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SUM OF SQUARES WITH Q-SERIES, GOSPER'S Q-TRIGONOMETRY, AN NEW IDENTITIES VIA AN EXTENDED BAILEY TRANSFORM

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Zina Al Houchan

This thesis is submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

Under the Supervision of Dr. Mohamed El Bachraoui

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Declaration of Original Work Declaration of Original Work

I, Zina Samir Al Houchan, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled *"Sum Of Squares With q-Series, Gosper's q-Trigonometry, And New Identities Via An Extended Bailey Transform "*, hereby, solemnly declare that this thesis is my own original research work that has been done and prepared by me under the supervision of Dr. Mohamed El Bachraoui, in the College of Science at UAEU. This work has not previously been presented or published, or formed the basis for the award of any academic degree, diploma or a similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my thesis have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this thesis.

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Abstract

This report is concerned about q-series and some of their applications. Firstly, Jacobi's q-series proof for Legendre's theorem on sums of four squares will be presented. By way of comparison, the classical approach of this result will be also discussed. Secondly, Gosper's q-trigonometry will be introduced using Jacobi's theta functions and the theory of elliptic functions shall be employed to confirm one of Gosper's conjectures. As an application, a proof for Fermat's theorem on the sums of squares will be provided. Thirdly, an extended version of Bailey's transform will be established and as a consequence a variety of new q-series identities will be proved. In some of these identities the q-binomial coefficients will be involved.

Keywords: q-series, q-trigonometry, Bailey transform, sums of squares, q-analogues.

Title and Abstract (in Arabic)

متوالية $q-$ مع تطبيقات لمعاملات ذات الحدين، تقسيم الاعداد الصحيحة، و مجموع المربعات

الملخص

في هذه الأطروحة ستركّز على سلاسل q وبعض تطبيقاتها. أولاً سيتم عرض اثبات سلسلة الخاصة بياكوبي لنظرية ليجاندر المعنيّة بمجموع المربّعات الأربعة، وكنوع من المقارنة سيتم عرض النهج الكلاسيكي لهذه النتيجة. ثانياً سيتم عرض حساب مثلثات q التابعة لغوسبر عن طريق دوال ثيتا الخاصة بياكوبي ونظرية الدوال الإهليلجيّة من أجل توظيفها في اثبات واحدة من تخمينات غوسبر. سيتم أيضاً عرض اثبات مبرهنة فيرما الخاصة بمجموع المربعات كنوع من التطبيقات. ثالثاً سيتم نشر النسخة الموسّعة من تحويلات بيلي ونتيجة لذلك يستم اثبات مجموعة متنوعة من هويات سلاسل q الجديدة. بعض هذه الهويّات تتضمّن معامِلات q ذو الحدّين.

مفاهيم البحث الرئيسية: متوالية $q-$ ، معاملات $q-$ ذات الحدين، معاملات ذات الحدين، تقسيم الاعداد الصحيحة، مجموع المربعات، نظائر $q-$.

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Dedication

To my beloved parents and family

Table of Contents

Chapter 1: Introduction

The theme of this report is basic hypergeometric series or also called q-series. This topic has a long history starting from Euler and Gauss through their work on Number Theory and Analysis. Among mathematicians of the 19th and 20th centuries who have worked with q-series Jacobi, Riemann, Rogers, Jackson, and Ramanujan are mentioned. Basic hypergeometric series is still a very popular area under contemporary mathematicians. As a result of their long history and wide range of applications, there are hundreds of books and articles written on the subject. Standard references on q-series and their applications include [4, 3, 6, 18]. In Chapter 2, Lagrange's famous four squares theorem on representing nonnegative integers as a sum of four integer squares will be presented. In details both the classical proof and Jacobi's proof based on q-series techniques will be discussed.

Chapter 3 is mainly concerned about Jacobi's theta functions and Gosper's q-trigonometry. A proof for one of Gosper's conjectures using the theory of elliptic functions through theta functions will be presented. As an application, Fermat's two squares theorem on the representation of non-negative integers as a sum of two integer squares is proven. Finally in Chapter 4, an extended version of the Bailey transform will be shown, and it will be employed to establish a variety of new q-series identities. Some of our identities involve the q-binomial coefficients.

Chapter 2: Application of q-series to sums of four squares

Throughout let $\mathbb{N}$ be the set of positive integers, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and let $\mathbb{Z}$ denote the set of integers. let k, l, m, n denote non-negative integers. In this chapter a very important application of q-series to the integer representations in Number Theory will be presented. A survey on Jacobi's extension by q-series of Lagrange's famous theorem on sums of four squares will be given. By way of comparison, both version of this theorem along with their proofs will be given. For a number theoretic approach for integer representations it refers to [28, 29] and for a q-series method it refers to [15].

2.1 Basics of q-series

Throughout this report q will denote a complex number which sometimes is assumed to be $|q| < 1$ in order to justify convergence facts. The following notation from the theory of basic hypergeometric series will be used, see Gasper-Rahman [18]. The q-shifted factorials for complex numbers $x, x_1, \dots, x_m$ are given by

$$(x;q)_0 = (x)_0 = 1, \quad (x;q)_n = (x)_n = \prod_{j=0}^{n-1}(1 - xq^j), \quad (x;q)_\infty = (x)_\infty = \prod_{j=0}^{\infty}(1 - xq^j),$$

$$(x_1, x_2, \dots, x_m; q)_n = \prod_{j=1}^{m}(x_j;q)_n \text{ and } (x_1, x_2, \dots, x_m; q)_\infty = \prod_{j=1}^{m}(x_j;q)_\infty.$$

Noting that

$$(x;q)_n = \frac{(x;q)_\infty}{(xq^n;q)_\infty},$$

The previous definition can be extended to negative integers $-n$ as follows

$$(x;q)_{-n} = \frac{(x;q)_\infty}{(xq^{-n};q)_\infty}.$$

Proposition 2.1.1. *Let x be a complex number and let n be a non-negative integer. Then*

$$\text{(a)}\quad (x;q)_{-n} = \frac{(-1)^n q^{\binom{n+1}{2}}}{x^n\left(\frac{q}{x};q\right)_n}.$$

$$\text{(b)}\quad \lim_{x\to 0} x^n(x^{-1};q^2)_n = (-1)^n q^{n(n-1)}.$$

Proof. (a) This part follows from the basic fact

$$(x;q)_{-n} = \frac{(x;q)_\infty}{(xq^{-n};q)_\infty} = \frac{1}{(xq^{-n};q)_n}$$

combined with

$$\begin{aligned}
(xq^{-n};q)_n &= (1-xq^{-n})(1-xq^{-n+1})\cdots(1-xq^{-2})(1-xq^{-1}) \\
&= \big(1-\frac{x}{q}\big)\big(1-\frac{x}{q^2}\big)\cdots\big(1-\frac{x}{q^{n-1}}\big)\big(1-\frac{x}{q^n}\big) \\
&= \frac{x}{q}\big(\frac{q}{x}-1\big)\frac{x}{q^2}\big(\frac{q^2}{x}-1\big)\cdots\frac{x}{q^{n-1}}\big(\frac{q^{n-1}}{x}-1\big)\frac{x}{q^n}\big(\frac{q^n}{x}-1\big) \\
&= \frac{x^n}{q^{\frac{n(n+1)}{2}}}(-1)^n\big(\frac{q}{x};q\big)_n.
\end{aligned}$$

(b) As for this part, note that

$$x^n(x^{-1};q^2)_n = x^n(1-\frac{1}{x})(1-\frac{1}{x}q^2)\cdots(1-\frac{1}{x}q^{2n-2}) = (x-1)(x-q^2)\cdots(x-q^{2n-2})$$

Now take limits as $x\to 0$ to obtain

$$\begin{aligned}
\lim_{x\to 0} x^n(x^{-1};q^2)_n &= (-1)(-q^2)\cdots(-q^{2n-2}) \\
&= (-1)^n q^2\cdot q^4\cdots q^{2n-2} = (-1)^n q^{2+4+\cdots+(2n-2)} \\
&= (-1)^n q^{n(n-1)}
\end{aligned}$$

as desired.

The q-binomial Theorem [15] which is stated without proof is needed.

Theorem 2.1.2. *For any complex numbers x and z such that $|z| < 1$ there holds*

$$\sum_{n=0}^{\infty} \frac{(x;q)_n}{(q;q)_n} z^n = \frac{(xz;q)_\infty}{(z;q)_\infty}.$$

2.2 Lagrange's theorem and its classical proof

In this section Lagrange's theorem on sums of squares and its classical proof is presented.

Theorem 2.2.1. *Every non-negative integer is the sum of four integer squares. That is, the Diophantine equation*

$$n = z_1^2 + z_2^2 + z_3^2 + z_4^2$$

has solutions for any $n \in \mathbb{N}_0$ and any $z_1, z_2, z_3, z_4 \in \mathbb{Z}$.

Proof. Observe that if $m = s_1^2 + s_2^2 + s_3^2 + s_4^2$ and $n = r_1^2 + r_2^2 + r_3^2 + r_3^2$, then $mn = z_1^2 + z_2^2 + z_3^2 + z_4^2$ where

$$\begin{aligned}
z_1 &= s_1 r_1 + s_2 r_2 + s_3 r_3 + s_4 r_4 \\
z_2 &= s_1 r_2 - s_2 r_1 - s_3 r_4 + s_4 r_3 \\
z_3 &= s_1 r_3 - s_3 r_1 + s_2 r_4 - s_4 r_2 \\
z_4 &= s_1 r_4 - s_4 r_1 - s_2 r_3 + s_3 r_2.
\end{aligned} \tag{2.1}$$

This observation and the fact that every $n \in \mathbb{N}$ is the product of prime numbers imply that it is enough to prove the result for prime numbers p. As $2 = 1^2 + 1^2 + 0^2 + 0^2$, it will be enough to prove the result for prime $p > 2$. Note that if a and b are distinct elements from the set $\{0,1,\dots,\frac{p-1}{2}\}$, then $a-b<p$ and $a+b<p$ from which it follows that $a \not\equiv b \pmod p$ and $a \not\equiv -b \pmod p$ and so $(a-b)(a+b) \not\equiv 0 \pmod p$. Then, the set $\{a^2|a=0,1,\dots,\frac{p-1}{2}\}$ contains $\frac{p+1}{2}$ elements. Similarly, the set $\{-b^2-1|b = 0,1,\dots,\frac{p-1}{2}\}$ contains $\frac{p+1}{2}$ elements. But there are only p incongruent elements modulo p and so, there exist integers $a,b \in \{0,1,\dots,\frac{p-1}{2}\}$ such that $a^2 \equiv -b^2-1 \pmod p$. Now writing $a^2+b^2+1=np$ for some integers $n \geqslant 1$ gives

$$\begin{aligned} p \leq np &= a^2+b^2+1 \\ &\leq (\frac{p-1}{2})^2+(\frac{p-1}{2})^2+1 \\ &= 2\Big(\frac{p-1}{2}\Big)^2+1 < 2\frac{p^2}{4}+1 \\ &< p^2, \end{aligned}$$

implying that $1 \leq n < p$. As $np = a^2+b^2+1^2+0^2$ then, the set

$$\{k \geq 1 : \ kp \text{ is the sum of four squares}\}$$

has a least element by the well-ordering's principal. Call this least element m. Thus, there exist $s_1,s_2,s_3,s_4 \in \mathbb{Z}$ such that

$$mp = s_1^2+s_2^2+s_3^2+s_4^2 \text{ and } 1 \leqslant m \leqslant n < p.$$

Note that this will be done if it showed that $m=1$. Suppose for a contradiction that $m>1$. Let for $i=1,2,3,4$, the number r_i be the unique integer such that $r_i \equiv s_i \pmod m$ and $\frac{-m}{2} < r_i < \frac{m}{2}$. Then $r_1^2+r_2^2+r_3^2+r_4^2 \equiv s_1^2+s_2^2+s_3^2+s_4^2 = mp \equiv 0 \pmod m$ and so there is $r \in \mathbb{N}_0$ such that $r_1^2 + r_2^2 + r_3^2 + r_4^2 = rm$. If $r = 0$, then

$$r_1^2 = r_2^2 = r_3^2 = r_4^2 = 0 \text{ and } s_i^2 \equiv r_i^2 \equiv 0 \pmod{m^2}$$

from which it follows that $mp = s_1^2 + s_2^2 + s_3^2 + s_4^2 \equiv 0 \pmod{m^2}$. That is, $p \equiv 0 \pmod{m}$ which is impossible as $1 < m < p$. Suppose now that $r \geq 1$. Then

$$mr = r_1^2 + r_2^2 + r_3^2 + r_4^2 \leqslant 4(\frac{m}{2})^2 = m^2$$

Note that $r = m \Rightarrow r_i = \frac{m}{2}$ and conversely, if $r_i = \frac{m}{2}$ and m is even, then implies that

$$mr = r_1^2 + r_2^2 + r_3^2 + r_4^2 = 4(\frac{m}{2})^2 = m^2$$

and so $r = m$. Thus, shown that $r = m$ if and only if m is even and $r_i = \frac{m}{2}$ for $i = 1, 2, 3, 4$. So, if $r = m$, then $s_i \equiv \frac{m}{2} \pmod{m}$ and

$$mp = s_1^2 + s_2^2 + s_3^2 + s_4^2 = 4(\frac{m}{2})^2 = m^2 \equiv 0 \pmod{m^2},$$

from which it follows that $p \equiv 0 \pmod{m}$, which is a contradiction. Suppose that $1 \leqslant r < m$. Then

$$\begin{aligned} m^2 rp &= (mp)(mr) \\ &= (s_1^2 + s_2^2 + s_3^2 + s_4^2)(r_1^2 + r_2^2 + r_3^2 + r_4^2) \qquad (2.2) \\ &= z_1^2 + z_2^2 + z_3^2 + z_4^2 \end{aligned}$$

where $s_i \equiv r_i \pmod{m}$ and z_i are as in (2.1). Then,

$$z_1 = s_1r_1 + s_2r_2 + s_3 + r_3 + s_4r_4 \equiv s_1^2 + s_2^2 + s_3^2 + s_4^2 \pmod{m} \equiv 0 \pmod{m},$$

$$z_2 = s_1r_2 - s_2r_1 - s_3r_4 + s_4r_3 \equiv s_1s_2 - s_2s_1 - s_3s_4 + s_4s_3 \pmod{m} \equiv 0 \pmod{m},$$

and similarly $z_3 \equiv z_4 \equiv 0 \pmod{m}$. Letting $w_i = \frac{z_i}{m} \in \mathbb{Z}$, deduce from (2.2) that

$$rp = \frac{z_1^2}{m^2} + \frac{z_2^2}{m^2} + \frac{z_3^2}{m^2} + \frac{z_4^2}{m^2} = w_1^2 + w_2^2 + w_3^2 + w_4^2$$

which contradicts the minimality of m. Consequently $m = 1$ and the proof is complete.

2.3 Ramanujan's ${}_1\psi_1$ sum

The main result in this section is the so-called Ramanujan's ${}_1\psi_1$ sum which is needed to give Jacobi's proof for his extension of Theorem 2.2.1.

Theorem 2.3.1. *(Ramanujan) If a, b and c are complex numbers such that $|\frac{b}{a}| < |z| < 1$, then*

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n = \frac{(az;q)_\infty(\frac{q}{az};q)_\infty(q;q)_\infty(\frac{b}{a};q)_\infty}{(z;q)_\infty(\frac{b}{az};q)_\infty(b;q)_\infty(\frac{q}{a};q)_\infty}.$$

Proof. Letting for simplicity $f(b) = \sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n$, then

$$\begin{aligned} f(b) &= \sum_{n=0}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n + \sum_{n=1}^{\infty} \frac{(a;q)_{-n}}{(b;q)_{-n}} z^{-n} \\ &= \sum_{n=0}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n + \sum_{n=1}^{\infty} \frac{(\frac{q}{b};q)_n}{(\frac{q}{a};q)_n} (\frac{b}{az})^n, \end{aligned}$$

where the previous two series converge absolutely if $|\frac{b}{az}| < 1$ and $|z| < 1$ by Ratio test.

Furthermore,

$$\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n - a\sum_{n-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}(qz)^n = \sum_{n-\infty}^{\infty}(\frac{(a;q)_n}{(b;q)_n} - \frac{aq^n(a;q)_n}{(b;q)_n})z^n$$
$$= \sum_{n=-\infty}^{\infty}\frac{(1-aq^n)(a;q)_n}{(b;q)_n}z^n$$
$$= \sum_{n=-\infty}^{\infty}\frac{(a;q)_{n+1}}{(b;q)_n}z^n$$
$$= \frac{1-\frac{b}{q}}{z}\sum_{n=-\infty}^{\infty}\frac{(a;q)_{n+1}}{(\frac{b}{q};q)_{n+1}}z^{n+1}$$

which means that

$$f(b) - z^{-1}(1-\frac{b}{q})f(\frac{b}{q}) = a\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}(qz)^n$$

and so by substituting $b \to bq$ yields

$$f(bq) - z^{-1}(1-b)f(b) = a\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(bq;q)_n}(qz)^n \tag{2.3}$$

On the other hand,

$$a\sum_{n-\infty}^{\infty}\frac{(a;q)_n}{(bq;q)_n}(qz)^n = -ab^{-1}\sum_{n-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_{n+1}}((1-bq^n)-1)(1-b)z^n$$
$$= -ab^{-1}(1-b)\sum_{n-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n + ab^{-1}\sum_{n-\infty}^{\infty}\frac{(a;q)_n}{(bq;q)_n}z^n \tag{2.4}$$
$$= -ab^{-1}(1-b)f(b) + ab^{-1}f(bq).$$

Then (2.3) and (2.3) give

$$(1-ab^{-1})f(bq) = z^{-1}(1-b)f(b) - ab^{-1}(1-b)f(b)$$
$$= (z^{-1}-ab^{-1})(1-b)f(b)$$

and therefore,

$$f(b) = \frac{(1-\frac{a}{b})f(bq)}{(\frac{1}{z}-\frac{a}{b})(1-b)}f(bq).$$

Repeatedly application of the previous identity yields

$$f(b) = \frac{\left(\frac{b}{a};q\right)_\infty}{(b;q)_n(\frac{b}{az};q)_n}f(bq^n)$$

where by letting $n \to \infty$ to obtain

$$f(b) = \frac{(\frac{b}{a};q)_\infty}{(b,\frac{b}{az};q)_\infty}f(0). \tag{2.5}$$

Back to the fact that

$$f(b) = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n + \sum_{n=1}^{\infty}\frac{(\frac{q}{b};q)_n}{(\frac{q}{a};q)_n}(\frac{b}{az}),$$

With the help of Theorem 2.1.2

$$f(q) = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty}.$$

Now by letting $b \to q$ in (2.5) and using the foregoing formula the following holds

$$\begin{aligned} f(0) &= f(q)\frac{(\frac{q}{az},q;q)_\infty}{(\frac{q}{a};q)_\infty} \\ &= \frac{(az,\frac{q}{az},q;q)_\infty}{(\frac{q}{a},z;q)_\infty}. \end{aligned}$$

Then by putting this value of $f(0)$ in (2.5) implies

$$f(b) = \frac{\left(\frac{b}{a}, az, \frac{q}{az}, q; q\right)_\infty}{\left(b, \frac{b}{az}, \frac{q}{a}, z; q\right)_\infty}.$$

This completes the proof.

Having the following very useful consequence of Theorem 2.3.1 which is due to Jacobi [23].

Corollary 2.3.2. *(Jacobi's triple product identity) For any complex number z there holds*

$$\sum_{n-\infty}^{\infty} (-z)^n q^{n^2} = (zq, \frac{q}{z}, q^2; q^2)_\infty.$$

Proof. Substitute in Theorem 2.3.1 $(a,z,q) \to (a^{-1}, az, 0)$ to get

$$\sum_{n-\infty}^{\infty} (a^{-1};q)_n a^n z^n = \frac{(z, q/z, q; q)_\infty}{(az, qa; q)_\infty}.$$

Substitute in the foregoing identity $(q,z) \to (q^2, zq)$ to find

$$\sum_{n-\infty}^{\infty} (a^{-1};q^2)_n a^n z^n q^n = \frac{(zq, q/z, q^2; q^2)_\infty}{(azq, q^2 a; q^2)_\infty}.$$

Now taking limits as $a \to 0$ and using Proposition 2.1.1(b) yields

$$\sum_{n-\infty}^{\infty} (-1)^n q^{n(n-1)+n} z^n = (zq, \frac{q}{z}, q^2; q^2)_\infty,$$

or equivalently

$$\sum_{n-\infty}^{\infty} (-z)^n q^{n^2} = (zq, \frac{q}{z}, q^2; q^2)_\infty,$$

as desired.

2.4 Jacobi's version for sums of four squares

In this section a q-series proof for Jacobi's extension for Lagrange's theorem on sums of squares will be presented. It is worth to note that there many proofs given by different authors. The proof presented here is due to Hirschhorn [22]. For a survey on integer representations through q-series it refers to the book by Berndt [15].

Definition 2.4.1. For any non-negative integer n let

$$r_4(n) = \#\{(x_1, x_2, x_3, x_4) \in \mathbb{Z} : \; x_1^2 + x_2^2 + x_3^2 + x_4^2 = n\}.$$

For instance, $r_4(0) = 1, \; r_4(1) = 8, \; r_4(2) = 24, \; r_4(3) = 32$, and $r_4(4) = 24$.

Letting $\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$, it's clearly can be seen that $\varphi^4(q)$ is the generating function for $r_4(n)$. That is,

$$\sum_{n=0}^{\infty} r_4(n) q^n = \varphi^4(q) = \Big(\sum_{n=-\infty}^{\infty} q^{n^2} \Big)^4. \tag{2.6}$$

Recall the following properties of $\varphi(q)$.

Proposition 2.4.1. *There holds*

(a) $\varphi^4(-q^2) = \dfrac{(q^2;q^2)_\infty^4}{(-q^2;q^2)_\infty^4} = \Big(1 + 4\sum_{n=1}^{\infty} \dfrac{q^n \cos n\theta}{1+q^{2n}}\Big)\Big(1 + 4\sum_{n=1}^{\infty} \dfrac{(-1)^n q^n \cos n\theta}{1+q^{2n}}\Big).$

(b) $\varphi^4(q) = 1 + 8\sum_{n=1}^{\infty} \dfrac{nq^n}{1+(-q)^n}.$

Proof. (a) Letting $z = -1$ in Corollary 2.3.2 gives

$$\varphi(q) = (-q;q^2)_\infty{}^2 (q^2;q^2)_\infty$$

implying that

$$\varphi(-q) = (q;q^2)_\infty{}^2(q^2;q^2)_\infty = (q;q)_\infty(q;q^2)_\infty = \frac{(q^2;q^2)_\infty}{(-q;q)_\infty}(q;q^2)_\infty = \frac{(q;q)_\infty}{(-q;q)_\infty}$$

which gives the first formula in this part upon replacing q by q^2 and taking the fourth powers. As for the second formula, note first that

$$\frac{q^n e^{in\theta}}{1+q^{2n}} + \frac{q^{-n}e^{-in\theta}}{1+q^{-2n}} = \frac{q^n(e^{in\theta}+e^{-in\theta})}{1+q^{2n}} = \frac{2q^n\cos n\theta}{1+q^{2n}}$$

and so,

$$1+4\sum_{n=1}^{\infty}\frac{q^n\cos n\theta}{1+q^{2n}} = 1+2\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{q^n e^{in\theta}}{1+q^{2n}} = \sum_{n=-\infty}^{\infty}\frac{(-1;q^2)_n}{(-q^2;q^2)_n}q^n e^{in\theta}$$
$$= \frac{(-qe^{i\theta},-qe^{-i\theta},q^2,q^2;q^2)_\infty}{(qe^{i\theta},qe^{-i\theta},-q^2,-q^2;q^2)_\infty}$$

where in the last step the Theorem 2.3.1 was used. Replacing in the previous identity θ by $\pi-\theta$ and multiplying gives

$$\Big(1+4\sum_{n=1}^{\infty}\frac{q^n\cos n\theta}{1+q^{2n}}\Big)\Big(1+4\sum_{n=1}^{\infty}\frac{(-1)^n q^n\cos n\theta}{1+q^{2n}}\Big)$$

$$= \frac{(-qe^{i\theta},-qe^{-i\theta},qe^{-i\theta},qe^{i\theta};q^2)(q^2;q^2)_\infty^4}{(qe^{i\theta},qe^{-i\theta},-qe^{-i\theta},-qe^{i\theta};q^2)(-q^2;q^2)_\infty^4} = \frac{(q^2;q^2)_\infty^4}{(-q^2;q^2)_\infty^4} = \varphi^4(-q^2).$$

(b) Recall that

$$\int_{-\pi}^{\pi}\cos m\theta\cos n\theta\, d\theta = \begin{cases}\pi, \text{if } m=n,\\[2ex] 0, \text{if } m\neq n.\end{cases}$$

Then using this fact and integrating in part (a), then after simplification

$$2\pi+16\pi\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n}}{(1+q^{2n})^2}=2\pi\varphi^4(-q^2)$$

where upon substituting $-q^2$ by q and rewriting then deriving

$$\varphi^4(q)=1+8\sum_{n=1}^{\infty}\frac{q^n}{\left(1+(-q)^n\right)^2}=1+8\sum_{n=1}^{\infty}\frac{nq^n}{1+(-q)^n}$$

where in the last step employing the basic facts

$$\frac{1}{(1+x)^2}=\left(-\frac{1}{1+x}\right)'=\sum_{n=1}^{\infty}(-1)^{n-1}nx^{n-1}.$$

This finishes the proof.

The main result of this section will now be stated.

Theorem 2.4.2. *(Jacobi) For any positive integer n :*

$$r_4(n)=8\sum_{\substack{d|n\\ 4\nmid d}}d.$$

Proof. Recall that $\varphi^4(q)=\sum_{n=0}^{\infty}r_4(n)q^n$. Then by Proposition 2.4.1(b) and series manipulations, it leads to

$$\varphi^4(q)=1+8\left(\sum_{m=1}^{\infty}\frac{mq^m}{1+(-q)^m}\right)=1+8\sum_{n=1}^{\infty}\frac{2nq^{2n}}{1+q^{2n}}+\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}$$

$$=1+8\sum_{n=1}^{\infty}\left(\frac{2nq^{2n}}{1-q^{2n}}+\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}\right)-8\sum_{n=1}^{\infty}\left(\frac{2nq^{2n}}{1-q^{2n}}-\frac{2nq^{2n}}{1+q^{2n}}\right)$$

$$= 1 + 8\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - 8\sum_{n=1}^{\infty} \frac{4nq^{4n}}{1-q^{4n}} = 1 + 8\sum_{n=1, 4\nmid n}^{\infty} \frac{nq^n}{1-q^n}$$

$$= 1 + 8\sum_{n=1, 4\nmid n}^{\infty} \frac{nq^n}{1-q^n} = 1 + 8\sum_{n=1, 4\nmid n}^{\infty} nq^n . \sum_{m=0}^{\infty} q^{nm}$$

$$= 1 + 8\sum_{m=1}^{\infty} \sum_{d=1, 4\nmid d}^{\infty} dq^{dm} = 1 + 8\sum_{n=1}^{\infty} \sum_{d=1, 4\nmid d}^{\infty} dq^n .$$

This competes the proof.

Remark 2.4.1. For any non-negative integer n let $r_2(n)$ be defined as follows

$$r_2(n) = \#\{(x_1, x_2) \in \mathbb{Z} : \ x_1^2 + x_2^2 = n\}.$$

Then clearly, $\varphi^2(q)$ is the generating function for $r_2(n)$. In particular, if p is an odd prime, then by a statement due to Fermat, p is a sum of two squares if and only if $p \equiv 1 \bmod 4$. Jacobi [23] gave the following extension

$$r_2(n) = 4\Big(\sum_{\substack{d|n \\ d\equiv 1 \bmod 4}} 1 - \sum_{\substack{d|n \\ d\equiv 3 \bmod 4}} 1 \Big). \tag{2.7}$$

There are many proofs known for this result which are based on q-series, see for example [15, 17, 21]. In Theorem 3.4.5 a different proof will be given for (2.7) based on Gosper's q-trigonometry.

Chapter 3: Gosper's q-trigonometry and some applications

3.1 Introduction

In this chapter Gosper's q-analogues for the trigonometric functions $\sin z$ and $\cos z$ are represented. Recently many conjectures of Gosper on q-trigonometry have been confirmed using the theory of elliptic functions, see for instance [1, 10, 11, 12, 20]. As an illustration of this approach, one on these proofs found in [10] will be presented. Moreover, the connection between q-trigonometric functions and the q-gamma function shall be used to prove Fermat's theorem on sums of two squares. Throughout, let $q = e^{\pi i\tau}$ with $\mathrm{Im}(\tau) > 0$, let $\tau' = -\frac{1}{\tau}$, and let $p = e^{\pi i\tau'}$. Note that as $\mathrm{Im}(\tau) > 0$ then $|q| < 1$ and $|p| < 1$. Jacobi's theta functions are very important in this chapter. They will be introduced briefly with their properties which is needed. Refer to Whittaker and Watson [31] for more information on theta functions.

Definition 3.1.1. Jacobi's first and second theta functions are defined as follows:

$$\theta_1(z,q) = \theta_1(z \mid \tau) = 2\sum_{n=0}^{\infty}(-1)^n q^{\frac{(2n+1)^2}{4}} \sin(2n+1)z,$$
$$\theta_2(z,q) = \theta_2(z \mid \tau) = 2\sum_{n=0}^{\infty} q^{\frac{(2n+1)^2}{4}} \cos(2n+1)z.$$

It is clear that the function θ_1 is odd and the function θ_2 is even.

The following identities can be easily verified from the definitions.

$$\begin{aligned}
\theta_1(z \mid \tau) &= \theta_2\big(z - \frac{\pi}{2} \mid \tau\big) \\
\theta_1(k\pi \mid \tau) &= 0 \ (k \in \mathbb{Z}) \\
\theta_1(z+\pi \mid \tau) &= -\theta_1(z \mid \tau), \\
\theta_1(z+\pi\tau \mid \tau) &= -q^{-1}e^{-2iz}\theta_1(z \mid \tau)
\end{aligned} \tag{3.1}$$

Finally, it is worth to mention that the functions θ_1 and θ_2 can be expressed as infinite products as follows

$$
\begin{aligned}
\theta_1(z \mid \tau) &= iq^{\frac{1}{4}}e^{-iz}(q^2e^{-2iz}, e^{2iz}, q^2; q^2)_\infty \\
\theta_2(z \mid \tau) &= q^{\frac{1}{4}}e^{-iz}(-q^2e^{-2iz}, -e^{2iz}, q^2; q^2)_\infty.
\end{aligned}
\tag{3.2}
$$

3.2 The functions $\sin_q z$ and $\cos_q z$

In this section Gosper's q-analogues for the trigonometric functions $\sin z$ and $\cos z$ and some of their properties will be introduced. To this end, Jackson's q-gamma and the q-digamma function will be needed.

Definition 3.2.1. For any complex number z the q-gamma function is given by

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}.$$

and the q-digamma function is given by

$$\psi_q(z) = (\log \Gamma_q(z))' = \frac{\Gamma_q'(z)}{\Gamma_q(z)}.$$

Definition 3.2.2. Gosper [19] defined the functions $\sin_q z$ and $\cos_q z$ as follows

$$
\begin{aligned}
\sin_q \pi z &= q^{\frac{1}{4}}\Gamma_{q^2}^2\left(\frac{1}{2}\right)\frac{q^{z(z-1)}}{\Gamma_{q^2}(z)\Gamma_{q^2}(1-z)} \\
\cos_q \pi z &= \Gamma_{q^2}^2\left(\frac{1}{2}\right)\frac{q^{z^2}}{\Gamma_{q^2}\left(\frac{1}{2}-z\right)\Gamma_{q^2}\left(\frac{1}{2}+z\right)}
\end{aligned}
$$

and showed that these two functions can also be defined via Jacobi theta functions as follows.

$$\sin_q(z) = \frac{\theta_1(z|\tau')}{\theta_1(\frac{\pi}{2}|\tau')} \text{ and } \cos_q(z) = \frac{\theta_1(z+\frac{\pi}{2}|\tau')}{\theta_1(\frac{\pi}{2}|\tau')}. \tag{3.3}$$

From the well-known fact [18, p. 21] $\lim_{q\to 1}\Gamma_q(z)=\Gamma(z)$, one can easily check that

$$\lim_{q\to 1}\sin_q \pi z=\sin\pi z \text{ and } \lim_{q\to 1}\cos_q \pi z=\cos\pi z,$$

showing that $\sin_q z$ and $\cos_q z$ are q-analogue for $\sin z$ and $\cos z$ respectively.

Using the properties of Jacobi's theta function (3.1), the following basic facts on $\sin_q z$ and $\cos_q z$ are recorded.

Proposition 3.2.1. *There holds*

$$\begin{aligned}
\sin_q(\frac{\pi}{2}-z)&=\cos_q(z),\ \sin_q k\pi=0\ (k\in\mathbb{Z}),\\
\sin_q(\pi+z)&=-\sin_q(z)=\sin_q(-z),\\
-\cos_q(\pi+z)&=\cos_q(z)=\cos_q(-z),\\
\sin_q'(\frac{\pi}{2}-z)&=-\cos_q'(z),\ \cos_q'(z-\frac{\pi}{2})=\sin_q'(z),\\
\sin_q'(\pi-z)&=-\sin_q'(z),\ \cos_q'(\pi-z)=\cos_q'(z).
\end{aligned}$$

Moreover, Gosper [19] established the following relations.

Proposition 3.2.2. *There holds*

$$\begin{aligned}
&\sin_{q^2}\frac{\pi}{4}=\cos_{q^2}\frac{\pi}{4}=\frac{\Pi_{q^2}^{\frac{1}{2}}}{\Pi_q^{\frac{1}{2}}}\\
&\sin_q'(0)=\cos_q'(\frac{\pi}{2})=-\frac{2\log(q)}{\pi}\Pi_q.
\end{aligned} \tag{3.4}$$

3.3 A proof for a conjecture of Gosper

Gosper [19, p. 93] as a q-analogue for the formula

$$\sin 2z = 2\sin z \cos z$$

showed that

$$\sin_q(2z) = \frac{\Pi_q}{\Pi_{q^2}} \sin_{q^2}(z) \cos_{q^2}(z) \tag{3.5}$$

where

$$\Pi_q = q^{\frac{1}{4}} \frac{(q^2, q^2)_\infty^2}{(q, q^2)_\infty^2} \tag{3.6}$$

and as a q-analogue for

$$\cos 2z = \cos^2 z - \sin^2 z,$$

he conjectured that

$$\cos_q(2z) = (\cos_{q^2} z)^2 - (\sin_{q^2} z)^2. \tag{3.7}$$

A different proof for (3.5) based on logarithmic differentiation was given by Mező [27]. El Bachraoui [10] using elliptic functions confirmed (3.7) and gave another proof for (3.5). In this section the proof of (3.7) using the theory of elliptic functions is given as presented in [10]. Note that by (3.3), the identity (3.7) is equivalent with

$$\frac{\theta_1\left(2z + \frac{\pi}{2} \mid \tau'\right)}{\theta_1\left(\frac{\pi}{2} \mid \tau'\right)} = \left(\frac{\theta_1\left(z + \frac{\pi}{2} \mid \frac{\tau'}{2}\right)}{\theta_1\left(\frac{\pi}{2} \mid \frac{\tau'}{2}\right)}\right)^2 - \left(\frac{\theta_1\left(z \mid \frac{\tau'}{2}\right)}{\theta_1\left(\frac{\pi}{2} \mid \frac{\tau'}{2}\right)}\right)^2,$$

which after basic simplifications means

$$\theta_1\left(2z+\frac{\pi}{2}\mid\tau'\right)\theta_1^2\left(\frac{\pi}{2}\mid\frac{\tau'}{2}\right) \tag{3.8}$$

$$=\theta_1\left(\frac{\pi}{2}\mid\tau'\right)\theta_1^2\left(z+\frac{\pi}{2}\mid\frac{\tau'}{2}\right)-\theta_1\left(\frac{\pi}{2}\mid\tau'\right)\theta_1^2\left(z\mid\frac{\tau'}{2}\right).$$

Proving that (3.8) is true. Showing some preparation which started by recalling elliptic functions as in Apostol [8].

Definition 3.3.1. A meromorphic function $f(z)$ is called elliptic if it is doubly periodic. That is, if there are complex numbers w_1, w_2 whose ratio is not a real number such that for all $z \in \mathbb{C}$ then

$$f(z+w_1) = f(z+w_2) = f(z).$$

It is clear that if w_1 and w_2 are periods of $f(z)$, then so is $aw_1 + bw_2$ for any integers a and b. If every period of $f(z)$ is of the form $aw_1 + bw_2$ where $a, b \in \mathbb{Z}$, then (w_1, w_2) is called a fundamental pair. In this case, the set

$$\{aw_1 + bw_2 : \ a, b \in \mathbb{Z}\}$$

is called a period parallelogram. The following well-known result which is referred to as the residue theorem for elliptic functions is:

Theorem 3.3.1. *[8, Theorem 1.7] The sum of the residues of an elliptic function is zero at its poles in any period parallelogram.*

Proving the following result.

Theorem 3.3.2. *Let $f(z)$ be an entire function such that*

$$f(z+\pi) = -f(z) \quad \text{and } f\left(z+\frac{\pi\tau}{2}\right) = q^{-\frac{1}{2}}e^{-2iz}f(z).$$

Then for all complex numbers x_1, x_2, and x_3 :

$$\frac{\theta_1(x_3-x_2-x_1 \mid \tau)f(x_3)}{\theta_1\left(x_1-x_3 \mid \frac{\tau}{2}\right)\theta_1\left(x_2-x_3 \mid \frac{\tau}{2}\right)} = \frac{\theta_1(x_2-x_1-x_3 \mid \tau)f(x_2)}{\theta_1\left(x_1-x_2 \mid \frac{\tau}{2}\right)\theta_1\left(x_2-x_3 \mid \frac{\tau}{2}\right)}$$

$$-\frac{\theta_1(x_1-x_2-x_3 \mid \tau)f(x_1)}{\theta_1\left(x_1-x_2 \mid \frac{\tau}{2}\right)\theta_1\left(x_1-x_3 \mid \frac{\tau}{2}\right)}. \tag{3.9}$$

Proof. Let

$$g(z) = \frac{\theta_1(2z-x_1-x_2-x_3 \mid \tau)f(z)}{\theta_1(z-x_1 \mid \frac{\tau}{2})\theta_1(z-x_2 \mid \frac{\tau}{2})\theta_1(z-x_3 \mid \frac{\tau}{2})},$$

where x_1, x_2, and x_3 are different from the zeros of the function $\theta_1(2z-x_1-x_2-x_3 \mid \tau)f(z)$. Suppose that $0 < x_1, x_2, x_3 < \pi$. Then by the properties (3.1) and the assumptions on the function $f(z)$ this can be easily checked

$$g(z+\pi) = g(z) \quad \text{and } g\left(z+\frac{\pi\tau}{2}\right) = g(z),$$

showing that $g(z)$ is an elliptic function with periods π and $\frac{\pi\tau}{2}$. Moreover, it is clear that $g(z)$ has simple poles at x_1, x_2, and x_3 in the fundamental parallelogram $0, \pi, \frac{\pi\tau}{2}, \pi+\frac{\pi\tau}{2}$. computing the residues.

$$\begin{aligned}\operatorname{Res}(g; x_1) &= \lim_{z\to x_1} \frac{z-x_1}{\theta_1\left(z-x_1 \mid \frac{\tau}{2}\right)} \cdot \frac{\theta_1(x_1-x_2-x_3 \mid \tau)f(x_1)}{\theta_1\left(x_1-x_2 \mid \frac{\tau}{2}\right)\theta_1\left(x_1-x_3 \mid \frac{\tau}{2}\right)} \\ &= \frac{\theta_1(x_1-x_2-x_3 \mid \tau)f(x_1)}{\theta'\left(0 \mid \frac{\tau}{2}\right)\theta_1\left(x_1-x_2 \mid \frac{\tau}{2}\right)\theta_1\left(x_1-x_3 \mid \frac{\tau}{2}\right)},\end{aligned} \tag{3.10}$$

and similarly,

$$\begin{aligned}\operatorname{Res}(g;x_2) &= \frac{\theta_1(x_2-x_1-x_3 \mid \tau)f(x_2)}{\theta'\left(0 \mid \frac{\tau}{2}\right)\theta_1\left(x_2-x_1 \mid \frac{\tau}{2}\right)\theta_1\left(x_2-x_3 \mid \frac{\tau}{2}\right)} \\ \operatorname{Res}(g;x_3) &= \frac{\theta_1(x_3-x_1-x_2 \mid \tau)f(x_3)}{\theta'\left(0 \mid \frac{\tau}{2}\right)\theta_1\left(x_3-x_1 \mid \frac{\tau}{2}\right)\theta_1\left(x_3-x_2 \mid \frac{\tau}{2}\right)}.\end{aligned} \tag{3.11}$$

Hence by Theorem 3.3.1 combined with (3.10) and (3.11) deriving (3.9). As this true for all $0 < x_1, x_2, x_3 < \pi$, the identity (3.9) holds for all complex x_1, x_2, and x_3 by the principal of analytic continuation [26].

Theorem 3.3.3. *For all complex number x_1, x_2, and x_3 :*

$$\begin{aligned}&\theta_1(x_3-x_1-x_2 \mid \tau)\theta_1\left(x_1-x_2 \mid \frac{\tau}{2}\right)\theta_1\left(x_3+\frac{\pi}{2} \mid \frac{\tau}{2}\right) \\ &= \theta_1(x_2-x_1-x_3 \mid \tau)\theta_1\left(x_1-x_3 \mid \frac{\tau}{2}\right)\theta_1\left(x_2+\frac{\pi}{2} \mid \frac{\tau}{2}\right) \\ &-\theta_1(x_1-x_2-x_3 \mid \tau)\theta_1\left(x_2-x_3 \mid \frac{\tau}{2}\right)\theta_1\left(x_1+\frac{\pi}{2} \mid \frac{\tau}{2}\right).\end{aligned}$$

Proof. Let $f(z) = \theta_2(z \mid \frac{\tau}{2})$. Then it is easily verified by the properties (3.1) that the function $f(z)$ satisfies the two conditions of Theorem 3.3.2 and so,

$$\begin{aligned}\frac{\theta_1(x_3-x_1-x_1 \mid \tau)\theta_2\left(x_3 \mid \frac{\tau}{2}\right)}{\theta_1\left(x_1-x_3 \mid \frac{\tau}{2}\right)\theta_1\left(x_2-x_3 \mid \frac{\tau}{2}\right)} &= \frac{\theta_1(x_2-x_1-x_3 \mid \tau)\theta_2\left(x_2 \mid \frac{\tau}{2}\right)}{\theta_1\left(x_1-x_2 \mid \frac{\tau}{2}\right)\theta_1\left(x_2-x_3 \mid \frac{\tau}{2}\right)} \\ &- \frac{\theta_1(x_1-x_2-x_3 \mid \tau)\theta_2\left(x_1 \mid \frac{\tau}{2}\right)}{\theta_1\left(x_1-x_2 \mid \frac{\tau}{2}\right)\theta_1\left(x_1-x_3 \mid \frac{\tau}{2}\right)}.\end{aligned}$$

Now rearranging and using the basic fact $\theta_2(z \mid \tau) = \theta_1\left(z+\frac{\pi}{2} \mid \tau\right)$, the previous formula yields

$$\begin{aligned}
&\theta_1(x_3-x_1-x_2 \mid \tau)\theta_1\left(x_1-x_2 \mid \frac{\tau}{2}\right)\theta_1\left(x_3+\frac{\pi}{2} \mid \frac{\tau}{2}\right)\\
&=\theta_1(x_2-x_1-x_3 \mid \tau)\theta_1\left(x_1-x_3 \mid \frac{\tau}{2}\right)\theta_1\left(x_2+\frac{\pi}{2} \mid \frac{\tau}{2}\right)\\
&-\theta_1(x_1-x_2-x_3 \mid \tau)\theta_1\left(x_2-x_3 \mid \frac{\tau}{2}\right)\theta_1\left(x_1+\frac{\pi}{2} \mid \frac{\tau}{2}\right),
\end{aligned}$$

as desired.

Proving the main result of this section.

Theorem 3.3.4. *Identity* (3.8) *is true.*

Proof. Letting in Theorem 3.3.3, $x_1-x_3=x_2-\frac{\pi}{2}$, $x_2-x_3=x_1-\frac{3\pi}{2}$, and so $x_3=\pi$, gives

$$\begin{aligned}
-\theta_1^2\left(\frac{\pi}{2} \mid \frac{\tau}{2}\right)\theta_1\left(\pi-2x_2-\frac{\pi}{2} \mid \tau\right) &= -\theta_1\left(-\frac{3\pi}{2} \mid \tau\right)\theta_1^2\left(x_2+\frac{\pi}{2} \mid \frac{\tau}{2}\right)\\
&-\theta_1\left(-\frac{\pi}{2} \mid \tau\right)\theta_1\left(x_2-\pi \mid \frac{\tau}{2}\right)\theta_1\left(x_2+\frac{\pi}{2} \mid \frac{\tau}{2}\right)
\end{aligned}$$

which by the basic properties (3.1) is equivalent to

$$-\theta_1^2\left(\frac{\pi}{2} \mid \frac{\tau}{2}\right)\theta_1\left(2x_2-\frac{\pi}{2} \mid \tau\right) = \theta_1(\frac{\pi}{2} \mid \tau)\theta_1^2\left(x_2+\frac{\pi}{2} \mid \frac{\tau}{2}\right)$$

$$-\theta_1\left(\frac{\pi}{2} \mid \tau\right)\theta_1^2\left(x_2 \mid \frac{\tau}{2}\right).$$

Then the desired formula follows upon letting $z := x_2-\frac{\pi}{2}$ in the previous identity.

3.4 Sums of two squares

To prove our next result, the following lemma is required.

Lemma 3.4.1. *There holds*

$$\psi_{q^2}(z) = -\log(1-q^2) + 2(\log q)\sum_{n=0}^{\infty}\frac{q^{2z+2n}}{1-q^{2z+2n}}.$$

Proof. having

$$\begin{aligned}\log\Gamma_{q^2}(z) &= \log(q^2;q^2)_\infty + (1-z)\log(1-q^2) - \log(q^{2z};q^2)_\infty \\ &= \log(q^2;q^2)_\infty + (1-z)\log(1-q^2) - \log\prod_{n=0}^{\infty}(1-q^{2z+2n}) \\ &= \log(q^2;q^2) + (1-z)\log(1-q^2) - \sum_{n=0}^{\infty}\log(1-q^{2z+2n}).\end{aligned}$$

Taking derivatives with respect to z in the previous identity gives

$$\begin{aligned}\frac{d}{dz}(\log\Gamma_q^2(z)) &= -\log(1-q^2) - \sum_{n=0}^{\infty}\frac{(1-q^{2z+2n})'}{1-q^{2z+2n}} \\ &= -\log(1-q^2) + \sum_{n=0}^{\infty}\frac{2(\log q)q^{2z+2n}}{1-q^{2z+2n}},\end{aligned}$$

which means that

$$\psi_q^2(z) = -\log(1-q^2) + 2(\log q)\sum_{n=0}^{\infty}\frac{q^{2z+2n}}{1-q^{2z+2n}},$$

as desired.

Proposition 3.4.2. *then*

$$\sin_q' \pi z = \frac{\sin_q \pi z}{\pi}\Big((2z-1)\log q - \big(\psi_{q^2}(z) - \psi_{q^2}(1-z)\big)\Big),$$

where the derivative is with respect to the variable z.

Proof. From the definition, having clearly

$$\log\sin_q \pi z = \frac{1}{4}\log q + 2\log\Gamma_{q^2}(\frac{1}{2}) + (z^2 - z)\log q - \log_{q^2}\Gamma(z) - \log_{q^2}\Gamma(1-z).$$

Then differentiating with respect to z gives

$$\pi\frac{\sin_q'(\pi z)}{\sin_q(\pi z)} = (2z-1)\log q - \psi_{q^2}(z) - \psi_{q^2}(1-z),$$

and so

$$\sin_q' \pi z = \frac{\sin_q \pi z}{\pi}\Big((2z-1)\log q - \big(\psi_{q^2}(z) - \psi_{q^2}(1-z)\big)\Big),$$

which is the desired formula.

Now giving important identities on derivatives at $\frac{\pi}{4}$.

Proposition 3.4.3. *There holds*

$$\sin_{q^2}' \frac{\pi}{4} = -\cos_{q^2}' \frac{\pi}{4} = \frac{-\log(q)}{\pi}\frac{\Pi_q^{\frac{3}{2}}}{\Pi_{q^2}^{\frac{1}{2}}}.$$

Proof. Note that from the basic fact $\sin_q'(\frac{\pi}{2} - z) = -\cos_q'(z)$, finding by letting $z = \frac{\pi}{4}$ $\sin_q' \frac{\pi}{4} = -\cos_q' \frac{\pi}{4}$, showing the first identity. As for the second identity, by (3.7) deducing by differentiating with respect to z

$$2\cos_q'(2z) = 2\cos_{q^2}(z)\cos_{q^2}'(z) - 2\sin_{q^2}(z)\sin_{q^2}'(z).$$

Then by letting $z = \frac{\pi}{4}$:

$$\begin{aligned}2\cos'_q(\frac{\pi}{2}) &= 2\cos_{q^2}\frac{\pi}{4}\cos'_{q^2}(\frac{\pi}{4}) - 2\sin_{q^2}\frac{\pi}{4}\sin'_{q^2}(\frac{\pi}{4}) \\ &= -4\sin_{q^2}(\frac{\pi}{4})\sin'_{q^2}(\frac{\pi}{4}).\end{aligned} \tag{3.12}$$

Solving in the foregoing formula for $\sin'_{q^2}(\frac{\pi}{4})$ and using Proposition 3.2.2 to obtain

$$\sin'_{q^2}(\frac{\pi}{4}) = -\frac{\cos'_q \frac{\pi}{2}}{2\sin_{q^2}\frac{\pi}{4}} = -\frac{(\log q)}{\pi}.\frac{\Pi_q \Pi_q^{\frac{1}{2}}}{\Pi_{q^2}^{\frac{1}{2}}} = \frac{-\log q}{\pi}\frac{\Pi_q^{\frac{3}{2}}}{\Pi_{q^2}^{\frac{1}{2}}}.$$

Theorem 3.4.4. *There holds*

$$1 + 4\sum_{n=0}^{\infty}\Big(\frac{q^{4n+1}}{1-q^{4n+1}} - \frac{q^{4n+3}}{1-q^{4n+3}}\Big) = \varphi^2(q),$$

where $\varphi(q) = (q^2;q^2)_\infty(-q;q^2)^2_\infty$.

Proof. Evaluating $\sin'_q\left(\frac{\pi}{4}\right)$ in two different ways. On the one hand, put $z = \frac{\pi}{4}$ in Proposition 3.4.2 to obtain

$$\sin'_q\frac{\pi}{4} = \frac{\sin_q\frac{\pi}{4}}{\pi}\Big(-\frac{1}{2}\log q - \big(\psi_{q^2}(\frac{1}{4}) - \psi_{q^2}(\frac{3}{4})\big)\Big). \tag{3.13}$$

Moreover by Lemma 3.4.1 then

$$\psi_{q^2}(\frac{1}{4}) - \psi_{q^2}(\frac{3}{4}) = (2\log q)\sum_{n=0}^{\infty}\Big(\frac{q^{2n+\frac{1}{2}}}{1-q^{2n+\frac{1}{2}}} - \frac{q^{2n+\frac{3}{2}}}{1-q^{2n+\frac{3}{2}}}\Big),$$

and since by Proposition 3.2.2 $\sin_q\frac{\pi}{4} = \frac{\Pi_q^{\frac{1}{2}}}{\Pi^{\frac{1}{2}}_{q^{\frac{1}{2}}}}$

deducing that (3.13) becomes

$$\sin'_q \frac{\pi}{4} = \frac{\Pi_q^{\frac{1}{2}}}{\Pi^{\frac{1}{2}}_{q^{\frac{1}{2}}}} \frac{1}{\pi} \left(-\frac{1}{2}\log q - 2(\log q) \sum_{n=0}^{\infty} \Big(\frac{q^{2n+\frac{1}{2}}}{1-q^{2n+\frac{1}{2}}} - \frac{q^{2n+\frac{3}{2}}}{1-q^{2n+\frac{3}{2}}} \Big) \right). \tag{3.14}$$

On the other hand, by virtue of Proposition 3.4.3:

$$\sin'_q \frac{\pi}{4} = -\frac{1}{2} \frac{\log q}{\pi} \frac{\Pi^{\frac{3}{2}}_{q^{\frac{1}{2}}}}{\Pi_q^{\frac{1}{2}}}. \tag{3.15}$$

Combining (3.14) and (3.15) yields

$$\frac{\Pi_q^{\frac{1}{2}}}{\Pi^{\frac{1}{2}}_{q^{\frac{1}{2}}}} \Big(\frac{1}{2} + 2 \sum_{n=0}^{\infty} \big(\frac{q^{2n+\frac{1}{2}}}{1-q^{2n+\frac{1}{2}}} - \frac{q^{2n+\frac{3}{2}}}{1-q^{2n+\frac{3}{2}}} \big) \Big) = \frac{1}{2} \frac{\Pi^{\frac{3}{2}}_{q^{\frac{1}{2}}}}{\Pi_q^{\frac{1}{2}}},$$

or equivalently,

$$1 + 4\sum_{n=0}^{\infty} \Big(\frac{q^{2n+\frac{1}{2}}}{1-q^{2n+\frac{1}{2}}} - \frac{q^{2n+\frac{3}{2}}}{1-q^{2n+\frac{3}{2}}} \Big) = \frac{\Pi^2_{q^{\frac{1}{2}}}}{\Pi_q}.$$

Replacing in the previous identity q by q^2,to find

$$1 + 4\sum_{n=0}^{\infty} \big(\frac{q^{4n+1}}{1-q^{4n+1}} - \frac{q^{4n+3}}{1-q^{4n+3}} \big) = \frac{\Pi_q^2}{\Pi_{q^2}}.$$

Finally note that by definition (3.6) the right hand-side of the foregoing formula is

$$\frac{\left(q^{\frac{1}{4}}\right)^2}{(q^2)^{\frac{1}{4}}} \frac{(q^2;q^2)^4_\infty}{(q;q^2)^4_\infty} \frac{(q^2;q^4)^2_\infty}{(q^4;q^4)^2_\infty} = \frac{(q^2;q^2)^2_\infty (q^2;q^4)^4_\infty}{(q;q^2)^4_\infty}$$

$$= (q^2;q^2)_\infty^2(-q;q^2)_\infty^4 = \varphi^2(q).$$

This completes the proof.

Now proof of Fermat's theorem on sums of two squares will be stated.

Theorem 3.4.5. *For any positive integer n :*

$$r_2(n) = 4\Big(\sum_{\substack{d|n \\ d\equiv 1 \bmod 4}} 1 - \sum_{\substack{d|n \\ d\equiv 3 \bmod 4}} 1\Big).$$

Proof. By virtue of Theorem 3.4.4,

$$\varphi^2(q) = 1 + 4\sum_{n=0}^{\infty}\Big(\frac{q^{4n+1}}{1-q^{4n+1}} - \frac{q^{4n+3}}{1-q^{4n+3}}\Big)$$

which means that

$$\begin{aligned}
1 + \sum_{n=1}^{\infty} r_2(n)q^n &= 1 + 4\sum_{n=0}^{\infty}\Big(\frac{q^{4n+1}}{1-q^{4n+1}} - \frac{q^{4n+3}}{1-q^{4n+3}}\Big) \\
&= 1 + 4\Big(\sum_{n=0}^{\infty} q^{4n+1}\sum_{m=0}^{\infty} q^{m(4n+1)} - \sum_{n=0}^{\infty} q^{4n+3}\sum_{m=0}^{\infty} q^{m(4n+3)}\Big) \\
&= 1 + 4\Big(\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} q^{m(4n+1)} - \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} q^{m(4n+3)}\Big) \\
&= 1 + 4\left(\sum_{n=1}^{\infty}\Big(\sum_{\substack{d|n \\ d\equiv 1 \bmod 4}} 1\Big) - \sum_{n=1}^{\infty}\Big(\sum_{\substack{d|n \\ d\equiv 3 \bmod 4}} 1\Big)\right).
\end{aligned}$$

This completes the proof.

Chapter 4: Extended Bailey transforms and applications

In this chapter an extended Bailey transform will be introduced and will be used to derive new formulas of q-series. Among others, applying this transform to some recent identities of Area et al. [9, p. 157] and Tcheutia [30]. As an intermediate result,a formula for x^n will be extended due to Area et al. [9, p. 157]. Noting that the results in this chapter can be found in AboTouk-AlHouchan-ElBachraoui [2].

4.1 An extended Bailey transform and background

Lemma 4.1.1. *Let a, s and z be positive integers and let $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$, $\{C_n\}_{n=0}^{\infty}$, $\{D_n\}_{n=0}^{\infty}$, $\{U_n\}_{n=0}^{\infty}$, $\{V_n\}_{n=0}^{\infty}$, $\{X_n\}_{n=0}^{\infty}$, and $\{Y_n\}_{n=0}^{\infty}$ be sequences of complex numbers. Assuming convergence of the series, if*

$$B_n = \sum_{i=0}^{sn} A_i U_{asn-ia} V_{asn+ia} X_{zn-ia} Y_{zn+ia}$$

and

$$C_n = \sum_{i=\lceil \frac{n}{s} \rceil}^{\infty} D_i U_{asi-na} V_{asi+na} X_{zi-na} Y_{zi+na}$$

where $\lceil x \rceil$ denotes the least integer which is greater than or equal x, then

$$\sum_{n=0}^{\infty} A_n C_n = \sum_{n=0}^{\infty} B_n D_n.$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty} A_n C_n &= \sum_{n=0}^{\infty} A_n \sum_{i=\lceil \frac{n}{s} \rceil}^{\infty} D_i U_{asi-na} V_{asi+na} X_{zi-na} Y_{zi+na} \\ &= \sum_{i=0}^{\infty} D_i \sum_{n=0}^{si} U_{asi-na} V_{asi+na} X_{zi-na} Y_{zi+na} \\ &= \sum_{i=0}^{\infty} D_i B_i,\end{aligned}$$

which proves the result.

Note that the case $z = s = a = 1$ with (X_n) and (Y_n) being the constant sequences with value 1 in Lemma 4.1.1 gives the classical Bailey transform [13]. Throughout, this shall be used often without reference, elementary properties such as

$$\frac{(x;q)_{n+k}}{(x;q)_n} = (xq^n;q)_k, \ \frac{(x;q)_{n-k}}{(x;q)_n} = \frac{(-x^{-1}q)^k q^{\binom{k}{2}-nk}}{(x^{-1}q^{1-n};q)_k},$$

$$\frac{(x^{-1}q^{1-n};q)_n}{(x;q)_n} = (-1)^n x^{-n} q^{\binom{n}{2}}, \ (x;q)_{2n} = (x;q^2)_n (xq;q^2)_n, \tag{4.1}$$

$$(x;q)_\infty = (x;q)_n (xq^n;q)_\infty, \text{ and } (-q;q)_\infty = \frac{1}{(q;q^2)_\infty}.$$

Among other formulas needed in this paper, the q-binomial formula and its consequence were respectively recorded [18, (1.3.2) and (1.6.2)]

$$\sum_{n\geq 0} \frac{(a;q)_n}{(q;q)_n} x^n = \frac{(ax;q)_\infty}{(x;q)_\infty} \text{ and } \sum_{n\geq 0} \frac{(-1)^n q^{n^2} x^n}{(q^2;q^2)_n} = (xq;q^2)_\infty, \tag{4.2}$$

Euler's formula (cf. [4, Corollary 2.2])

$$(-x,q)_\infty = \sum_{n\geq 0} \frac{q^{\binom{n}{2}} x^n}{(q;q)_n} \tag{4.3}$$

Cauchy's identities (cf. [4, Theorem 3.3])

$$(x;q)_n = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} x^k \text{ and } \frac{1}{(x;q)_{n+1}} = \sum_{k\geq 0} \begin{bmatrix} n+k \\ k \end{bmatrix} x^k, \tag{4.4}$$

where $\begin{bmatrix} N \\ M \end{bmatrix}_q$ is the q-binomial coefficient which is given for any nonnegative integers M and N by

$$\begin{bmatrix} N \\ M \end{bmatrix}_q = \begin{bmatrix} N \\ M \end{bmatrix} = \begin{cases} \frac{(q;q)_N}{(q;q)_M (q;q)_{N-M}} & \text{if } N \geq M, \\ 0 & \text{otherwise.} \end{cases}$$

Also the following formula will be needed [18, Exercise 1.16]

$$\sum_{n\geq 0} \frac{(x;q)_n q^{\binom{n+1}{2}}}{(q;q)_n} = (-q;q)_\infty (xq;q^2)_\infty. \tag{4.5}$$

4.2 An extension formula for x^n and an application

By Area et al. [9, p. 157] (cf. [30, Lemma 3.18]) :

$$x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^k q^{\binom{k+1}{2}} (x;q)_k}{q^{kn}}. \tag{4.6}$$

The following more general form of this identity were proved.

Theorem 4.2.1. *There holds*

$$x^n \frac{(y, yq/x;q)_n}{(xy, yq;q)_n} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^k q^{\binom{k+1}{2} - kn}}{(xy, yq;q)_k} (x, y^2 q^n; q)_k.$$

Proof. then

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(-1)^k q^{\binom{k+1}{2}-kn}}{(xy,yq;q)_k}(x,y^2q^n;q)_k = \sum_{k=0}^{n}\frac{(q^{-n},x,y^2q^n;q)_k q^k}{(xy,yq,q;q)_k}$$
$$= {}_3\phi_2\left[\begin{matrix}x,\ y^2q^n,\ q^{-n}\\ xy,\ yq\end{matrix};q,\ q\right]$$
$$= \frac{(y,xy^{-1}q^{-n};q)_n}{(xy,y^{-1}q^{-n};q)_n}$$
$$= x^n\frac{(y,yq/x;q)_n}{(xy,yq;q)_n},$$

where the third formula follows by the q-Pfaff-Saalschütz sum [18, II.12]

$$ {}_3\phi_2\left[\begin{matrix}a,\ b,\ q^{-n}\\ c,\ abc^{-1}q^{1-n}\end{matrix};q,\ q\right] = \frac{(c/a,c/b;q)_n}{(c,c/(ab);q)_n}$$

and the remaining formulas follow by basic manipulations (4.1). This completes the proof.

Observe that the case $y=0$ in the Theorem 4.2.1 yields (4.6).

Theorem 4.2.2. *:*

$$\sum_{n\geq 0}\frac{(a;q)_n x^n q^{\binom{n+1}{2}}}{(q;q)_n} = (-q;q)_\infty \sum_{n\geq 0}\frac{(-1)^n q^n (x,a;q)_n (aq^{n+1};q^2)_\infty}{(q;q)_n}. \tag{4.7}$$

Proof. It is clear that (4.6) means

$$\sum_{k=0}^{n}\frac{(-1)^k q^{\frac{k}{2}+\frac{(n-k)^2}{2}}(x)_k}{(q;q)_k(q;q)_{n-k}} = \frac{x^n q^{\frac{n^2}{2}}}{(q;q)_n}.$$

Applying Lemma 4.1.1 with $z=s=a=1$ to

$$A_n = \frac{(-1)^n q^{\frac{n}{2}}(x;q)_n}{(q;q)_n},\ B_n = \frac{x^n q^{\frac{n^2}{2}}}{(q;q)_n},\ U_n = \frac{q^{\frac{n^2}{2}}}{(q;q)_n},\ D_n = q^{\frac{n}{2}}(a;q)_n$$

and V, X and Y being the constant sequences with value 1. Then

$$
\begin{aligned}
C_n &= \sum_{k=n}^{\infty} \frac{q^{\frac{k}{2}}(a;q)_k q^{\frac{(k-n)^2}{2}}}{(q;q)_{k-n}} \\
&= q^{\frac{n}{2}}(a)_n \sum_{k=0}^{\infty} \frac{(aq^n;q)_k q^{\binom{k+1}{2}}}{(q;q)_k} \\
&= q^{\frac{n}{2}}(a;q)_n(-q;q)_\infty(aq^{n+1};q^2)_\infty,
\end{aligned}
$$

where the last formula follows from (4.5). Then

$$
\sum_{n\geq 0} A_n C_n = (-q;q)_\infty \sum_{n\geq 0} \frac{(-1)^n q^n (x,a;q)_n (aq^{n+1};q^2)_\infty}{(q;q)_n}
$$

$$
= \sum_{n\geq 0} B_n D_n = \sum_{n\geq 0} \frac{(a;q)_n x^n q^{\binom{n+1}{2}}}{(q;q)_n}
$$

as desired.

Corollary 4.2.3. *:*

(a) $\displaystyle \sum_{n\geq 0} \frac{(a;q)_{n+1} q^{\binom{n+1}{2}}}{(q;q)_n} = \sum_{n\geq 0} \frac{(aq^{-1};q)_n q^{\binom{n+1}{2}}}{(q;q)_n} = (-q;q)_\infty (a;q^2)_\infty.$

(b) $\displaystyle \sum_{n\geq 0} \frac{(-1)^n q^n (aq^k;q)_n (aq^{n+k+1};q^2)_\infty}{(q;q)_n} = (q;q^2)_\infty.$

(c) $\displaystyle \sum_{n\geq 0} (-1)^n q^n \begin{bmatrix} n+k \\ k \end{bmatrix} (q^{n+k+2};q^2)_\infty = (q;q^2)_\infty.$

Proof. The second identity of part (a) clearly follows from (4.5). As for the first identity, letting in (4.7) $x = q^{-1}$ and using (4.5) arriving at

$$
\begin{aligned}
\sum_{n\geq 0} \frac{(a;q)_n q^{\binom{n}{2}}}{(q;q)_n} &= (-q;q)_\infty (a;q^2)_\infty + (-q;q)_\infty (aq;q^2)_\infty \\
&= \sum_{n\geq 0} \frac{(aq^{-1};q)_n q^{\binom{n+1}{2}}}{(q;q)_n} + \sum_{n\geq 0} \frac{(a;q)_n q^{\binom{n+1}{2}}}{(q;q)_n},
\end{aligned}
$$

which by rearranging implies that

$$\sum_{n\geq 0}\frac{(a;q)_n q^{\binom{n}{2}}(1-q^n)}{(q;q)_n}=\sum_{n\geq 0}\frac{(aq^{-1};q)_n q^{\binom{n+1}{2}}}{(q;q)_n},$$

or equivalently,

$$\sum_{n\geq 1}\frac{(a;q)_n q^{\binom{n}{2}}}{(q;q)_{n-1}}=\sum_{n\geq 0}\frac{(aq^{-1};q)_n q^{\binom{n+1}{2}}}{(q;q)_n},$$

Now shift the left hand-side of the foregoing formula to deduce the desired identity. Part (c) is the case $a=q$ of part (b). To prove part (b), firstly use (4.4) to express the right hand-side of (4.7) as powers series in x. Then

$$\begin{aligned}
\text{R.H.S of (4.7)} &= (-q;q)_\infty \sum_{n\geq 0}\frac{(-q)^n(a;q)_n(aq^{n+1};q^2)_\infty}{(q;q)_n}\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix} n\\ k\end{bmatrix}x^k\\
&= (-q;q)_\infty \sum_{k\geq 0}\frac{(-1)^k q^{\binom{k}{2}}x^k}{(q;q)_k}\sum_{n=k}^{\infty}\frac{(-1)^n q^n(a;q)_n(aq^{n+1};q^2)_\infty}{(q;q)_{n-k}}\\
&= (-q;q)_\infty \sum_{k\geq 0}\frac{q^{\binom{k+1}{2}}x^k}{(q;q)_k}\sum_{n=0}^{\infty}\frac{(-1)^n q^n(a;q)_{k+n}(aq^{k+n+1};q^2)_\infty}{(q;q)_n}\\
&= (-q;q)_\infty \sum_{k\geq 0}\left(\sum_{n\geq 0}\frac{(-1)^n q^{n+\binom{k+1}{2}}(a;q)_{k+n}(aq^{k+n+1};q^2)_\infty}{(q;q)_k(q;q)_n}\right)x^k.
\end{aligned}$$

Now equate the terms corresponding to x^k in (4.7) to obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n+\binom{k+1}{2}}(a;q)_{k+n}(aq^{k+n+1};q^2)_\infty}{(q;q)_k(q;q)_n}=\frac{(a;q)_k q^{\binom{k+1}{2}}}{(-q;q)_\infty(q;q)_k}$$

which gives the desired identity after straightforward simplifications.

4.3 Finite sums involving q-binomial coefficients

In this section Lemma 4.1.1 will be employed to establish a series transformation from which two finite sums involving the q-binomial coefficients will be evaluated.

Theorem 4.3.1. *There holds*

$$\sum_{n\geq 1}\frac{a^nq^{n^2+n}}{(q;q)_n(q;q)_{n-1}}=\frac{(aq^2;q)_\infty}{2}\sum_{n\geq 0}\frac{a^nq^n(1-aq-q^{2n})}{(q;q)_n^2}. \tag{4.8}$$

Proof. Note first that from $(x;q)_n=(1-x)(1-xq)\cdots(1-xq^{n-1})$ lead

$$\frac{d}{dx}(x;q)_n\Big|_{x=0}=-\sum_{j=0}^{n-1}q^j=-\frac{1-q^n}{1-q}\text{ and }\frac{d}{dx}(x;q)_n^2\Big|_{x=0}=\frac{-2(1-q^n)}{1-q}.$$

With the help of the following identity of Tcheutia which appears in his PhD thesis [30]:

$$(x;q)_m(x;q)_n=\sum_{k=\max\{m,n\}}^{m+n}\frac{(q;q)_m(q;q)_n(-1)^{m+n-k}q^{\binom{m+n-k}{2}-mn}}{(q;q)_{k-m}(q;q)_{k-n}(q;q)_{m+n-k}}(x;q)_k \tag{4.9}$$

Then taking derivatives with respect to x on both sides of (4.9) (with $m=n$) and using the above two identities :

$$\frac{-2(1-q^n)}{1-q}=\sum_{k=n}^{2n}\frac{(q;q)_n^2(-1)^{2n-k}q^{\binom{2n-k}{2}-n^2}}{(q;q)_{k-n}^2(q;q)_{2n-k}}\Big(-\frac{1-q^k}{1-q}\Big)$$

which yields after shifting the summation and simplifying

$$\sum_{k=0}^{n}\frac{(-1)^{n-k}q^{\binom{n-k}{2}}(1-q^{n+k})}{(q;q)_k^2(q;q)_{n-k}}=\frac{2(1-q^n)q^{n^2}}{(q;q)_n^2}.$$

Let in Lemma 4.1.1 with $z = s = a = 1$,

$$A_n = \frac{1}{(q;q)_n^2},\ B_n = \frac{2(1-q^n)q^{n^2}}{(q;q)_n^2}, U_n = \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n},\ V_n = 1-q^n,\ D_n = a^n q^n$$

and let X and Y be the constant sequences with value 1. Then

$$\begin{aligned}
C_n &= \sum_{k=n}^{\infty} \frac{a^k q^k (-1)^{k-n} q^{\binom{k-n}{2}} (1-q^{k+n})}{(q;q)_{k-n}} \\
&= a^n q^n \sum_{k=0}^{\infty} \frac{(-1)^k a^k q^k q^{\binom{k}{2}} (1-q^{k+2n})}{(q;q)_k} \\
&= a^n q^n \sum_{k=0}^{\infty} \frac{q^{\frac{k^2}{2}} (-aq^{\frac{1}{2}})^k}{(q;q)_k} - a^n q^{3n} \sum_{k=0}^{\infty} \frac{q^{\frac{k^2}{2}} (-aq^{\frac{3}{2}})^k}{(q;q)_k} \\
&= a^n q^n (aq;q)_\infty - a^n q^{3n} (aq^2;q)_\infty \\
&= a^n q^n (aq^2;q)_\infty (1 - aq - q^{2n})
\end{aligned}$$

where in the penultimate formula the second formula in (4.2) was used. Then by virtue of Lemma 4.1.1

$$\sum_{n\geq 0} A_n C_n = (aq^2;q)_\infty \sum_{n\geq 0} \frac{a^n q^n (1-aq-q^{2n})}{(q;q)_n^2}$$

$$= \sum_{n\geq 0} B_n D_n = 2\sum_{n\geq 0} \frac{a^n q^{n^2+n}(1-q^n)}{(q;q)_n^2} = 2\sum_{n\geq 1} \frac{a^n q^{n^2+n}}{(q;q)_n (q;q)_{n-1}}$$

which completes the proof.

Corollary 4.3.2. *then*

(a) $\displaystyle\sum_{k=0}^{n} \frac{q^{k(k+1)}}{(q;q)_{k+1}} \begin{bmatrix} n \\ k \end{bmatrix} = \frac{1}{(q)_{n+1}}.$

(b) $\displaystyle\sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_{n-k+1}} \begin{bmatrix} n \\ k \end{bmatrix} = \frac{q^{n^2}}{(q)_{n+1}}.$

Proof. Part (a), first rewrite (4.8) as follows

$$\frac{2}{(aq^2;q)_\infty}\sum_{n\geq 0}\frac{q^{(n+1)(n+2)}a^{n+1}}{(q;q)_n(q;q)_{n+1}}=\sum_{n\geq 0}\frac{q^n(1-q^{2n})a^n}{(q;q)_n^2}-\sum_{n\geq 0}\frac{q^{n+1}a^{n+1}}{(q;q)_n^2}$$

or equivalently by (4.2)

$$2\sum_{n\geq 0}\frac{q^{2n}a^n}{(q;q)_n}\sum_{n\geq 0}\frac{q^{(n+1)(n+2)}a^{n+1}}{(q;q)_n(q;q)_{n+1}}=\sum_{n\geq 0}\frac{q^n(1-q^{2n})a^n}{(q;q)_n^2}-\sum_{n\geq 0}\frac{q^{n+1}a^{n+1}}{(q;q)_n^2}.$$

Equating terms of the corresponding powers of a achieves

$$2\sum_{k=0}^{n-1}\frac{q^{2k}}{(q)_k}\frac{q^{(n-k)(n-k+1)}}{(q;q)_{n-k-1}(q;q)_{n-k}}=\frac{q^n(1-q^{2n})}{(q;q)_n^2}-\frac{q^n}{(q;q)_{n-1}^2}$$

which by taking the sum up to n and after some basic simplifications boils down to

$$\sum_{k=0}^{n}\frac{q^{k^2-2nk-k}}{(q;q)_{n+1-k}}\begin{bmatrix}n\\k\end{bmatrix}=\frac{q^{-n(n+1)}}{(q;q)_{n+1}}$$

or equivalently,

$$\sum_{k=0}^{n}\frac{q^{(n-k)^2-k}}{(q;q)_{n+1-k}}\begin{bmatrix}n\\k\end{bmatrix}=\frac{q^{-n}}{(q;q)_{n+1}}.$$

Now the desired formula follows readily by the substitution $k\to n-k$ in the previous summation.

Now establish part (b) as follows. From (4.8) and (4.3) :

$$2\sum_{n\geq 0}\frac{q^{(n+1)(n+2)}a^{n+1}}{(q;q)_n(q;q)_{n+1}} = (aq^2;q)_\infty \sum_{n\geq 0}\frac{q^n(1-q^{2n})a^n - q^{n+1}a^{n+1}}{(q;q)_n^2}$$
$$= \sum_{n\geq 0}\frac{(-1)^n q^{2n+\binom{n}{2}}a^n}{(q;q)_n}\Big(\sum_{n\geq 0}\frac{q^n(1-q^{2n})a^n}{(q;q)_n^2} - \sum_{n\geq 0}\frac{q^{n+1}a^{n+1}}{(q;q)_n^2}\Big)$$

which means that

$$\sum_{k=0}^{n}\frac{(-1)^k q^{2k+\binom{k}{2}}}{(q;q)_k}\frac{q^{n-k}(1-q^{2n-2k})}{(q;q)_{n-k}^2} - \sum_{k=0}^{n-1}\frac{(-1)^k q^{2k+\binom{k}{2}}}{(q;q)_k}\frac{q^{n-k}}{(q;q)_{n-k-1}^2} = 2\frac{q^{n(n+1)}}{(q;q)_n(q;q)_{n-1}}.$$

After simplification this becomes

$$\sum_{k=0}^{n-1}\frac{(-1)^k q^{\binom{k}{2}+2n}}{(q;q)_k(q;q)_{n-k}(q;q)_{n-k-1}} = \frac{q^{n(n+1)}}{(q;q)_n(q;q)_{n-1}}.$$

Now take in the foregoing formula the sum up to n and multiply both sides by $(q;q)_n$ to complete the proof.

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