Spring 4-2022

NUMERICAL AND THEORETICAL INVESTIGATIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS

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United Arab Emirates University

MASTER THESIS NO. 2022: 55

College of Science

Department of Mathematical Sciences

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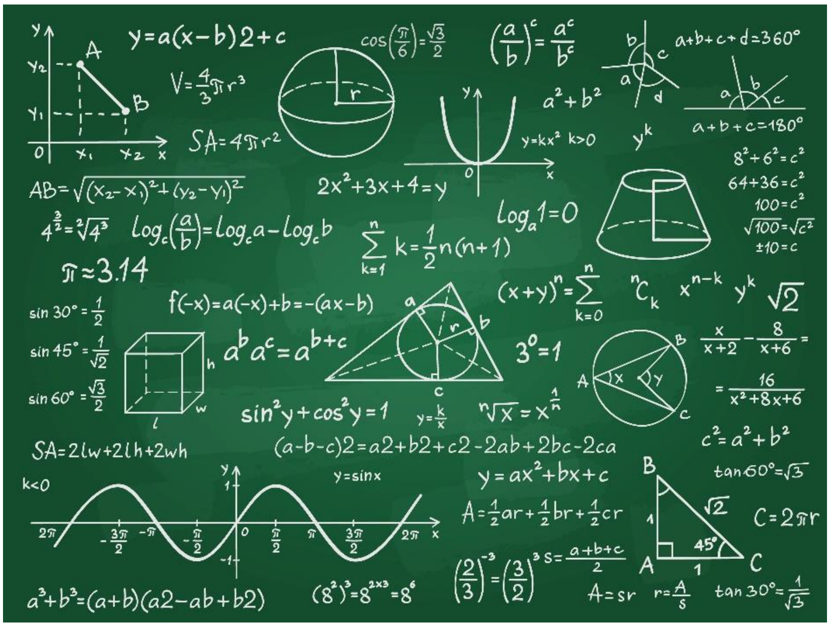

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Sara Rafiq Al Fahel

This thesis is submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

Under the Supervision of Prof. Qasem M. Al-Mdallal

April 2022

Declaration of Original Work

I, Sara Rafiq Al Fahel, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled *"Numerical and Theoretical Investigations of Fractional Differential Equations"*, hereby, solemnly declare that this thesis is an original research work that has been done and prepared by me under the supervision of Prof. Qasem M. Al-Mdallal, in the College of Science at UAEU. This work has not been previously formed as the basis for the award of any academic degree, diploma or a similar title at this or any other university. The materials borrowed from other sources and included in my thesis have been properly cited and acknowledged.

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Abstract

Fractional calculus has been recently received huge attention from Mathematicians and engineers due to its importance in many real-life applications such as: fluid mechanics, electromagnetic, acoustics, chemistry, biology, physics and material sciences. In this thesis, we present numerical algorithms for solving fractional IVPs and system of fractional IVPs where two types of fractional derivatives are used: Caputo-Fabrizio, and Atangana-Baleanu-Caputo derivatives. These algorithms are developed based on modified Adams-Bashforth method. In addition, we discuss the theoretical solution of special class of fractional IVPs. Several examples are discussed to illustrate the efficiency and accuracy of the present schemes.

Keywords: Fractional initial value problems; Fractional system of initial value problems; Caputo-Fabrizio derivative; Atangana-Baleanu-Caputo derivative.

Title and Abstract (in Arabic)

التحقيقات العددية والنظرية للمعادلات التفاضلية الكسرية

الملخص

حظي حساب التفاضل والتكامل الجزئي مؤخرًا باهتمام كبير من علماء الرياضيات والمهندسين نظرًا لأهميته في العديد من تطبيقات الحياة الواقعية مثل: ميكانيكا الموائع والكهرومغناطيسية والصوتيات والكيمياء والبيولوجيا والفيزياء وعلوم المواد. في هذه الأطروحة، نقدم خوارزميات عددية لحل مشاكل القيمة الأولية الجزئية ونظام مشاكل القيمة الأولية الجزئية حيث يتم استخدام نوعين من المشتقات الكسرية: مشتقات Caputo-Fabrizio وAtangana-Baleanu-Caputo. تم تطوير هذه الخوارزميات بناءً على طريقة Adams-Bashforth المعدلة. بالإضافة إلى ذلك نناقش الحل النظري لفئة خاصة من مشاكل القيمة الأولية الجزئية. تمت مناقشة العديد من الأمثلة لتوضيح كفاءة ودقة المخططات الحالية.

مفاهيم البحث الرئيسية: مشاكل القيمة الأولية الجزئية، نظام مشاكل القيمة الأولية الجزئية، مشتقات Caputo-Fabrizio، مشتقات Atangana-Baleanu-Caputo.

Acknowledgements

I sincerely give my deepest gratitude to my advisor, Prof. Qasem M. Al-Mdallal, for his enormous efforts, patience, professionalism, and understanding for the last year. His guidance enhance my mathematical skills, and learn new practical programming.

Special thanks go to the Department of Mathematical Sciences at UAE University for providing the computing facilities that this research heavily depends upon.

I also would like to express my sincere appreciation to the Examining Committee Members, Professor Thabet Abdeljawad and Dr. Youssef Abdul Monehm El Khatib, for their fruitful comments and suggestions.

Finally from the bottom of my heart I wish to thank my family for their patience and support throughout this work.

Dedication

To my family without whom I could not start my graduate degree

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Chapter 1: Introduction

Many researchers and scholars have deeply studied the subject of at fractional calculus in recent decades. The fact that fractional calculus considers derivatives and integrals of non-integer orders gives extra degrees of freedom and that owes to the core principle underpinning it. Fractional calculus has also caught the interest of numerous researchers due to the rapid development and improvement of nanotechnology. We can explore various real-world problems in more precise and significant ways by using arbitrary order derivatives and integrals. In addition, when compared to classical calculus, fractional calculus may be used to study the characteristic behaviours, heredity, and memory qualities of various processes and phenomena [2, 3]. The aforementioned field is more suitable for the properties of the real-world problems. In a letter to L'Hospital in 1665, Leibnitz proposed the concept of fractional calculus. The differentiation of order $\frac{1}{2}$ was described in the aforementioned letter [4]. Later, in 1819, Lacroix [5] conducted a thorough investigation into this concept. For instance, the half order derivative of a function $f(t) = t$ is given by

$$D_t^{\frac{1}{2}}[f(t)] = 2\sqrt{\frac{t}{\pi}}$$

Following that, several researchers have shown a strong desire to conduct research in this area [6]. Fourier, Abel, Liouville, Riemann, Grunwald, Letnikov, Hadamard, and others made significant contributions to fractional derivatives in the early years [7, 8]. Here, we point out that a derivative with fractional order does not have a unique definition, but can be defined in a variety of ways. Riemann-Liouville gave the first significant definition in 1832. After that, a modification was made to the fractional operator mentioned. Then, in 1967, Caputo provided a new definition, which was mostly used in dealing with several real-world problems [7]. The authors of [9, 10] show the analysis of various problems using the Caputo fractional differential operator. It is worthy mentioning herein that Riemann and Liouville were able to define the so-called fractional integral of arbitrary order as follows:

Definition 1.0.1. [6] The left sided Riemann-Liouville fractional integral operator of order α is defined by

$$I^{\alpha}y(t)=\frac{1}{\Gamma(\alpha)}\int_{a}^{t}(t-\tau)^{\alpha-1}y(\tau)d\tau,\quad \alpha\in\mathbb{R}^{+}, \tag{1.1}$$

where $y\in L_1(a,b):=\left\{z:[a,b]\to\mathbb{R}\mid \int_a^b z(t)dt<\infty\right\}$.

Notice that $\Gamma(x)$ generalizes the factorial $n!$ and allows n to take even non-integer and complex values. The Gamma function is defined by

$$\Gamma(x)=\int_{0}^{\infty}e^{-t}t^{x-1}dt,$$

for all $x\in\mathbb{R}^{+}$, provided that the integral exists. It should be noted that the definition of I^{α} given in (1.1), is used by Caputo to define the left sided Caputo fractional derivative, $D^{\alpha}y(t)$ for $y\in L_1[a,b]$ as follows:

$$({}_{a}^{C}D^{\alpha}y)(t)=I^{n-\alpha}y^{(n)}(t)=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-\tau)^{n-\alpha-1}y^{(n)}(\tau)d\tau,$$

where $n=\lceil\alpha\rceil$ is the ceiling of α.

The Caputo fractional derivative satisfies the following properties for $f\in L_1[0,1]$ $\alpha,\beta\geqslant 0$ and $n=\lceil\alpha\rceil$:

1. $D^{\alpha}I^{\alpha}f(t)=f(t)$.
2. $I^{\alpha}D^{\alpha}=f(t)-\sum_{k=0}^{n-1}f^{(k)}(0^{+})(t^{k}/k!)$.
3. $D^{\alpha}c=0$, where c is a constant.
4. $D^{\alpha}t^{\gamma}=\left\{\begin{array}{lll}0 & , & \gamma<\alpha,\ \gamma\in\{0,1,2,...\}\\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}t^{\gamma-\alpha} & , & \text{otherwise}\end{array}\right\}$.
5. $D^{\alpha}\left(\sum_{i=0}^{m}c_if_i(t)\right)=\sum_{i=0}^{m}c_iD^{\alpha}f_i(t)$, where $c_1,c_2,...,c_m$ are constants.

In 2015, Caputo and Fabrizio introduced a new definition, the Caputo-Fabrizio (CF) derivative by replacing the singular kernel in the prior definition with a non-singular one

[11]. This definition is given by

$$({}_{a}^{CF}D^{\alpha}f)(t) = \frac{M(\alpha)}{(1-\alpha)}\int_{a}^{t} f'(\tau)e^{-\frac{\alpha}{1-\alpha}(t-\tau)}d\tau$$

where $M(\alpha) > 0$ is a normalization function such that $M(0) = M(1) = 1$.

Later, in 2016, Atangana and Baleanu broadened the CF-derivative to include ABC type derivative given by The $\mathbb{ABC}$ derivative of a function $G(t)$ under the condition $G(t) \in \mathscr{H}^1(0,T)$ is defined as follows:

$${}^{\mathbb{ABC}}{}_{0}D_{t}^{\alpha}G(t) = \frac{1}{1-\alpha}\int_{0}^{t} G'(\tau)E_{\alpha}[\frac{-\alpha}{1-\alpha}(t-\tau)^{\alpha}]d\tau,$$

where, E_{α} represents the special function known as Mittag-Leffler function. This new fractional operator has proven to be quite useful in a variety of mathematical modeling and real-world problems [12]. Fractional order differential equations (FODEs) have been proven to be more realistic than integer order in studies of mathematical models connected to physical and biological problems [13]. As a result, FODEs are increasingly being used in fields of science and engineering, such as blood flow, electrodynamics of complex media, signal and image processing, control theory, economics with management, chemistry with polymer rheology, physics and its subbranches such as aerodynamics, and so on [14, 15, 16]. In [17], for example, the study of the fractional order linear triatomic molecules model was investigated. Recently, a fractional order mathematical model of measles with the best control method was studied [18]. From the perspective of medical engineering, mathematical models of fractional order derivative for several infectious diseases, as well as their examination for stability, optimization, approximate solution, and qualitative analysis, have proven particularly interesting [19].

As a result, scholars have focused their efforts on studying FODEs from a variety of perspectives, including qualitative, numerical, and stability analysis. For finding the semi-analytical, analytical, and series solutions to FODEs, different techniques have been used [20]. Predictor-corrector methods have been utilized extensively, just like Runge-Kutta methods of various orders, modified Euler methods, and Adams-Bashforth methods, (see

[21]). For the semi-analytical solution of FODEs, transform methods such as Laplace, Laplace Adomian decomposition, Fourier transform, Z-transform, differential transform, double and triple Laplace transform, and Sumudu transform have been used [22]. For series solution, as well as stability and convergence analysis of FODEs, the Homotopy perturbation, improved homotopy perturbation, Homotopy analysis, and Taylor's series methods were used (for instance, see [23]).

Different methodologies were used to examine the FODEs for stability. Keeping in mind that several types of stabilities, such as exponential, Mittage-Lefiler, Laypunove, and Local asymptotic stability of equilibrium points, global stability, stability by first approximation approach, stability by Routh-Hurwitz criteria, and so on, are available in the literature [24]. Ulam-Hyers stability, named after Ulam in 1940, is one of the most important types of stability studied recently for linear and non-linear FODEs. Hyers used Banach spaces to explain the said stability in 1941.

In this thesis, we have taken a single and general system of three fractional differential equations under the non-singular and non-local fractional differential operators known as CF and ABC, as described in the article in [25]. Many research publications have simplified the process of determining the existence and uniqueness of a solution, system stability, and numerical analysis by converting various models and systems to a single equation. We compared nine fractional order outcomes to integer order results in the numerical simulation part. As fractional orders are increased (approaches 1.0), the solution curves tend to the curves of integer order 1. As a result, it can be concluded that for integer order systems, only one discrete curve of order 1 can be obtained, whereas in fractional analysis, continuous spectrum curves between 0 and 1 can be obtained. Furthermore, this is a general system of fractional order analysis, and using the same method, we can take many specific systems or models representing various real-world phenomena under different fractional operators in order to check their dynamics at a non-integer order as studied in [26, 27, 28].

The knowledge of the entire spectrum for each dynamical system lying between any two integer values has been offered by modern calculus [29]. Various real-world problems, such as mathematical fractional order model for small-organism population, logistic non-linear model for human population, tuberculosis model, disease models like hepatitis B, C, and the basic Lotka-Volterra models, have been studied using arbitrary order differential or integral equations [30, 31]. Using various methodologies, the FODEs were also examined for numerical, semi-analytical, and analytical solutions. Euler, Taylor, Adams-Bashforth, predictor-corrector, and numerous transformation approaches are some of the well-known methods that have been employed so far (see [32]). We discovered certain areas that require further exploration and discussion after reviewing the existing literature on FODEs. We also get numerical findings using Adam's Bashforth's fractional order approach and exhibit them graphically using Mathematica. To support the entire study, interesting examples are offered. We should also mention that while exponential laws are a typical approach for studying the dynamics of population densities in diverse phenomena/processes, the dynamics in certain systems can be quicker or slower. The use of fractional calculus is the best solution in such situations where anomalous changes in the dynamics occur. Recently, some well-organized results in this area have been reported (see [33, 34, 35]).

The rest of the thesis is organized as follows:
In chapter 2, we present preliminary results and main theorems related to the Caputo-Fabrizio fractional derivative and integral.
In chapter 3, we discuss in details the numerical and theoretical solutions of fractional initial value problems with the Caputo-Fabrizio fractional derivative sense of the form

$$\left({}_{a}^{CF}D^{\alpha} y\right)(t) = f\left(t, y(t)\right), \;\; y(0) = y_0 \quad t \in (0, T].$$

More precisely, we discussed the quadratic and cubic fractional logistic models.

In chapter 4, we discussed the numerical solutions of system of fractional initial value problems of the form

$$\begin{cases} {}_{0}^{ABC}D^{\alpha}(x(t)) = G_1\Big(t, x(t), y(t), z(t)\Big), \\ {}_{0}^{ABC}D^{\alpha}(y(t)) = G_2\Big(t, x(t), y(t), z(t)\Big), \\ {}_{0}^{ABC}D^{\alpha}(z(t)) = G_3\Big(t, x(t), y(t), z(t)\Big), \\ x(0) = x_0, \;\; y(0) = y_0, \;\; z(0) = z_0, \;\; t \in [0, T], \;\; 0 < \alpha \leq 1, \end{cases} \tag{1.2}$$

where $G_{\mathtt{i}} : [0, T] \times \mathrm{R}^3 \to \mathrm{R}$ are continuous functions and $\mathtt{i} = 1, 2, 3$. Note that dynamical systems involve *ABC* fractional order derivative suffer from initialization. In order to solve this, we assume that functions on the right side vanish at $t = 0$. In other words, if $t = 0$, we have $G_i(t, x(t), y(t), z(t)) = 0$, for $i = 1, 2, 3$.

Chapter 2: Caputo-Fabrizio Fractional Derivative and Integral

In this chapter, we discuss in details the Caputo-Fabrizio fractional derivative and its properties. Several theoretical results will be discussed.

Among the several definitions of fractional derivative; the most well-known one is Caputo fractional derivative (CFD) summarized by the following definition.

Definition 2.0.1. For a function $f \in H^1(a,b)$, with $b > a$, $\alpha \in (0,1)$; the left-sided Caputo fractional derivative (CFD) is given by

$$({}_a^C D^\alpha f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau \tag{2.1}$$

It should be noted that the integral (2.1) involves a singular kernel $k(t,\tau) = (t-\tau)^{-\alpha}$ at $t = \tau$.

Caputo and Fabrizio [36] defined a new fractional derivative by changing the kernel $(t-\tau)^{-\alpha}$ with the function $e^{-\frac{\alpha}{1-\alpha}(t-\tau)}$ and $\frac{1}{\Gamma(1-\alpha)}$ with $\frac{M(\alpha)}{(1-\alpha)}$ in (2.1) as follows:

Definition 2.0.2. For a smooth function $f(t) : [a,\infty) \to R$ with $b > a$, $\alpha \in [0,1]$; the Caputo-Fabrizio fractional derivative is given by

$$({}_a^{CF} D^\alpha f)(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t f'(\tau) e^{-\frac{\alpha}{1-\alpha}(t-\tau)} d\tau \tag{2.2}$$

where $M(\alpha) > 0$ is a normalization function such that $M(0) = M(1) = 1$.

Notice that, the kernel of Caputo-Fabrizio fractional operator given in (2.2) is nonsingular. To better understanding of the differences between, integer and Caputo-Fabrizio derivatives; we constructed the below example.

Example 2.0.1. Consider the function $f(t) = t^3 + 2t$, $a = -3$ and $M(\alpha) = 1$. Then, the

C-F derivative is given by

$$
\begin{aligned}
{}^{CF}_{-3}D^{\alpha}(t^3+2t) &= \frac{}{(1-\alpha)}\int_{-3}^{t}(3\tau^2+2)e^{-\frac{\alpha}{1-\alpha}(t-\tau)}d\tau \\
&= \frac{\alpha^2(3t(t+2)+8)-6\alpha(t+2)-(\alpha(17\alpha+6)+6)e^{\frac{\alpha(t+3)}{\alpha-1}}+6}{\alpha^3}.
\end{aligned}
$$

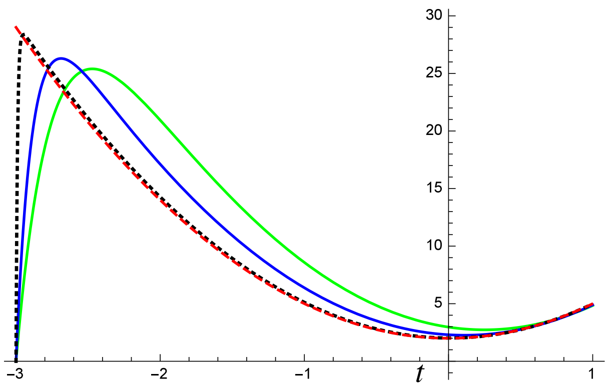

Figure 2.1: Graphical representation $f'(t)$ $(---)$ of C-F fractional derivatives at: $\alpha = 0.8$ (—); $\alpha = 0.9$ (—) and $\alpha = 0.99$ $(\cdots)$ for example 1.

It is clearly seen that the C-F operator $({}^{CF}_{a}D^{\alpha}f)(t)$ converges uniformly to $f'(t)$ for all $t \neq a$.

2.1 Properties of Caputo-Fabrizio Fractional Operator

In this subsection, we discuss the properties of Caputo-Fabrizio fractional operator ${}^{CF}_{a}D^{\alpha}$. First we will recall the definition of the delta function.

Definition 2.1.1. The Dirac delta function is defined by

$$
\delta(t-\tau) = \begin{cases} \infty, & \text{if;} \quad t=\tau \\ 0, & \text{O.W.} \end{cases}
$$

Lemma 2.1.1. *[36] For a smooth function $f(t) : [a,\infty) \to R$,*

$$
\lim_{\alpha\to 1}({}^{CF}_{a}D^{\alpha}f)(t) = f'(t).
$$

Proof. Using the definition 2.2,

$$\begin{aligned}
\lim_{\alpha\to 1}({}_{a}^{CF}D^{\alpha}f)(t) &= \lim_{\alpha\to 1}\frac{M(\alpha)}{(1-\alpha)}\int_a^t f'(\tau)\exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right)d\tau \\
&= \int_a^t f'(\tau)\lim_{\alpha\to 1}\frac{M(\alpha)}{(1-\alpha)}\exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right)d\tau \\
&= \int_a^t f'(\tau)\delta(t-\tau)d\tau \\
&= f'(t)\left(\theta(a-t,t-a)+\delta(a)-2\theta(a-t)+1\right) \\
&= f'(t)
\end{aligned}$$

Notice that θ is Heaviside theta function. □

Lemma 2.1.2. *[36] For a smooth function $f(t):[a,\infty)\to R$,*

$$\lim_{\alpha\to 0}({}_{a}^{CF}D^{\alpha}f)(t) = f(t)-f(a).$$

Proof. Using the definition 2.2,

$$\begin{aligned}
\lim_{\alpha\to 0}({}_{a}^{CF}D^{\alpha}f)(t) &= \lim_{\alpha\to 0}\frac{M(\alpha)}{(1-\alpha)}\int_a^t f'(\tau)\exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right)d\tau \\
&= \int_a^t f'(\tau)d\tau \\
&= f(t)-f(a).
\end{aligned}$$

The higher-order Caputo-Fabrizio fractional derivative of order $\beta=\alpha+n>1$ where $n\in\mathbb{N}$ and $\alpha\in(0,1)$ is defined as:

$$({}_{a}^{CF}D^{\beta}f)(t) = ({}_{a}^{CF}D^{\alpha+n}f)(t) = {}_{a}^{CF}D^{\alpha}\left(D_t^n f(t)\right),$$

where $D_t^n=\dfrac{d^n}{dt^n}$ represents the integer derivative of order n. □

Theorem 2.1.3. *[36] If the function $f(t)$ is such that*

$$f^{(s)}(a)=0,\ s=1,2,...,n$$

then, we have

$$D^n\left({}_{a}^{CF}D_t^{\alpha} f(t)\right) = {}_{a}^{CF}D^{\alpha}\left(D_t^n f(t)\right).$$

Lemma 2.1.4. *For a smooth function $f(t) : [a,\infty) \to R$, consider the simple Caputo-Fabrizio fractional differential equation*

$$({}_{a}^{CF}D^{\alpha} f)(t) = 0 \quad \textit{if and only if} \quad f(t) = constant. \tag{2.3}$$

Proof. $(\longleftarrow)$ If $f(t)$ is constant then showing that $({}_{a}^{CF}D^{\alpha} f)(t) = 0$ is trivial.

$(\longrightarrow)$ Applying the definition of C-F Fractional Derivative, we obtain

$$({}_{a}^{CF}D^{\alpha} f)(t) := \frac{1}{1-\alpha} e^{-\frac{\alpha}{1-\alpha}t} \int_0^t e^{-\frac{\alpha}{1-\alpha}s} f'(s)ds = 0, \quad t \geq a. \tag{2.4}$$

Differentiating Equation (2.4) with respect to t; one may obtain

$$-\alpha({}_{a}^{CF}D^{\alpha} f)(t) + f'(t) = 0, \quad t \geq a.$$

Therefore, $f'(t) = 0$ for $t \geq a$, which means that $f(t)$ is constant for $t \geq a$. □

2.2 The Laplace Transform

Unfortunately, the Laplace transform of Caputo-Fabrizio fractional derivative is valid only when $a = 0$. Therefore, we will discuss briefly this transform.

$$\begin{aligned}
\mathscr{L}\left\{({}_{0}^{CF}D^{\alpha} f)(t)\right\} &= \mathscr{L}\left\{\frac{M(\alpha)}{(1-\alpha)} \int_0^t f'(\tau) e^{-\frac{\alpha}{1-\alpha}(t-\tau)} d\tau\right\} \\
&= \frac{M(\alpha)}{(1-\alpha)} \mathscr{L}\left\{f'(t)\right\} \mathscr{L}\left\{e^{-\frac{\alpha}{1-\alpha}t}\right\} \quad \text{(Convolution Theorem)} \\
&= \frac{s\mathscr{L}\left\{f(t)\right\} - f(0)}{s + \alpha(1-s)}.
\end{aligned}$$

The above results can be easily generalized for higher-order derivatives such as

$$\begin{aligned}
\mathscr{L}\left\{\left({}_{0}^{CF}D^{\alpha+n}f\right)(t)\right\} &= \frac{1}{(1-\alpha)}\mathscr{L}\left\{f^{(n+1)}(t)\right\}\mathscr{L}\left\{e^{-\frac{\alpha}{1-\alpha}t}\right\} \\
&= \frac{s^{n+1}\mathscr{L}\{f(t)\}-s^{n}f(0)-s^{n-1}f'(0)\ldots-f^{(n)}(0))}{s+\alpha(1-s)}.
\end{aligned}$$

2.3 Caputo-Fabrizio Fractional Integral

Let $f(t):[a,\infty)\to R$ be integrable function. Then, it is well-known that the first order integral is $F(t)=\left[I^{1}f\right](t)=\int_{0}^{t}f(s)ds$. However, the definition of fractional integral associated with the fractional derivative was not easy task. In fact, several mathematicians such as Euler [37], Laplace [38], Lacroix [39], Fourier [40], Liouville [41] and Rienmann [42] defined different types of fractional derivatives. For example, the Rienmann-Liouville fractional derivative of order $\alpha\in(0,1)$ which is defined by

$$ {}_{0}^{RL}D^{\alpha}f(t)=\frac{1}{\Gamma(\alpha-1)}\frac{d}{dt}\int_{0}^{t}(t-s)^{\alpha}f(s)ds \tag{2.5}$$

suggested by the relation

$$ {}_{0}^{RL}D^{\alpha}f(t)=[D^{1}I^{1-\alpha}]f=\frac{d}{dt}I^{1-\alpha}f.$$

Therefore, it was suggested that the Riemann-Liouville integral can be defined by

$$I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_{a}^{t}f(\tau)(t-\tau)^{\alpha-1}d\tau,\;\; Real(\alpha)\geq 0,$$

where Γ is the gamma function.

In the twenty century, Caputo introduced new definition of fractional derivative defined by

$$ {}_{0}^{C}D^{\alpha}f(t)=\frac{1}{\Gamma(\alpha-1)}\int_{0}^{t}(t-\tau)^{\alpha}f'(\tau)d\tau. \tag{2.6}$$

In fact, it can be easily shown that the relation ${}_{0}^{C}D^{\alpha} = I^{1-\alpha} \cdot D^{1}$ is implemented.

The Caputo-Fabrizio fractional integral associated to Caputo-Fabrizio fractional derivative was determined by [44] which is given by the following lemma.

Lemma 2.3.1. *For a smooth function $g : [a, \infty) \to \mathbb{R}$, the C-F fractional integral of order $\alpha \in (0,1)$ associated with the C-F fractional derivative is defined by*

$$({}_{a}^{CF}I^{\alpha} g)(t) = (1-\alpha)[g(t) - g(0)] + \alpha \int_{0}^{t} g(s)ds$$

with $g_0 = \int_a^0 e^{-\frac{\alpha}{1-\alpha}s} f'(s)ds$. In addition, the following two properties are satisfied

(a) $({}_{a}^{CF}D^{\alpha} \; {}^{CF}I^{\alpha}) f(t) = f(t) - f(a) e^{\frac{-\alpha(t-a)}{1-\alpha}}$

(b) $({}^{CF}I^{\alpha} \; {}_{a}^{CF}D^{\alpha}) f(t) = f(t) - f(0)$.

Note that, if $a = 0$, then $g_0 = 0$ and $({}_{a}^{CF}I^{\alpha} g)(t) = (1-\alpha) g(t) + \alpha \int_0^t g(s)ds$.

Proof. See [44]. □

Chapter 3:Numerical and Theoretical Investigating of Fractional Initial Value Problems: Caputo-Fabrizio Derivative Sense

In this chapter, we discuss the numerical and theoretical solutions of fractional initial value problems with the Caputo-Fabrizio fractional derivative sense of the form

$$\left({}_{a}^{CF}D^{\alpha}y\right)(t) = f\left(t, y(t)\right), \quad t \in (0,T]. \tag{3.1}$$

Below we establish existence results.

Lemma 3.0.1. *Let $x \in L[0,T]$ and $y \in A \subset [0,T]$, then the solution of*

$$\begin{cases} {}_{a}^{CF}D^{\alpha}y(t) = x(t), \; 0 < \alpha \leq 1 \quad t \in [0,T], \\ y(0) = y_0, \end{cases} \tag{3.2}$$

is

$$y(t) = y(0) + (1-\alpha)\left(x(t) - x(0)\right) + \alpha \int_0^t x(\tau)d\tau. \tag{3.3}$$

Proof. The proof is direct by applying the Caputo-Fabrizio fractional integral of order α on both sides of (3.2). □

3.1 Numerical Solution of (3.1)

Applying the Caputo-Fabrizio fractional integral of order α; i.e.

$$\left({}^{CF}I^{\alpha}g\right)(t) = (1-\alpha)\left(g(t) - g(0)\right) + \alpha \int_0^t g(\tau)d\tau$$

on Equation (3.1), we obtain

$$\begin{aligned} \left({}^{CF}I^{\alpha}\, {}_{a}^{CF}D^{\alpha}y\right)(t) &= {}^{CF}I^{\alpha} f\left(t, y(t)\right) & (3.4) \\ y(t) - y(0) &= (1-\alpha)\left(f(t,y(t)) - f(0,y(0))\right) + \alpha \int_0^t f(\tau, y(\tau))d\tau. & (3.5) \end{aligned}$$

The domain $[0,T]$ is divided into $N-$subintervals with the grid points $t_n = nh, \; n = 0, \cdots, N$. Here h represents a uniform step size; $h = T/N$. At $t = t_n$, Equation (3.4) has the form

$$y(t_n) = y(0) + (1-\alpha)\left(f(t_n, y(t_n)) - f(0, y(0))\right) + \alpha \int_0^{t_n} f(\tau, y(\tau))d\tau, \tag{3.6}$$

or in the following form

$$y(t_n) = y(0) + (1-\alpha)\left(f(t_n, y(t_n)) - f(0, y(0))\right) + \alpha \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f(\tau, y(\tau))d\tau. \tag{3.7}$$

The integral in right hand side is approximated by a Lagrange polynomial using the nodes t_k and t_{k+1}; i.e.

$$\begin{aligned} f(\tau, y(\tau)) = P_k(\tau) = & \frac{(\tau - t_{k+1})}{-h} f(t_k, y(t_k)) + \frac{(\tau - t_k)}{h} f(t_{k+1}, y(t_{k+1})) \\ & + \frac{f''(\xi(\tau))}{2!}(t - t_k)(t - t_{k+1}). \end{aligned}$$

For simplicity, assume that $f_k = f(t_k, y(t_k))$ and $y_n = y(t_n)$. Therefore, neglecting the error term, Equation (3.7) can be rewritten as

$$\begin{aligned} y_n &= y_0 + (1-\alpha)(f_n - f_0)) + \alpha \sum_{k=0}^{n-1} \left(\frac{h}{2} f_k + \frac{h}{2} f_{k+1} \right), \\ &= y_0 + \left(\alpha - 1 + \frac{h}{2}\alpha \right) f_0 + \left(1 - \alpha + \frac{h}{2}\alpha \right) f_n + \alpha \sum_{k=1}^{n-1} \frac{h}{2} f_k \end{aligned}$$

In the proceeding subsections we discuss, respectively, the numerical and theoretical solutions of two well-known examples: quadratic and cubic logistic models.

3.2 Numerical Solutions of Logistic Models

Example 3.2.1.

$$({}_{a}^{CF}D^{\alpha} y)(t) = ry(t)(1 - y(t)/K), \ \ t > 0, \ y(0) = y_0, \tag{3.8}$$

with $r = 0.5$ and $K = 2$. The targets of this example are to discuss the effects of the parameters y_0 and α on the solution trajectories.

It is clearly observed that Equation (3.8) has two equilibria given by $y_1 = 0$ and $y_2 = 2$. Figure 3.1 shows the solution trajectories as the initial point, at y_0, changes in the set

$\{1,3\}$ for $\alpha = 0.7 - 1.0$. One can clearly see that the solution trajectories converge to $y_2 = K = 2$ asymptotically for any y_0. Thus, we conclude that $y_2 = 2$ is asymptotically stable equilibrium solution whereas $y_1 = 0$ is unstable equilibrium solution. It is also noted that the effect of increasing α will slow the required time for solution trajectories to reach the equilibrium solution. It is worthy mentioning that as $\alpha \to 1$, the solution of the problem converges smoothly to the classical problem at $\alpha = 1$.

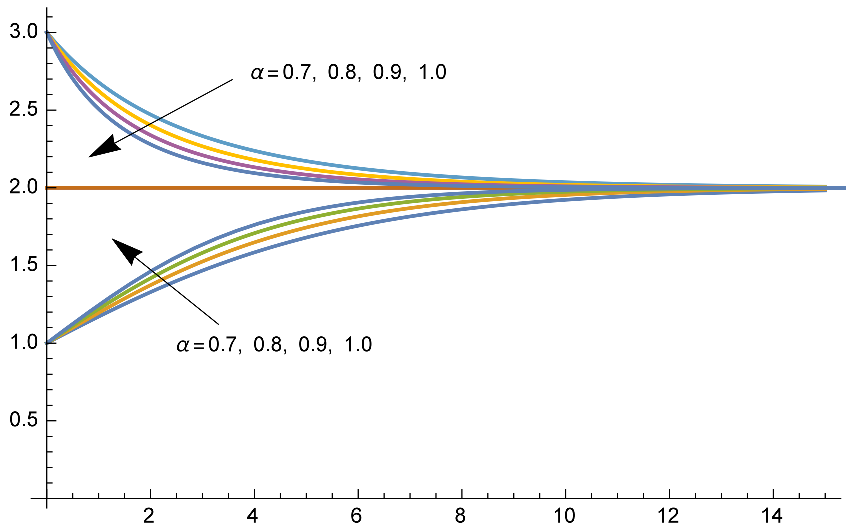

Figure 3.1: Graphs of the the solution trajectories for Example 1 at $y_0 = 1.0$, $y_0 = 3.0$ for different values of α.

Example 3.2.2.

$$({}_{a}^{CF}D^{\alpha}y)(t) = ry(t)(1 - \frac{y(t)}{K})(y(t) - m), \;\; t > 0, \; y(0) = y_0, \tag{3.9}$$

with $r = 0.5$, $m = 1$ and $K = 2$. The targets of this example are to discuss the effects of the parameters y_0 and α on the solution trajectories.

It is clearly observed that Equation (3.9) has three equilibria given by $y_1 = 0$, $y_2 = 1$ and $y_3 = 2$. Figure 3.2 shows the solution trajectories as the initial point, at y_0, changes in the sets $\{0.5, 1.5, 3\}$ for $\alpha = 0.7 - 1.0$. One can clearly see that the solution trajectories converge to $y_3 = K = 2$ asymptotically for any y_0. Thus, we conclude that $y_3 = 2$ is asymptotically stable equilibrium solution whereas $y_1 = 0$ and $y_2 = 1$ are unstable equilibrium solutions. It is also noted that the effect of increasing α will slow the required time for solution trajectories to reach the equilibrium solution. It is worthy mentioning that as $\alpha \to 1$, the solution of the problem converges smoothly to the classical problem at $\alpha = 1$.

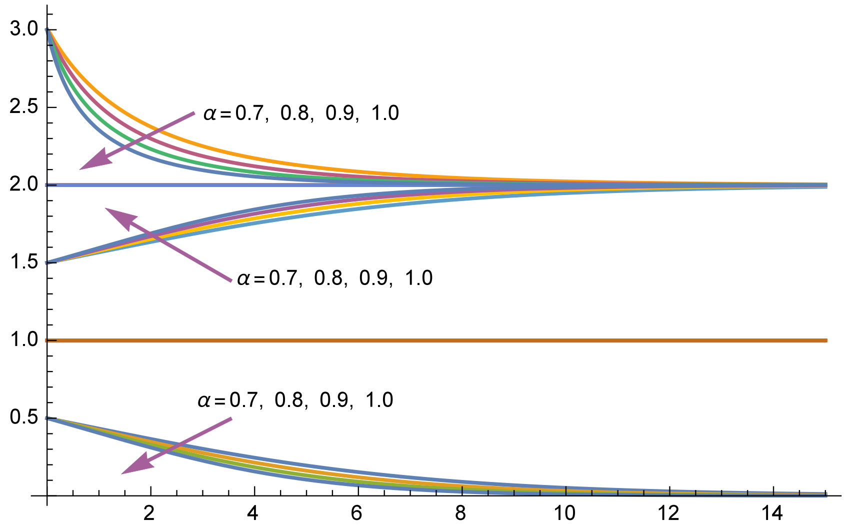

Figure 3.2: Graphs of the the solution trajectories for Example 2 at $y_0 = 1.0$, $y_0 = 3.0$ for different values of α.

3.3 Theoretical Solutions of Logistic Models

In this section, we solve theoretically the quadratic and cubic fractional logistic models where the derivative is considered in Caputo-Fabrizio sense.

Example 3.3.1. Consider the following quadratic fractional logistic model

$$({}^{CF}_{a}D^{\alpha}y)(t) = ry(t)\left(1 - \frac{y(t)}{K}\right), \ \ t > 0, \ y(0) = y_0. \tag{3.10}$$

If $\alpha = 1$, the fractional equation is simplified to the classical differential equation with exact solution given by

$$y(t) = -\frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

and we can observe that as $t \to \infty$, $y(t) \to K$.

Applying the fractional operator, ${}^{CF}I^{\alpha}$, on equation (3.10); one obtains

$$({}^{CF}I^{\alpha}\,{}^{CF}_{a}D^{\alpha}y)(t) = {}^{CF}I^{\alpha}\left(ry(t)\left(1 - \frac{y(t)}{K}\right)\right),$$

or

$$y(t)-y(0)=(1-\alpha)\left(ry(t)\left(1-\frac{y(t)}{K}\right)-ry(0)\left(1-\frac{y(0)}{K}\right)\right)+\alpha\int_0^t ry(\tau)\left(1-\frac{y(\tau)}{K}\right)d\tau. \tag{3.11}$$

Sorting out (3.11), we get

$$y(t)=y_0+(\alpha-1)ry_0+(1-\alpha)\left(r(y(t))+r\frac{y_0^2-y(t)^2}{K}\right)+\alpha\int_0^t ry(\tau)\left(1-\frac{y(\tau)}{K}\right)d\tau. \tag{3.12}$$

Taking the first derivative of (3.12) gives

$$y'(t)=(1-\alpha)ry'(t)+(\alpha-1)\frac{2ry(t)y'(t)}{K}+\alpha ry(t)\left(1-\frac{y(t)}{K}\right). \tag{3.13}$$

Simple operations will help in writing Equation (3.13) in the following separable form

$$\frac{(K((\alpha-1)r+1)-2(\alpha-1)ry(t))}{y(t)(K-y(t))}dy=\alpha rdt, \tag{3.14}$$

or in the following form

$$\left(\frac{\alpha r-r-1}{y(t)-K}+\frac{\alpha r-r+1}{y(t)}\right)dy=\alpha rdt. \tag{3.15}$$

Integrating (3.15) gives

$$A_1\ln|y(t)|+A_2\ln|y(t)-K|=\alpha rt+C, \tag{3.16}$$

where $A_1=\alpha r-r+1, A_2=\alpha r-r-1$ and C is an arbitrary constant can be found using the initial condition $y(0)=y_0$. Hence, we obtain

$$C=A_1\ln|y_0|+A_2\ln|y_0-K|. \tag{3.17}$$

Therefore, combining (3.16) and (3.17) leads to implicit solution of the quadratic fractional logistic model (3.13).

Example 3.3.2. Consider the following cubic fractional logistic model

$$({}_{a}^{CF}D^{\alpha}y)(t) = ry(t)\left(1-\frac{y(t)}{K}\right)(y(t)-m), \;\; t>0, \; y(0)=y_0. \tag{3.18}$$

It should be noted herein that in the case of $\alpha = 1$, equation (3.18) reduces to the classical cubic logistic model with the following exact solution

$$\begin{aligned}&\frac{-m\ln(y(t)-K)+K\ln(y(t)-m)+(m-K)\ln(y(t))}{m(K-m)}\\ &\qquad = rt + \frac{-K\ln(y_0)+m\ln(y_0)-m\ln(y_0-K)+K\ln(y_0-m)}{m(K-m)}.\end{aligned}$$

To find the solution, $y(t)$, of (3.18) for $0 \le \alpha < 1$, we have to apply the operator ${}^{CF}I^{\alpha}$ on both sides of the equation

$$({}^{CF}I^{\alpha}\, {}_{a}^{CF}D^{\alpha}y)(t) = {}^{CF}I^{\alpha}\left(ry(t)\left(1-\frac{y(t)}{K}\right)(y(t)-m)\right),$$

which can be rewritten using (3.4) in the following form

$$\begin{aligned} y(t) &= y_0 + (1-\alpha)\left(ry(t)\left(1-\frac{y(t)}{K}\right)(y(t)-m) - ry_0\left(1-\frac{y_0}{K}\right)(y_0-m)\right) \\ &+ \alpha\int_0^t ry(\tau)\left(1-\frac{y(\tau)}{K}\right)(y(\tau)-m)d\tau, \end{aligned} \tag{3.19}$$

where $y_0 = y(0)$. Taking the first derivative with respect to t for both sides of (3.19), we obtain

$$y'(t) = \frac{(\alpha-1)ry'(t)\left(-2(K+m)y(t)+Km+3y(t)^2\right)+\alpha ry(t)(K-y(t))(y(t)-m)}{K},$$

which can be written in the following separable form

$$\left(\frac{\mathrm{A1}}{y(t)}+\frac{\mathrm{A2}}{y(t)-K}+\frac{\mathrm{A3}}{y(t)-m}\right)dy = \alpha rm(K-m)dt, \tag{3.20}$$

where $A1 = (K-m)(\alpha mr - mr - 1)$, $A2 = \alpha Kmr - Kmr - \alpha m^2 r + m^2 r - m$ and $A3 =$

$\alpha Kmr - Kmr + K - \alpha m^2 r + m^2 r$. Integrating 3.20 can be obtained

$$A_1 \ln|y(t)| + A_2 \ln|y(t) - K| + A_3 \ln|y(t) - m| = \alpha rm(K - m)t + C,$$

where C is an arbitrary constant that can be determined using the initial condition $y(0) = y_0$; its value is

$$C = A_1 \ln|y_0| + A_2 \ln|y_0 - K| + A_3 \ln|y_0 - m|.$$

Chapter 4:Numerical Investigation of Fractional Initial Value Problems: ABC Derivative Sense

In this chapter, we implement a modified definition of Caputo-Fabrizo differential operator called the $\mathbb{ABC}$ fractional derivative [48] given by.

Definition 4.0.1. The $\mathbb{ABC}$ derivative of a function $G(t)$ under the condition $G(t) \in \mathscr{H}^1(0,T)$ is defined as follows:

$$^{\mathbb{ABC}}{}_0D_t^{\alpha}G(t) = \frac{1}{1-\alpha}\int_0^t G'(\tau)E_{\alpha}[\frac{-\alpha}{1-\alpha}(t-\tau)^{\alpha}]d\tau, \tag{4.1}$$

where, E_{α} represents the special function known as Mittag-Leffler function. Further, it is to be noted that, for a constant C,

$$^{\mathbb{ABC}}{}_0D_t^{\alpha}[C] = 0.$$

Definition 4.0.2. [48] Let $G(t) \in L[0,T]$, then the associated fractional integral, $^{\mathbb{ABC}}{}_0I_t^{\alpha}$, in $\mathbb{ABC}$ sense is given by:

$$^{\mathbb{ABC}}{}_0I_t^{\alpha}G(t) = (1-\alpha)G(t) + \frac{\alpha}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}G(\tau)d\tau. \tag{4.2}$$

In this chapter, we focus on theoretical and numerical solutions to the following proposed system of fractional differential equations:

$$\begin{cases} {}_0^{ABC}D^{\alpha}(x(t)) = G_1\bigg(t,x(t),y(t),z(t)\bigg), \\ {}_0^{ABC}D^{\alpha}(y(t)) = G_2\bigg(t,x(t),y(t),z(t)\bigg), \\ {}_0^{ABC}D^{\alpha}(z(t)) = G_3\bigg(t,x(t),y(t),z(t)\bigg), \\ x(0)=x_0,\ \ y(0)=y_0,\ \ z(0)=z_0,\ \ t\in[0,T],\ \ 0<\alpha\leq 1, \end{cases} \tag{4.3}$$

where $G_{\mathtt{i}} : [0,T] \times \mathrm{R}^3 \to \mathrm{R}$ are continuous functions and $\mathtt{i} = 1,2,3$. The below lemma ensures the existence of a solution for the above system (4.3).

Lemma 4.0.1. *[Existence Result: Proposition 3, [48]] The solution of the given problem* *for* $0 < \alpha \leq 1$

$$\begin{aligned} {}^{\mathbb{ABC}}{}_{0}D_{t}^{\alpha}G(t) &= \Psi(t),\ t \in [0,T], \\ G(0) &= G_0, \end{aligned}$$

keeping in mind that right side vanish at $t = 0$ *is given by*

$$G(t) = G_0 + (1-\alpha)\Psi(t) + \frac{\alpha}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\Psi(\tau)d\tau.$$

For more theortical results related to the $\mathbb{ABC}$ fractional derivative; the reader is referred to [45, 46, 47, 48, 49, 50, 51, 52].

4.1 Numerical Solution

In this section, we are going to approximate the solutions of problem (4.3) using the so-called Adams-Bashforth (AB) iterative technique. We will derive the scheme for the first equation of (4.3), while the remaining equations can be discretised by the same manner. Upon integration of the first equation of (4.3) as

$$\begin{aligned} x(t) - x(0) =& (1-\alpha)\Big[G_1\Big(x(t), y(t), z(t), t\Big)\Big] \\ &+ \frac{\alpha}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}G_1\Big(x(\tau), y(\tau), z(\tau), \tau\Big)d\tau. \end{aligned}$$

Set $t = t_{j+1}$ for $j = 0, 1, 2\cdots$,

$$\begin{aligned} x(t_{j+1}) - x(0) &= (1-\alpha)\Big[G_1\Big(x(t_j), y(t_j), z(t_j), t_j\Big)\Big] \\ &+ \frac{\alpha}{\Gamma(\alpha)}\int_0^{t_{j+1}} (t_{j+1}-\tau)^{\alpha-1}G_1\Big(x(\tau), y(\tau), z(\tau), \tau\Big)d\tau, \\ &= (1-\alpha)\Big[G_1\Big(x(t_j), y(t_j), z(t_j), t_j\Big)\Big] \\ &+ \frac{\alpha}{\Gamma(\alpha)}\sum_{\ell=0}^{j}\int_{t_\ell}^{t_{\ell+1}} (t_{j+1}-\tau)^{\alpha-1}G_1\Big(x(\tau), y(\tau), z(\tau), \tau\Big)d\tau. \end{aligned}$$

Next, we interpolate the function G_1 on $[t_\ell, t_{\ell+1}]$ for $\ell = 1, 2, 3...$ using Lagrange interpolation as follows

$$G_1 \cong \frac{G_1}{h}(t - t_{\ell-1}) - \frac{G_1}{h}(t - t_\ell),$$

$$
\begin{aligned}
x(t_{j+1})-x(0) &= (1-\alpha)\Big[G_1\Big(x(t_j),y(t_j),z(t_j),t_j\Big)\Big] \\
&+ \frac{\alpha}{\Gamma(\alpha)}\sum_{\ell=0}^{j}\Bigg(\frac{G_1\Big(x(t_\ell),y(t_\ell),z(t_\ell),t_\ell)}{h}\int_{\ell}^{t_{\ell+1}}(t-t_{\ell-1})(t_{j+1}-t)^{\alpha-1}dt \\
&- \frac{G_1\Big(x(t_{\ell-1}),y(t_{\ell-1}),z(t_{\ell-1}),t_{\ell-1}\Big)}{h}\int_{\ell}^{t_{\ell+1}}(t-t_\ell)(t_{j+1}-t)^{\alpha-1}dt\Bigg),
\end{aligned}
$$

or

$$
\begin{aligned}
x(t_{j+1}) &= x(0)+(1-\alpha)\Big[G_1\Big(x(t_j),y(t_j),z(t_j),t_j\Big)\Big] \\
&+ \frac{\alpha}{\Gamma(\alpha)}\sum_{\ell=0}^{j}\Bigg(\frac{G_1\Big(x(t_\ell),y(t_\ell),z(t_\ell),t_\ell)}{h}\mathbf{I}_{\ell-1,\,\alpha} \\
&- \frac{G_1\Big(x(t_{\ell-1}),y(t_{\ell-1}),z(t_{\ell-1}),t_{\ell-1}\Big)}{h}\mathbf{I}_{\ell,\,\alpha}\Bigg). \qquad (4.4)
\end{aligned}
$$

Computing $I_{\ell-1,\,\alpha}$ and $I_{\ell,\,\alpha}$, we get

$$
\begin{aligned}
\mathbf{I}_{\ell-1,\,\alpha} &= \int_{\ell}^{t_{\ell+1}}(t-t_{\ell-1})(t_{j+1}-t)^{\alpha-1}dt \\
&= -\frac{1}{\alpha}\Big[(t_{\ell+1}-t_{\ell-1})(t_{j+1}-t_{\ell+1})^{\alpha}-(t_\ell-t_{\ell-1})(t_{j+1}-t_\ell)^{\alpha}\Big] \\
&\quad -\frac{1}{\alpha(\alpha-1)}\Big[(t_{j+1}-t_{\ell+1})^{\alpha+1}-(t_{j+1}-t_\ell)^{\ell+1}\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{I}_{\ell,\,\alpha} &= \int_{\ell}^{t_{\ell+1}}(t-t_\ell)(t_{j+1}-t)^{\alpha-1}dt \\
&= -\frac{1}{\alpha}\Big[(t_{\ell+1}-t_\ell)(t_{j+1}-t_{\ell+1})^{\alpha}\Big] \\
&\quad -\frac{1}{\alpha(\alpha-1)}\Big[(t_{j+1}-t_{\ell+1})^{\alpha+1}-(t_{j+1}-t_\ell)^{\alpha+1}\Big],
\end{aligned}
$$

put $t_\ell = \ell h$, we get

$$\begin{aligned}
\mathbf{I}_{\ell-1,\,\alpha} &= -\frac{h^{\alpha+1}}{\alpha}\Big[(\ell+1-(\ell-1))(j+1-(\ell+1))^{\alpha}-(\ell-(\ell-1))(j+1-\ell)^{\alpha}\Big] \\
&\quad -\frac{h^{\alpha+1}}{\alpha(\alpha-1)}\Big[(j+1-(\ell+1))^{\alpha+1}-(j+1-\ell)^{\alpha+1}\Big], \\
&= \frac{h^{\alpha+1}}{\alpha(\alpha-1)}\Big[-2(\alpha+1)(j-\ell)^{\alpha}+(\alpha+1)(j+1-\ell)^{\alpha}-(j-\ell)^{\alpha+1} \qquad (4.5) \\
&\quad +(j+1-\ell)^{\alpha+1}\Big], \\
&= \frac{h^{\alpha+1}}{\alpha(\alpha-1)}\Big[(j-\ell)^{\alpha}(-2(\alpha+1)-(j-\ell))+(j+1-\ell)^{\alpha}(\alpha+1+j+1-\ell)\Big], \\
&= \frac{h^{\alpha+1}}{\alpha(\alpha-1)}\Big[(j+1-\ell)^{\alpha}(j-\ell+2+\alpha)-(j-\ell)^{\alpha}(j-\ell+2+2\alpha)\Big] \qquad (4.6)
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{I}_{\ell,\,\alpha} &= -\frac{h^{\alpha+1}}{\alpha}\Big[(\ell+1-\ell)(j+1-(\ell+1))^{\alpha}\Big] \qquad (4.7) \\
&\quad -\frac{h^{\alpha+1}}{\alpha(\alpha-1)}\Big[(j+1-(\ell+1))^{\alpha+1}-(j+1-\ell)^{\alpha+1}\Big], \\
&= \frac{h^{\alpha+1}}{\alpha(\alpha-1)}\Big[-(\alpha+1)(j-\ell)^{\alpha}-(j-\ell)^{\alpha+1}+(j+1-\ell)^{\alpha+1}\Big], \\
&= \frac{h^{\alpha+1}}{\alpha(\alpha-1)}\Big[(j-\ell)^{\alpha}(-(\ell+1)-(j-\ell))+(j+1-\ell)^{\alpha+1}\Big], \\
&= \frac{h^{\alpha+1}}{\alpha(\alpha-1)}\Big[(j+1-\ell)^{\alpha+1}-(j-\ell)^{\alpha}(j-\ell+1+\alpha)\Big], \qquad (4.8)
\end{aligned}$$

substituting the values of (4.5) and (4.7) in (4.4), we get

$$x(t_{j+1}) = \begin{cases} x(0)+(1-\alpha)\Big[G_1\Big(x(t_j),y(t_j),z(t_j),t_j\Big)\Big] \\ +\dfrac{\alpha}{\Gamma(\alpha)}\displaystyle\sum_{\ell=0}^{j}\Bigg(\dfrac{G_1\Big(x(t_\ell),y(t_\ell),z(t_\ell),t_\ell\Big)}{h} \\ \times\dfrac{h^{\alpha+1}}{\alpha(\alpha-1)}\Big[(j+1-\ell)^{\alpha}(j-\ell+2+\alpha)-(j-\ell)^{\alpha}(j-\ell+2+2\alpha)\Big] \\ -\dfrac{G_1\Big(x(t_{\ell-1}),y(t_{\ell-1}),z(t_{\ell-1}),t_{\ell-1}\Big)}{h}\dfrac{h^{\alpha+1}}{\alpha(\alpha-1)} \\ \times\Big[(j+1-\ell)^{\alpha+1}-(j-\ell)^{\alpha}(j-\ell+1+\alpha)\Big]\Bigg). \end{cases} \qquad (4.9)$$

Similarly the other two compartments $y(t)$ and $z(t)$ can be computed by the same numerical scheme as

$$y(t_{j+1}) = \begin{cases} y(0) + (1-\alpha)\Big[G_2\Big(x(t_j), y(t_j), z(t_j), t_j\Big)\Big] \\ + \dfrac{\alpha}{\Gamma(\alpha)} \displaystyle\sum_{\ell=0}^{j} \Bigg(\dfrac{G_2\Big(x(t_\ell), y(t_\ell), z(t_\ell), t_\ell\Big)}{h} \\ \times \dfrac{h^{\alpha+1}}{\alpha(\alpha-1)} \Big[(j+1-\ell)^{\alpha}(j-\ell+2+\alpha) - (j-\ell)^{\alpha}(j-\ell+2+2\alpha)\Big] \\ - \dfrac{G_2\Big(x(t_{\ell-1}), y(t_{\ell-1}), z(t_{\ell-1}), t_{\ell-1}\Big)}{h} \dfrac{h^{\alpha+1}}{\alpha(\alpha-1)} \\ \times \Big[(j+1-\ell)^{\alpha+1} - (j-\ell)^{\alpha}(j-\ell+1+\alpha)\Big]\Bigg) \end{cases} \tag{4.10}$$

and

$$z(t_{j+1}) = \begin{cases} z(0) + (1-\alpha)\Big[G_3\Big(x(t_j), y(t_j), z(t_j), t_j\Big)\Big] \\ + \dfrac{\alpha}{\Gamma(\alpha)} \displaystyle\sum_{\ell=0}^{j} \Bigg(\dfrac{G_3\Big(x(t_\ell), y(t_\ell), z(t_\ell), t_\ell\Big)}{h} \\ \times \dfrac{h^{\alpha+1}}{\alpha(\alpha-1)} \Big[(j+1-\ell)^{\alpha}(j-\ell+2+\alpha) - (j-\ell)^{\alpha}(j-\ell+2+2\alpha)\Big] \\ - \dfrac{G_3\Big(x(t_{\ell-1}), y(t_{\ell-1}), z(t_{\ell-1}), t_{\ell-1}\Big)}{h} \dfrac{h^{\alpha+1}}{\alpha(\alpha-1)} \\ \times \Big[(j+1-\ell)^{\alpha+1} - (j-\ell)^{\alpha}(j-\ell+1+\alpha)\Big]\Bigg). \end{cases} \tag{4.11}$$

4.2 Numerical Examples

In this section, we discuss two numerical examples that are solved by using the above-discussed numerical scheme.

Example 4.2.1. The first example is

$$
\begin{cases}
{}^{\mathbb{ABC}}\mathrm{D}_t^{\alpha}(x(t)) = \left(\sqrt{x(t)} + y(t) + z(t)\right), \\
{}^{\mathbb{ABC}}\mathrm{D}_t^{\alpha}(y(t)) = \frac{e^{-\pi t}}{10+t}\left(x(t) + t\cos(y(t)) + z(t)\right), \\
{}^{\mathbb{ABC}}\mathrm{D}_t^{\alpha}(z(t)) = \frac{e^{-t}}{5+t}\left(x(t) + y(t) + \sin(z(t))\right), \\
x(0) = 0, \;\; y(0) = 0, \;\; z(0) = 0, \;\; t \in I := [0,1], \;\; 0 < \alpha \leq 1.
\end{cases}
\tag{4.12}
$$

Figures (4.1), (4.2) and (4.3) show the effect of changing $\alpha \in [0.9, 1.0]$ on functions x, y and z, respectively. It is clearly seen that the solutions, as α approaches 1.0, converge to the solution at $\alpha = 1$. It is worth mentioning herein that the convergence to the exact solution as α approaches 1.0 is faster and smoother when using the Caputo fractional derivative case. It can easily seen that the solution is getting larger as α decreases.

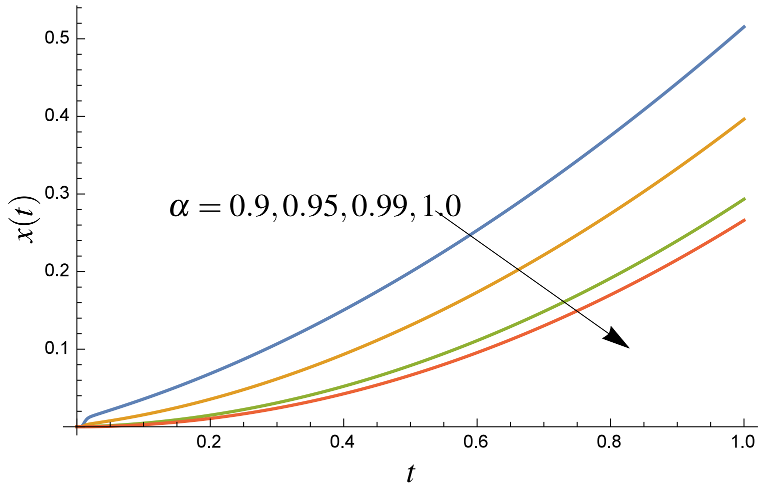

Figure 4.1: Graphs of the approximate solutions and the exact solution, x, at various values of α for Example 1.

Notice that, the exact solution of this problem is unknown, therefore, we measure the bound of the error using the residuals as follow:

$$
\begin{cases}
R_x(t) = {}^{\mathbb{ABC}}\mathrm{D}_t^{\alpha}(x(t)) - \left(\sqrt{x(t)} + y(t) + z(t)\right), \\
R_y(t) = {}^{\mathbb{ABC}}\mathrm{D}_t^{\alpha}(y(t)) - \frac{e^{-\pi t}}{10+t}\left(x(t) + t\cos(y(t)) + z(t)\right), \\
R_z(t) = {}^{\mathbb{ABC}}\mathrm{D}_t^{\alpha}(z(t)) - \frac{e^{-t}}{5+t}\left(x(t) + y(t) + \sin(z(t))\right).
\end{cases}
\tag{4.13}
$$

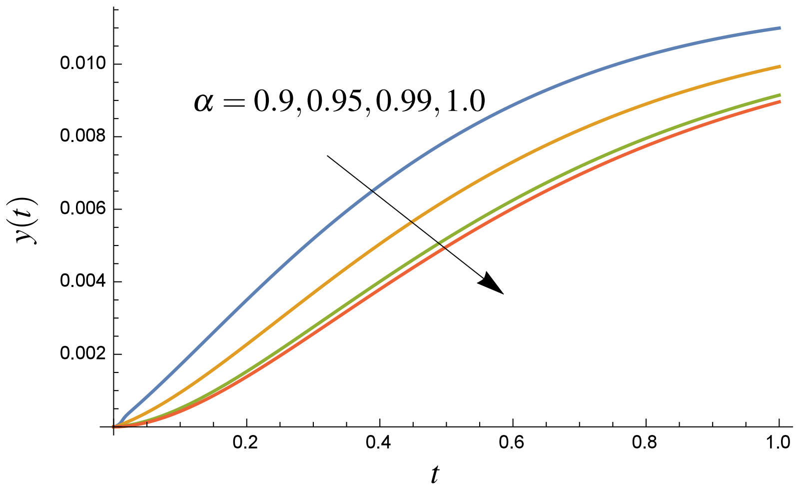

Figure 4.2: Graphs of the approximate solutions and the exact solution, y, at various values of α for Example 1.

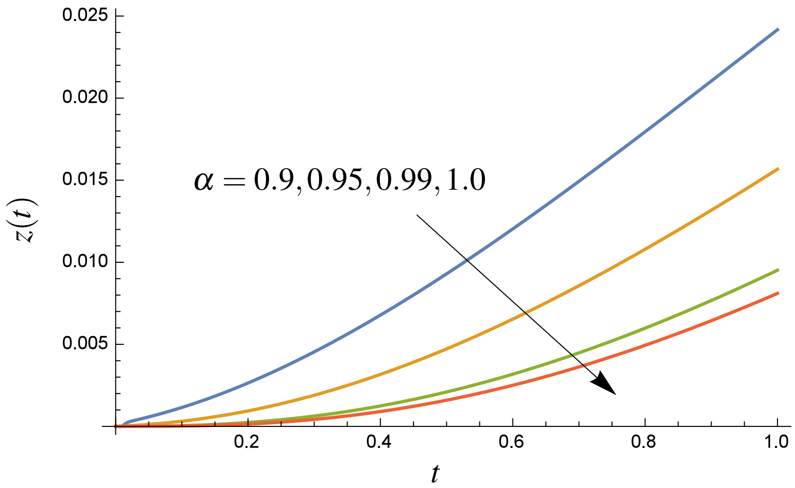

Figure 4.3: Graphs of the approximate solutions and the exact solution, z, at various values of α for Example 1.

Hence, the error is

$$E(t) = \max_{t \in I} \left\{ |R_x(t)|, |R_y(t)|, |R_z(t)| \right\}. \tag{4.14}$$

Table (4.1) displays the error bounds $E(t)$ at the points $t_j = 0.1j$, for $j = 1, \cdots, 10$ when $\alpha = 0.9$ which clearly indicates the accuracy of the present algorithm.

Table 4.1: Error bounds for Example 1 at $\alpha = 0.9$

t_j	$E(t_j)$
0.1	3.55949×10^{-4}
0.2	4.48364×10^{-4}
0.3	9.63994×10^{-4}
0.4	1.95481×10^{-4}
0.5	2.35699×10^{-4}
0.6	5.93055×10^{-3}
0.7	4.19320×10^{-3}
0.8	3.94933×10^{-3}
0.9	5.42097×10^{-3}
1.0	9.52593×10^{-3}

Example 4.2.2. The first example is

$$\begin{cases} {}^{\mathbb{ABC}}\mathbb{D}_t^{\alpha}(S(t)) = \Lambda - \beta SI - \lambda SA - \bar{d}S + \psi I, \\ {}^{\mathbb{ABC}}\mathbb{D}_t^{\alpha}(I(t)) = \beta SI - (\psi + \alpha + \bar{d})I, \\ {}^{\mathbb{ABC}}\mathbb{D}_t^{\alpha}(A(t)) = \mu I - \phi A, \\ S(0) = 13512, \;\; I(0) = 1, \;\; A(0) = 100, \;\; t > 0, \;\; 0 < \alpha \leq 1. \end{cases} \tag{4.15}$$

The model takes into account the following clue

1. The population $N(t)$ is generally split into three compartments; the general susceptible population $S(t)$, and the infective population $I(t)$. The cumulative density of the awareness programs is given by $A(t)$. The growth rate of the density of the awareness program is assumed to be proportional to the number of Infectives, while the parameters is well explained in the given table below:

Table 4.2: Description of the parameters.

Parameter	Description
$\Lambda = 400$	recruitment rate
$\beta = 0.0000157$	infection contact rate
$\lambda = 0.0002$	Dissemination rate of awareness
$\alpha_1 = 0.03275$	Death rate due to infection
$\bar{d} = 0.03275$	natural death rate
$\psi = 0.169788$	Rate of transfer of aware individuals to susceptible class
$\mu = 0.05$	Rate of implementation of the awareness program
$\phi = 0.0005$	Rate of depletion of the program due to social problems and ineffectiveness

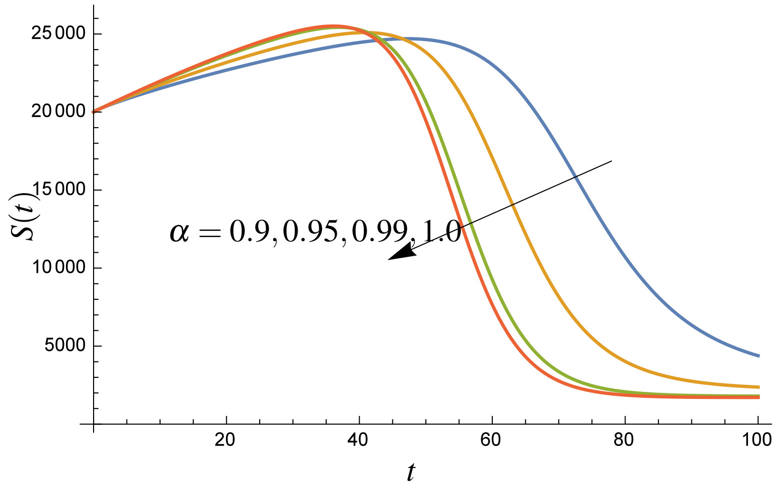

Figure 4.4: Graphs of the solution trajectories, $S(t)$, at various values of α for Example 2.

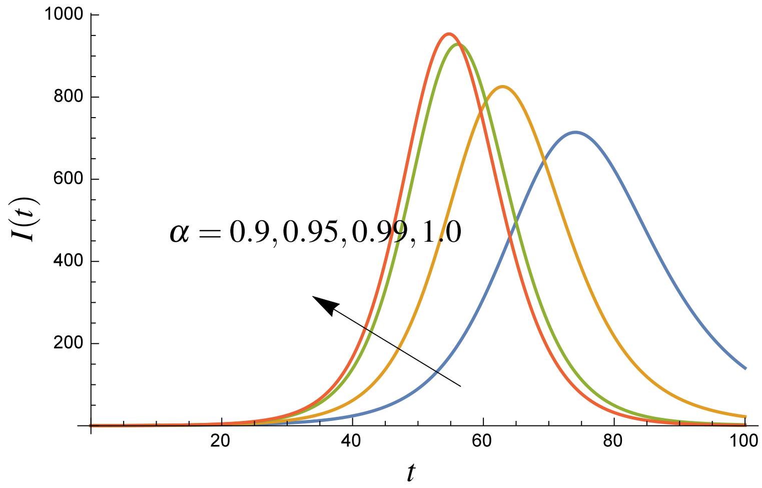

Figure 4.5: Graphs of the solution trajectories, $I(t)$, at various values of α for Example 2.

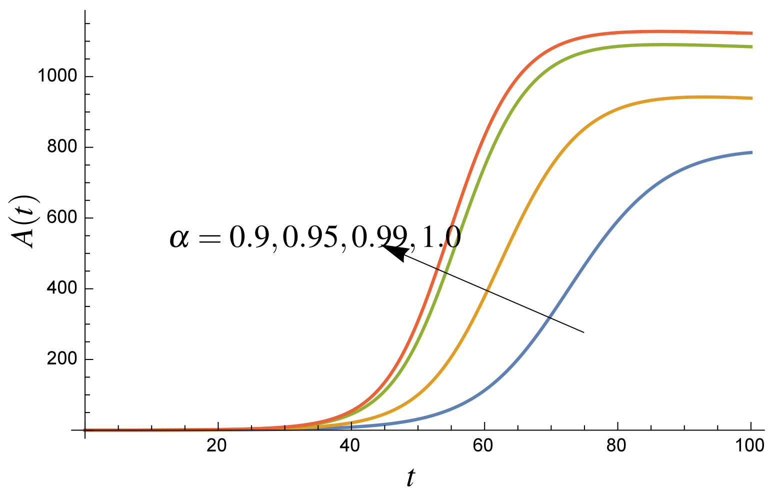

Figure 4.6: Graphs of the solution trajectories, $A(t)$, at various values of α for Example 2.

Chapter 5:Summary and Conclusions

The present thesis deals with numerical treatment of classes of nonlinear fractional initial value problems with CF fractional derivative and ABC fractional derivative. We used numerical algorithms based on modified Adams-Bashforth method to handle these problems. In addition, we discuss the theoretical solution of special class of fractional IVPs. The efficiency and accuracy of the present scheme is discussed via solving several examples and compare with other researchers.

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Fractional calculus has been recently received huge attention from Mathematicians and engineers due to its importance in many real-life applications such as: fluid mechanics, electromagnetic, acoustics, chemistry, biology, physics, and material sciences. In this thesis, we present numerical algorithms for solving fractional IVPs and system of fractional IVPs where two types of fractional derivatives are used: Caputo-Fabrizio, and Atangana-Baleanu-Caputo derivatives.

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