n}{2}-1} e^{\frac{-|x|^{\frac{2}{n}}}{t}}\right)(\lambda)$$
$$= |\lambda|^{\frac{-\nu}{n}}\left(\frac{2}{n}\right)^{\nu+1}\int_0^{+\infty}(a^{2(\nu+1)}e^{-y^2a^2})\, J_\nu(by)\, n\, y^{\nu+1}dy.$$

To find the value of the integral one needs to use the following formula

$$\int_0^{+\infty} J_v(by)y^{\nu+1}e^{-y^2a^2}dy = \frac{b^v}{(2a^2)^{\nu+1}}e^{-b^2/\left(4a^2\right)}$$

Which gives us,

$$\mathcal{F}_{k,n}\left(\left(\frac{2}{n}\right)^{kn-\frac{n}{2}+1} t^{-kn+\frac{n}{2}-1} e^{\frac{-|x|^{\frac{2}{n}}}{t}}\right)(\lambda)$$
$$= |\lambda|^{-k+\frac{1}{2}}\left(\frac{2}{n}\right)^{kn-\frac{n}{2}+1} t^{-kn+\frac{n}{2}-1} e^{\frac{-t|\lambda|^{\frac{2}{n}}n^2}{4}} \frac{n(n|\lambda|^{\frac{1}{n}})^{kn-\frac{n}{2}}}{(2(\frac{1}{t}))^{kn-\frac{n}{2}+1}}$$
$$= e^{\frac{-t|\lambda|^{\frac{2}{n}}n^2}{4}},$$

which is what needed to be proven for Lemma 6.3.6. □

Proof of Proposition 6.3.2. We have to consider two cases.

(i) First, assume that $|x|^{\frac{1}{n}} \leqslant 2r^{\frac{1}{n}}$, then, by Lemma 5.3.5 we have

$$|\tau_x^{k,n}\chi_r(y)| \leqslant \|\tau_x^{k,n}(\chi_r;\cdot)\|_{k,p} \lesssim \|\chi_r\|_{k,\infty} \leqslant \left(\frac{r}{|x|}\right)^{2k-1+\frac{1}{n}}$$

(ii) Secondly, assume that

$$|x|^{\frac{1}{n}} \geqslant 2r^{\frac{1}{n}} \text{ and } \left(|x|^{\frac{1}{n}} - y^{\frac{1}{n}}\right)^n \leqslant r,$$

so that

$$\tau_x^{k,n}\chi_r(y) \neq 0$$

due to the support of the translation operator, and from the Young inequality found in 5.3.8, we conclude that: $\chi_r \star_{k,n} q_t \in L^1(\mathbb{R}, d\mu_{k,n})$ and that the translation operator is bounded.

Therefore,

$$\tau_x^{k,n}(\chi_r \star_{k,n} q_t;\cdot) \in L^1(\mathbb{R}, d\mu_{k,n}), \tag{6.3.3}$$

for every $x \in \mathbb{R}$.

Now, using Hölder's inequality 2.2.5 and Plancherel's theorem 4.3.3, we get

$$\begin{aligned}\|\mathcal{F}_{k,n}(\chi_r \star_{k,n} q_t)\|_{k,1} &= \|\mathcal{F}_{k,n}(\chi_r)\mathcal{F}_{k,n}(q_t)\|_{k,1}\\ &\lesssim \|\mathcal{F}_{k,n}(\chi_r)\|_{k,2}\|\mathcal{F}_{k,n}(q_t)\|_{k,2}\\ &= \|\chi_r\|_{k,2}\|q_t\|_{k,2}\\ &< \infty,\end{aligned}$$

thus,

$$\mathcal{F}_{k,n}(\chi_r \star_{k,n} q_t) \in L^1(\mathbb{R}, d\mu_{k,n}) \tag{6.3.4}$$

and,

$$\mathcal{F}_{k,n}(\tau_x^{k,n}(\chi_r \star_{k,n} q_t;\cdot)) \in L^1(\mathbb{R}, d\mu_{k,n}). \tag{6.3.5}$$

From 6.3.4, 6.3.5 and 6.3.3, together with the inversion formula given in 4.3.4 we get:

$$\begin{aligned}&\tau_x^{k,n}(\chi_r \star_{k,n} q_t; y)\\ &\quad= \int_{\mathbb{R}} B_{k,n}((-1)^n x, z)B_{k,n}((-1)^n y, z)\mathcal{F}_{k,n}(\chi_r)(z)e^{\frac{-t|\lambda|^{\frac{2}{n}}n^2}{4}}d\mu_{k,n}(z)\\ &\quad= I^{(1)} + I^{(2)},\end{aligned}$$

where

$$\begin{aligned}I_1 &= \int_{\{z\in\mathbb{R}:|z|\leqslant\frac{1}{r}\}} B_{k,n}((-1)^n x, z)B_{k,n}((-1)^n y, z)\mathcal{F}_{k,n}(\chi_r)(z)e^{\frac{-t|\lambda|^{\frac{2}{n}}n^2}{4}}d\mu_{k,n}(z)\\ I_2 &= \int_{\{z\in\mathbb{R}:|z|\geqslant\frac{1}{r}\}} B_{k,n}((-1)^n x, z)B_{k,n}((-1)^n y, z)\mathcal{F}_{k,n}(\chi_r)(z)e^{\frac{-t|\lambda|^{\frac{2}{n}}n^2}{4}}d\mu_{k,n}(z).\end{aligned}$$

By Lemma 6.3.3 and the inequality 6.3.1 we get

$$\begin{aligned}|I_1| &\lesssim \int_0^{\frac{1}{r}} \frac{r^{2k+\frac{2}{n}-1}}{|xy|^{k+\frac{1}{2n}-\frac{1}{2}}} z^{-2k+1-\frac{1}{n}} z^{2k+\frac{2}{n}-2}dz\\ &= \int_0^{r^{-1}} \frac{r^{2k+\frac{2}{n}-1}}{|xy|^{k+\frac{1}{2n}-\frac{1}{2}}} z^{\frac{1}{n}-1}dz\\ &= n\frac{r^{2k+\frac{1}{n}-1}}{|xy|^{k+\frac{1}{2n}-\frac{1}{2}}}\end{aligned}$$

and since $|x|^{\frac{1}{n}} \geqslant 2r^{\frac{1}{n}}$ then,

$$|I_1| \lesssim (\frac{r}{|x|})^{2k+\frac{1}{n}-1}.$$

Now,

$$\begin{aligned}
|I_2| &\lesssim \int_{\frac{1}{r}}^{\infty} \frac{r^{k-\frac{1}{2}+\frac{1}{2n}}}{|xy|^{k+\frac{1}{2n}-\frac{1}{2}}} |z|^{-k+\frac{1}{2}-\frac{3}{2n}} z^{-2k+1-\frac{1}{n}} z^{2k+\frac{2}{n}-2} dz \\
&= \int_{\frac{1}{r}}^{\infty} \frac{r^{k-\frac{1}{2}+\frac{1}{2n}}}{|xy|^{k+\frac{1}{2n}-\frac{1}{2}}} |z|^{-k-\frac{1}{2}-\frac{1}{2n}} dz \\
&= k - \frac{1}{2} + \frac{1}{2n} \left(\frac{r^{k-\frac{1}{2}+\frac{1}{2n}} r^{k-\frac{1}{2}+\frac{1}{2n}}}{|xy|^{k+\frac{1}{2n}-\frac{1}{2}}} \right) \\
&= \frac{r^{2k-1+\frac{1}{n}}}{|xy|^{k+\frac{1}{2n}-\frac{1}{2}}}
\end{aligned}$$

thus,

$$|I_2| \lesssim \left(\frac{r}{|x|} \right)^{2k+\frac{1}{n}-1}.$$

Consequentially, for all $t > 0$

$$|\tau_x^{k,n}(\chi_r \star_{k,n} q_t; y) \lesssim \left(\frac{r}{|x|} \right)^{2k+\frac{1}{n}-1}$$

and since $\chi_r \star_{k,n} q_t \longrightarrow \chi_r$ as $t \longrightarrow 0$, then

$$|\tau_x^{k,n} \chi_r(y)| \lesssim \left(\frac{r}{|x|} \right)^{2k+\frac{1}{n}-1}$$

□

Proof of Theorem 6.3.1.

$$\begin{aligned}
\mu_{k,n}(]-r, r[) &= 2(M_{k,n})^{-1} \int_0^r |z|^{2k+\frac{2}{n}-2} dz \\
&= 2(M_{k,n})^{-1} \frac{r^{2k+\frac{2}{n}-1}}{2k+\frac{2}{n}-1}
\end{aligned}$$

if $|x|^{\frac{1}{n}} \leqslant r^{\frac{1}{n}}$

$$\begin{aligned}\mu_{k,n}(I(x,r)) &= \int_0^{(|x|^{\frac{1}{n}}+r^{\frac{1}{n}})^n} (M_{k,n})^{-1}|z|^{2k+\frac{2}{n}-2}dz\\ &\leqslant \int_0^{2^n r} (M_{k,n})^{-1}|z|^{2k+\frac{2}{n}-2}dz\\ &= (M_{k,n})^{-1}\frac{|2^n r|^{2k+\frac{2}{n}-1}}{2k+\frac{2}{n}-1}\\ &= (M_{k,n})^{-1}\frac{2^{2nk+2-n}}{2k+\frac{2}{n}-1}r^{2k+\frac{2}{n}-1}\\ &= 2^{2nk+1-n}\mu_{k,n}(]-r,r[)\end{aligned}$$

so, we can choose a constant C_k large enough so that

$$|\tau_x^{k,n}\chi_r(y)| \leqslant C_k\frac{\mu_{k,n}(]-r,r[)}{\mu_{k,n}(I(x,r))}$$

and if $|x|^{\frac{1}{n}} > r^{\frac{1}{n}}$, then

$$\begin{aligned}\mu_{k,n}(I(x,r)) &= \int_{(|x|^{\frac{1}{n}}-r^{\frac{1}{n}})^n}^{(|x|^{\frac{1}{n}}+r^{\frac{1}{n}})^n} (M_{k,n})^{-1}|z|^{2k+\frac{2}{n}-2}dz\\ &= (M_{k,n})^{-1}\int_{|x|^{\frac{1}{n}}-r^{\frac{1}{n}}}^{|x|^{\frac{1}{n}}+r^{\frac{1}{n}}} t^{2kn+2-2n}t^{n-1}dt\\ &= (M_{k,n})^{-1}\int_{|x|^{\frac{1}{n}}-r^{\frac{1}{n}}}^{|x|^{\frac{1}{n}}+r^{\frac{1}{n}}} t^{2kn+1-n}dt\\ &\lesssim (M_{k,n})^{-1}(|x|^{\frac{1}{n}}+r^{\frac{1}{n}})^{2kn-n+1}\int_{|x|^{\frac{1}{n}}-r^{\frac{1}{n}}}^{|x|^{\frac{1}{n}}+r^{\frac{1}{n}}} dt\\ &\lesssim (M_{k,n})^{-1}r^{\frac{1}{n}}(|x|^{\frac{1}{n}}+r^{\frac{1}{n}})^{2kn-n+1}\end{aligned}$$

and since $|x|^{\frac{1}{n}} > r^{\frac{1}{n}}$, and $\mu_{k,n}(]-r,r[) = 2(M_{k,n})^{-1}\frac{r^{2k+\frac{2}{n}-1}}{2k+\frac{2}{n}-1}$, then,

$$\begin{aligned}\mu_{k,n}(I(x,r)) &\lesssim (M_{k,n})^{-1}r^{\frac{1}{n}}|x|^{2k+\frac{1}{n}-1}\\ &= \left(k+\frac{1}{n}-\frac{1}{2}\right)\left(\frac{|x|}{r}\right)^{2k+\frac{1}{n}-1}\mu_{k,n}(]-r,r[).\end{aligned}$$

That is,

$$\left(\frac{r}{|x|}\right)^{2k+\frac{1}{n}-1} \lesssim \frac{\mu_{k,n}(]-r,r[)}{\mu_{k,n}(I(x,r))}.$$

Using proposition 6.3.2 the proof is concluded. □

6.4 Hardy-Littlewood-Type Maximal Theorem

We now define the generalized maximal function $\mathcal{M}_{k,n}f(x)$ in terms of the generalized translation operator $\tau_x^{k,n}$.

Definition 6.4.1. *For a locally integrable function f on $\mathbb{R}$ we define*

$$\mathcal{M}_{k,n}f(x) = \sup_{r>0} \frac{1}{\mu_{k,n}(]-r,r[)} \left| \int_{\mathbb{R}} f(y)\tau_x^{k,n}\chi_r(y)d\mu_{k,n}(y) \right|, x \in \mathbb{R}. \tag{6.4.1}$$

The following maximal theorem of Hardy–Littlewood-type for $\mathcal{M}_{k,n}$ is actually the main result of this thesis.

Theorem 6.4.2. *Let $f \in L^1_{loc}(\mathbb{R}, d\mu_{k,n})$, then,*

(1) (weak-type (1,1) estimate) if $f \in L^1(\mathbb{R}, d\mu_{k,n})$, then for every $\lambda > 0$,

$$\mu_{k,n}\left(\left\{x \in \mathbb{R} : \mathcal{M}_{k,n}f(x) > \lambda\right\}\right) \leqslant \frac{c_k}{\lambda}\|f\|_{k,1}, \tag{6.4.2}$$

where c_k is a constant independent of f and λ.

(2) (strong-type (p,p) estimate) if $f \in L^p(\mathbb{R}, d\mu_{k,n})$ with $1 < p \leqslant +\infty$, then $\mathcal{M}_{k,n}f \in L^p(\mathbb{R}, d\mu_{k,n})$ and

$$\|\mathcal{M}_{k,n}f\|_{k,p} \leqslant c_{k,p}\|f\|_{k,p}, \tag{6.4.3}$$

where the constant $c_{k,p}$ does not depend on f.

In order to achieve the proof of the above theorem, we must come up with a more convenient maximal operator $\mathbb{M}_{k,n}$.

Definition 6.4.3. *We define the maximal function $\mathbb{M}_{k,n}f$ by*

$$\mathbb{M}_{k,n}f(x) = \sup_{r>0} \frac{1}{\mu_{k,n}(I(x,r))} \int_{\{y\in\mathbb{R}:|y|\in I(x,r)\}} |f(y)|d\mu_{k,n}(y), \tag{6.4.4}$$

for every $x \in \mathbb{R}$, and every function f that is locally integrable on $\mathbb{R}$ with respect to $\mu_{k,n}$.

First, we have to prove that $\mathcal{M}_{k,n}f(x)$ is bounded up to a constant by $\mathbb{M}_{k,n}f(x)$.

Proposition 6.4.4. *Let f be a locally integrable function with respect to the measure $\mu_{k,n}$. Then, for every real number x, we have the following*

$$\mathcal{M}_{k,n}f(x) \lesssim \mathbb{M}_{k,n}f(x) \tag{6.4.5}$$

Proof. The statement is clearly true in the case of $x = 0$ since $\tau_0^{k,n}(f;y) = f(y)$ and $I(0,r) =]0,r[$. So, let us consider that $x \neq 0$. As a consequence of the support of $\tau_x^{k,n}$, we have the following

$$|y| \notin I(x,r) \longrightarrow \tau^{k,n}\chi_r(y) = 0.$$

The above fact together with Theorem 6.3.1 give us

$$\left| \int_{\mathbb{R}} f(y)\tau_x^{k,n}\chi_r(y)d\mu_{k,n}(y) \right| \lesssim \frac{\mu_{k,n}(]-r,r[)}{\mu_{k,n}((x,r))} \int_{\{y\in\mathbb{R}:|y|\in I(x,r)\}} |f(y)|d\mu_{k,n}(y).$$

The point-wise inequality follows. □

Now that we proved inequality 6.4.5, to get Theorem 6.4.2 we have to find its analogue for the new maximal operator $\mathbb{M}_{k,n}$. The covering Lemma 6.2.3 will play an important role here.

Theorem 6.4.5. *Let $f \in L^1_{Loc}(\mathbb{R}, d\mu_{k,n})$*

(1) (Weak-type $(1,1)$ estimate) If $f \in L^1(\mathbb{R}, d\mu_{k,n})$, then for every $\lambda > 0$,

$$\mu_{k,n}\left(\{x \in \mathbb{R} : \mathbb{M}_{k,n}f(x) > \lambda\}\right) \lesssim \frac{1}{\lambda}\|f\|_{k,1}$$

(2) (Strong-type (p,p) estimate) If $f \in L^p(\mathbb{R}, d\mu_{k,n})$ with $1 < p \leqslant +\infty$, then $\mathbb{M}_{k,n}f \in L^p(\mathbb{R}, d\mu_{k,n})$ and

$$\|\mathbb{M}_{k,n}f\|_{k,p} \lesssim \|f\|_{k,p}$$

Proof. Simply, we can see that $\mathbb{M}_{k,n}$ is bounded on $L^\infty(\mathbb{R}, d\mu_{k,n})$. For $1 < p < \infty$, we use the weak-type $(1,1)$ estimate, the L^∞-boundedness, and the Marcinkiewicz interpolation Theorem 2.3.7. Then, to achieve the proof we just need to show the first statement.

Let us define the following set for $\lambda > 0$,

$$\mathbb{R}_\lambda^+ := \left\{x \in \mathbb{R}_+^* : \mathbb{M}_{k,n} f(x) > \lambda\right\}$$

From the definition 6.4.4 of $\mathbb{M}_{k,n}$, it follows that $\forall\, x \in \mathbb{R}_\lambda^+ \;\exists\, r_x > 0$ such that

$$\mu_k\left(I\left(x, r_x\right)\right) < \frac{1}{\lambda} \int_{\{y \in \mathbb{R} : |y| \in I(x, r_x)\}} |f(y)| d\mu_{k,n}(y) \tag{6.4.6}$$

Let K be a compact subset of $\mathbb{R}_\lambda^+$. Since $K \subset \cup_{x \in K} I\left(x, r_x\right)$, then, by compactness there exists a finite subcover $I\left(x_1, r_1\right), \ldots I\left(x_m, r_m\right)$ of K. Using Lemma 6.2.3 we find a subcollection of pairwise disjoint intervals $I\left(x_{m_i}, r_{m_i}\right), \ldots, I\left(x_{m_i}, r_{m_i}\right)$ such that

$$\mu_{k,n}(K) \lesssim \sum_{\ell=1}^{i} \mu_{k,n}\left(I\left(x_{m_\ell}, r_{m_\ell}\right)\right) \tag{6.4.7}$$

Invoking the disjoint property of the intervals $\left(I\left(x_{n_\ell}, r_{n_\ell}\right)\right)_{1 \leqslant \ell \leqslant i}$, and using the fact that for every x_{m_i}, $\mu_{k,n}\left(I\left(x_{m_i}, r_{m_i}\right)\right)$ satisfies 6.4.6, then we may rewrite 6.4.7 as

$$\begin{aligned}
\mu_{k,n}(K) &\lesssim \frac{1}{\lambda} \sum_{\ell=1}^{i} \int_{\left\{y \in \mathbb{R} : |y| \in I\left(x_{m_\ell}, r_{m_\ell}\right)\right\}} |f(y)| d\mu_{k,n}(y) \\
&\lesssim \frac{1}{\lambda} \int_{\left\{y \in \mathbb{R} : |y| \in \cup_{\ell=1}^{i} I\left(x_{m_\ell}, r_{m_\ell}\right)\right\}} |f(y)| d\mu_{k,n}(y) \\
&\lesssim \frac{1}{\lambda} \|f\|_{k,1}
\end{aligned}$$

Since this inequality holds for every compact subset $K \subset \mathbb{R}_\lambda^+$, the inner regularity of the weighted Lebesgue measure gives us

$$\mu_{k,n}\left(\mathbb{R}_\lambda^+\right) \lesssim \frac{1}{\lambda} \|f\|_{k,1}$$

Finally, since the following fact holds.

$$\mu_{k,n}\left(\left\{x \in \mathbb{R} : \mathbb{M}_{k,n}f(x) > \lambda\right\}\right) \leqslant \mu_{k,n}\left(\mathbb{R}_\lambda^+\right) + \mu_{k,n}\left(\mathbb{R}_\lambda^-\right),$$

where

$$\mathbb{R}_\lambda^- := \left\{x \in \mathbb{R}_-^* : \mathbb{M}_{k,n}f(x) > \lambda\right\}$$

and since

$$\mathbb{M}_{k,n}f(-x) = \mathbb{M}_{k,n}f(x),$$

is true. Consequently, we have

$$\mu_{k,n}\left(\left\{x \in \mathbb{R} : \mathbb{M}_{k,n}f(x) > \lambda\right\}\right) \leqslant 2\mu_{k,n}\left(\mathbb{R}_\lambda^+\right) \lesssim \frac{1}{\lambda}\|f\|_{k,1}$$

□

Now, the first statement is proved. Therefore, Theorem 6.4.5 is proved.

On the other hand, Theorem 6.4.2 is a direct consequence from Proposition 6.4.4 together with Theorem 6.4.5.

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In this thesis we introduce and then study a maximal operator $M_{k,n}$ that generalizes the classical one introduced by Hardy and Littlewood in the rank one case. The main result is to prove the weak $(1,1)$ inequality and the strong (p,p) inequality for $M_{k,n}$, with $1 < p \leq \infty$. The approach uses geometric and analytic tools.

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