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Series Solutions of Multi-Layer Boundary Value Problems

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SERIES SOLUTIONS OF MULTI-LAYER BOUNDARY VALUE PROBLEMS

Amr Saad Hassan Bolbol

This thesis is submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

Under the Supervision of Dr. Mohamed Ali Hajji

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Declaration of Original Work

I, Amr Saad Hassan Bolbol, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled "*Series Solutions of Multi-layer Boundary Value Problems*", hereby, solemnly declare that this thesis is my own original research work that has been done and prepared by me under the supervision of Dr. Mohamed Ali Hajji in the Department of Mathematical Sciences, College of Science, UAEU. This work has not previously been presented or published, or formed the basis for the award of any academic degree, diploma or a similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my thesis have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this thesis.

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Approval of the Master Thesis

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Abstract

It is well-known that differential equations (DEs) play an important role in many sciences. They are mathematical representations of many physical systems. By studying such DEs, one gains a lot of important insights about the physical system. Solutions of DEs provide information on the physical system behavior. As many physical systems are nonlinear in nature, this naturally gives rise to nonlinear differential equations (NLDEs). Such NLDEs are, in most cases, hard or sometimes impossible to solve analytically. In such situations, we resort to numerical techniques to approximate the solutions. The purpose of this thesis is to consider nonlinear multi-layer boundary value problems and seek approximate solutions. Many methods exist in the literature to numerically solve nonlinear boundary value problems. However, only few papers dealt with nonlinear multi- layer boundary value problems. In this work, we employ the homotopy analysis method (HAM) as the method of choice. We consider a real physical system dealing with the fluid flow in multi-channel porous media whose governing equations is exactly a nonlinear multi-layer boundary value problem.

Keywords: Boundary value problems, Multi-layer boundary value problems, Finite difference method, Shooting method, Homotopy analysis method, Porous media.

Title and Abstract (in Arabic)

حل المعادلات التفاضلية الحدية المعرفة على فترات مختلفة باستخدام المتسلسلات

الملخص

من المعروف أن المعادلات التفاضلية تلعب دورا هاما في كثيرا من العلوم. وهي عبارة عن تطبيقات حسابية للعديد من النظم الفيزيائية. ومن خلال دراسة هذا المعادلات التفاضلية، نكتسب الكثير من الرؤى والتفسيرات المهمة ذات الطبيعة الفيزيائية. وتوفر حلول المعادلات التفاضلية معلومات عن سلوك النظام الفيزيائي محل الدراسة. حيث ان العديد من الأنظمة الفيزيائية تمثل مسائل تحتوي على معادلات تفاضلية لا خطية في معظم الحالات، لذلك من الصعب أو في بعض الأحيان من المستحيل إيجاد حل تحليلي في مثل هذه الحالات نلجأ إلى أساليب الطرق العددية لتقريب الحلول. والغرض من هذه الرسالة هو أن تنظر في مسائل القيمة الحدية اللا خطية متعددة الطبقات والبحث عن حلول تقريبية. وتوجد العديد من الأساليب والطرق في الرياضيات لحلول عددية لقيم المشاكل الحدية غير الخطية. ومع ذلك، تناولت بعض الأبحاث فقط مع مشاكل القيم الحدية اللا خطية لطبقات متعددة. في هذا العمل نحن نوظف أسلوب التحليل مثلية الهوموتوبي كأسلوب يتماشى مع نظام مادي حقيقي في التعامل مع تدفق السائل في مائع متعدد الطبقات تحت قيود معادلات منضبطة وهو بالضبط المشاكل الحدية عند تقابل الطبقات في المائع لسريان لا خطي .

مفاهيم البحث الرئيسية : المعادلات التفاضلية الحدية، المعادلات التفاضلية الحدية ذو الطبقات المتعددة، طريقةالفروق المحدودة، طريقة الرمي، طريقة الهوموتوبي، الموائع .

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I would like to thank my beloved family, beginning with my parents, whom without their love and encouragement, I would have never been where I am today. My thanks, with love, also go to my collaborator, my dear wife, for her continued and unfailing support throughout this long journey.

In addition, special thanks are extended to the Al Attas family for their assistance and supporting.

Dedication

To my beloved parents and family

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Chapter 1: Introduction

Differential equations (DEs) play an important part in many scientific areas of science and engineering [1, 3]. They are the mathematical representations of many physical systems [7, 8]. Studying differential equations and in particular seeking solutions to them is of great importance. On one hand, one do not need to run real experiments on the system, which may be costly and time consuming, in order to know the behavior of the system. Instead, by simulating the differential equations, one achieves great insights into the behavior of the physical system. Because many physical systems are naturally nonlinear, the differential equation governing the system would be nonlinear as well.

The theory of differential equations is well established in the literature [1, 15, 16, 17]. For linear differential equations, the theory is more established than that of nonlinear ones. Nonlinear differential equations are usually harder to deal with, in particular nonlinear boundary value problems. Boundary value problems have been extensively studied theoretically and numerically by many researchers. The theory of the existence and uniqueness of solutions of various types of differential equations have been studied and documented well in the literature [19, 15, 16, 17]. For instance, [19] contains a comprehensive survey.

A lot of research has been conducted to design numerical schemes to approximate the solution of boundary value problems [4]-[5]. Many of these methods use a finite difference approach by discretization and approximating the derivatives by finite difference formulas. Other schemes approximate the solution as a linear combination of some basis functions and then solve for the expansion coefficients. Other methods solve boundary value problems via solving other initial value problems such as the nonlinear shooting method [1]. In fact, this approach of considering initial value problems have been in the proof of prove existence and uniqueness of solutions to boundary value problems [13, 15].

In the present work, we are concerned with what is called multi-layer boundary value problem. In particular, we interested in following second-order problem

$$y'' = f_i(x,y,y'), \qquad x \in [x_{i-1}, x_i], \;\; i = 1,2,\ldots,N, \tag{1.1}$$

subject to the the following boundary conditions

$$y(x_0) = \alpha, \; y(x_N) = \beta \tag{1.2}$$

$$y(x_i^-) = y(x_i^+), \; i = 1,2,\ldots,N, \tag{1.3}$$

$$y'(x_i^-) = y'(x_i^+), \; i = 1,2,\ldots,N., \tag{1.4}$$

where is $[x_{i-1}, x_i]$ is the ith subdomain and the overall domain is $[a,b] = [x_0, x_N]$. The functions f_i, $1 \leq i \leq N$, are the functions that define the differential equation in the ith subdomain. The boundary conditions in (1.3) and (1.4) require the solution $y(x)$ to be continuous and smooth across the nodes x_i. Note that the nodes x_i need not be uniformly distributed in the interval $[a,b]$ as displayed in Figure 1.1.

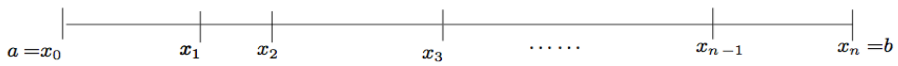

Figure 1.1: Domain decomposition of $[a,b]$

It is worth mentioning that although the literature is rich with papers dealing with boundary value problems defined over a single interval, only few papers dealt with problems like (1.1)-(1.4). This prompted us to consider such a problem. Moreover, we found that the boundary value problem (1.1)-(1.4) has many applications. One application, which we consider in this work, is in fluid flow through different layers of porous channels in which the boundary conditions (1.2) and (1.3) - (1.4) have real physical meaning.

The thesis is organized as follows. In Chapter 2, we review *regular* boundary value problems over a single interval and state results found in the literature concerning the existence and uniqueness of solutions. We also, review some most used numerical methods, and present some numerical examples. Chapter 3 will be concerned with the relatively new method, the homotopy analysis method [29], where we give, in details, its derivation. In Chapter 4, we consider multi-layer boundary problems and present three possible numerical methods that could be used for such boundary value problems, namely the finite difference method, the shooting method, and the homotopy analysis method. In Chapter 5, we apply the homotopy analysis method to solve multi-layer boundary problems arising in the modelling of fluid flow through multi-layer porous channels. Finally, concluding remarks and possible future work is mentioned in Chapter 6.

Chapter 2: Boundary Value Problems

In this chapter we shall review boundary value problems over a single interval. In particular, we consider second-order nonlinear boundary value problems. We state theorems pertaining to the existence and uniqueness of solutions and present some numerical methods used widely.

A second-order boundary problem has the form

$$y''(x) = f(x,y,y'), \qquad x \in \Omega, \tag{2.1}$$

subject to the boundary condition

$$\mathscr{B}(y(x)) = 0, \qquad x \in \partial\Omega, \tag{2.2}$$

where Ω is the domain of interest and $\partial\Omega$ is the boundary of Ω.

2.1 Existence and Uniqueness Theorems

It is well known that many differential equations under boundary conditions may admit no solutions, a unique solution, or many solutions. The problem of determining whether a given boundary problem admits a solution (a least one) or admits a unique solution is not an easy task. For example, the following boundary value problem [18]

$$u''(x) + \sin(u(x)) = 0, \qquad a < x < b, \tag{2.3}$$

$$u(a) = \alpha,\ u(b) = \beta, \tag{2.4}$$

admits more than one solution for $|b-a| > \pi$. This example shows that existence does not imply uniqueness. Finding sufficient and/or necessary conditions for the existence (and uniqueness) of solutions to nonlinear boundary value problems have caught the attention of many researchers in this field. In the 1960s, a series of papers [9]-[18], and references therein, have studied the problem. Many of them have given only sufficient conditions for the existence of unique solution, and others have given necessary and sufficient conditions.

In this section we limit our discussion about existence and uniqueness by stating the following theorem which can be found in many books and articles in the literature [1, 13]. The theorem gives a sufficient condition for a boundary value problem of the form

$$y''(x) = f(x,y,y'), \qquad a < x < b, \tag{2.5}$$

$$\begin{aligned} a_0 y(a) - a_1 y'(a) &= \alpha, \\ b_0 y(b) + b_1 y'(b) &= \beta, \end{aligned} \tag{2.6}$$

to have a unique solution, where $a_0 a_1 \neq 0$ and $b_0 b_1 \neq 0$.

Theorem 2.1.1. *[35] Suppose that the function $f(x,y,y')$ in (2.5) and its partial derivatives f_y and $f_{y'}$ are continuous on the set*

$$\mathscr{D} = \{(x,y,y') \,|\, a < x < b, -\infty < y, y' < \infty\}.$$

If

$$\frac{\partial f}{\partial y} > 0 \qquad \textit{and} \qquad \left|\frac{\partial f}{\partial y'}\right| \leq M < \infty, \tag{2.7}$$

then the BVP (2.5)-(2.6) has a unique solution for any α and β, where $a_i \geq 0,\ i = 0,1$ and $b_i \geq 0,\ i = 0,1$, with $a_0 + b_0 > 0$.

The proof of Theorem 2.1.1 can be found in [35]. For the convenience of the reader, we provide the proof, in details. The proof is based on the following theorem and lemma both of which can be found in [35].

Theorem 2.1.2. *Let* $\mathscr{R} = \{(t,u_1,u_2)\,|\,t \in [a,b], u_1,u_2 \in R\}$. *suppose the function* $f(t,u_1,u_2)$ *is a uniformly Lipschitz in* u_1,u_2*, i.e.,*

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq M_1|u_1 - v_1| + M_2|u_2 - v_2|,$$

the initial value problem

$$y'' = f(x,y,y'), \qquad a < x < b, \tag{2.8}$$

$$\begin{aligned} a_0 y(a) - a_1 y'(a) &= \alpha, \\ c_0 y(a) - c_1 y'(a) &= s, \end{aligned} \tag{2.9}$$

admits a unique solution, $y(x;s)$*, for any* s*, where* $a_1c_0 - a_0c_1 \neq 0$.

Without loss of generality, we may assume that $a_1c_0 - a_0c_1 = 1$.

Lemma 2.1.3. *Suppose that a function* $\phi : R \to R$ *is such that* ϕ' *is bounded away from 0, i.e.,* $\phi'(x) \geq c > 0$ *or* $\phi'(x) \leq c < 0$*, for some* c*, then there exists a unique* s^**, such that* $\phi(s^*) = 0$.

Proof of Theorem 2.1.1

The main idea of the proof is to show that there exists only one s such that the solution $y(x;s)$ of the initial value problem (2.8)–(2.9) satisfied the second boundary condition (at $x = b$) in (2.6), since it already satisfies the first boundary condition (at $x = a$) in (2.6). This is done by proving that the function

$$\phi(s) = b_0 y(b;s) + b_1 y'(b;s) - \beta = 0 \tag{2.10}$$

admits only one solution s^*. This is in turn proven, using Lemma 2.1.3, by showing that $\phi'(s) = b_0 \frac{\partial y(b;s)}{\partial s} + b_1 \frac{\partial y'(b;s)}{\partial s}$ is bounded away from zero. To this end, let $\zeta(x) = \dfrac{\partial y(x;s)}{\partial s}$. Then by differentiating (2.8) with respect to s, we get

$$\zeta''(x) = p(x)\zeta'(x) + q(x)\zeta(x), \tag{2.11}$$

with

$$p(x) = \frac{\partial f(x,y,y')}{\partial y'}, \qquad q(x) = \frac{\partial f(x,y,y')}{\partial y}. \tag{2.12}$$

Similarly, differentiating the initial condition (2.9) with respect to s, we get

$$a_0\zeta(a) - a_1\zeta'(a) = 0, \tag{2.13}$$

$$c_0\zeta(a) - c_1\zeta'(a) = 1. \tag{2.14}$$

By the assumption $a_1c_0 - a_0c_1 = 1$, it is easy to see that $\zeta(a) = a_1$ and $\zeta'(a) = a_0$. Now we have the claim that $\zeta(x) > 0$ for all $a \le x \le b$. To prove this, note that if $a_1 = 0$, then $a_0 > 0$. This means that $\zeta(a) = 0$ and $\zeta'(a) > 0$ which means that $z(x) > 0$ in $(a, a+\varepsilon)$ for some ε. Similarly if $a_1 > 0$ (and thus $a_0 \ge 0$), also we have $z(x) > 0$ in $(a, a+\varepsilon)$ for some ε. Now assume the contrary that $\zeta(x) \not> 0$ for all $x \in [a,b]$. Then there exists x^* such that $\zeta(x^*) \le 0$. This means that $\zeta(x)$ have a positive maximum in (a, x^*). If $a_0 > 0$, then $\zeta'(a) = a_0 = 0$, then the positive maximum cannot occur at a. If $a_0 = 0$, then and $\zeta'(a) = 0$ and $\zeta(a) = a_1 > 0$ and from (2.11), we have

$$\zeta''(a) = p(a)\zeta'(a) + q(a)\zeta(a) > q(a)\zeta(a), \tag{2.15}$$

by assumption of Theorem 2.1.1 that $q(x) > 0$. This means that $z''(a) > 0$ and the positive maximum of $\zeta(x)$ cannot occur at a. Therefore, the positive maximum occurs in the interior of (a, x^*), say at some $x_0 \in (a, x^*)$, where $\zeta(x_0) > 0$, $\zeta'(x_0) = 0$, and $\zeta''(x_0) < 0$. But from (2.11), we have

$$\zeta''(x_0) = p(x_0)\zeta'(x_0) + q(x_0)\zeta(x_0) = q(x_0)\zeta(x_0) > 0,$$

which is a contradiction. Therefore there does not exist an x^* such that $\zeta(x^*) \le 0$, and the claim $\zeta(x) > 0$ for all $x \in [a,b]$. Now from (2.11), $\zeta(x)$ and $q(x) > 0$ for $x \in [a,b]$, we have

$$\zeta''(x) \ge p(x)\zeta'(x) \tag{2.16}$$

Using the positive integrating factor $\mu(x) = e^{\int_a^x p(t)dt}$, it is easy to see that

$$\zeta(x) = a_1 + a_0 \int_a^x e^{\int_a^t p(r)dr} dt \tag{2.17}$$

Now from the hypothesis of 2.1.1 that $|p(x)| = \left|\frac{\partial f}{\partial y'}\right| \leq M$, we have $p(x) \geq -M$. It can then be easily deduced that

$$\zeta(x) > a_1 + a_0 \left(\frac{1 - e^{-M(x-a)}}{M} \right) > 0 \tag{2.18}$$

This completes that proof that $\zeta(x) > 0$ and then its is easy to see that

$$\phi'(s) = b_0 \zeta(b) + b_1 \zeta'(b) > 0, \tag{2.19}$$

since $\zeta(b) > 0,\ \zeta'(b) \geq 0,\ b_0 \geq 0,\ b_1 \geq 0$ with $b_0 b_1 \neq 0$. This proves that $\phi'(s)$ is bounded away from from zero. Thus, by Lemma 2.1.3,

$$\phi(s) = b_0 y(b;s) + b_1 y'(b;s) - \beta = 0 \tag{2.20}$$

has only one solution s^* and consequently $y(b, s^*)$ is the unique solution to (2.5)-(2.6). This completes the proof of Theorem 2.1.1.

2.2 Method of Solutions

In this section, we shall review two main numerical methods for solving nonlinear boundary value problems with Dirichlet boundary conditions

$$y''(x) = f(x, y, y'), \qquad a < x < b, \tag{2.21}$$

subject to

$$y(a) = \alpha, \qquad y(b) = \beta. \tag{2.22}$$

Of course, needless to mention that there are many other methods to solve nonlinear boundary value problems. However, in this thesis, we shall consider the finite difference method (Section 2.2.1) and the shooting method (Section 2.2.2) as being the most popular methods. It is important to mention that each method has its merits and disadvantages.

2.2.1 The Finite Difference Method

The finite different method works by discretizing the differential equation using nodes $x_i \in [a,b]$ and transforming the boundary value problem into a system of equations in the unknowns $y(x_i)$. Let $x_i,\ i = 0,1,\ldots,x_n$, be a uniform discritization of the interval $[a,b]$, with

$$x_i = a + hi, \qquad i = 0,1,\ldots,n,$$

where $h = \frac{b-a}{n}$ is the step size and n the number of subintervals. Note that $x_0 = a$ and $x_n = b$.

The first and second derivatives, y' and y'', at a node x_i, is approximated using some quadrature. Using central differences, we have

$$\begin{aligned} y'(x_i) &\approx \frac{y_{i+1} - y_{i-1}}{2h}, & i &= 1,2,\ldots,n-1, \\ y''(x_i) &\approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, & i &= 1,2,\ldots,n-1. \end{aligned} \tag{2.23}$$

Discritizing (2.21), we obtain

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = f(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}),\ i = 1,2,\ldots,n-1. \tag{2.24}$$

The boundary conditions (2.22) imply

$$y_0 = \alpha, \qquad y_n = \beta. \tag{2.25}$$

Now (2.24) with (2.25) together give a system of $n-1$ equations in the unknowns

$y_i,\ i = 1, 2, \ldots, n-1$:

$$\begin{cases} y_2 - 2y_1 - h^2 f(x_1, y_1, \frac{y_2-\alpha}{2h}) + \alpha = 0, \\ y_3 - 2y_2 + y_1 - h^2 f(x_2, y_2, \frac{y_3-y_1}{2h}) = 0, \\ \qquad\qquad \vdots \\ y_{n-2} - 2y_{n-1} - h^2 f(x_{n-1}, y_{n-1}, \frac{\beta-y_{n-2}}{2h}) + \beta = 0. \end{cases} \tag{2.26}$$

If f is a linear function, then the system is linear and if f is a nonlinear function, the system is nonlinear. For the linear case, it is easy to solve the system using Gaussian eliminations. However, in the nonlinear case, one resorts to suitable methods for nonlinear system such as the multidimensional Newton's method.

It is important to mention that accuracy of the finite difference method depends on the step size h. The smaller the step size, the more accurate the results. However, making the step size very small can produce an ill-conditioned system.

Next section reviews another widely used method for boundary value problems, namely, the shooting method.

2.2.2 The Shooting Method

The shooting method is a method for solving boundary value problems by solving, iteratively, a sequence of initial value problems whose solution converges to the solution of the boundary value problem.

Derivation of the shooting method algorithm

Recall that we are interested in solving the Dirichlet boundary value problem

$$y''(x) = f(x, y, y'), \qquad a < x < b, \tag{2.27}$$

$$y(a) = \alpha, \qquad y(b) = \beta. \tag{2.28}$$

The idea of the shooting method is to solve the following initial value problem

$$y''(x) = f(x,y,y'), \qquad a < x < b, \tag{2.29}$$

$$y(a) = \alpha, \qquad y'(a) = \lambda, \tag{2.30}$$

where λ is a parameter to be determined such that the solution of the IVP (2.29)–(2.30) satisfies the boundary conditions (2.28). Let $y(x;\lambda)$ be the solution of (2.29)-(2.30), where we make clear the dependence of the solution on the parameter λ. Then we search for the appropriate parameter λ^* such $y(x;\lambda^*)$ satisfies (2.28). Now $y(x;\lambda)$ satisfies (2.28) if and only if

$$y(b;\lambda) - \beta = 0. \tag{2.31}$$

Equation (2.31) is regarded as a function of λ and we seek its zero λ^*, i.e., $y(b;\lambda^*) - \beta = 0$. A convenient method to solve for λ^* is Newton's method, by solving, iteratively,

$$\lambda_{k+1} = \lambda_k - \frac{y(b;\lambda_k)}{\dfrac{\partial y(b;\lambda_k)}{\partial t}}, \qquad k \geq 0. \tag{2.32}$$

If Newton's method converges, then $\lim\limits_{k\to\infty} \lambda_k = \lambda^*$.

Clearly to operate (2.32), we require $\dfrac{\partial y(b;\lambda_k)}{\partial \lambda}$. To this end, let $z(x;\lambda) = \dfrac{\partial y(x;\lambda)}{\partial \lambda}$. Then differentiation of (2.29) with respect to λ, we find that $z(x;\lambda)$ satisfies

$$z''(x,\lambda) = \frac{\partial f}{\partial y} z(x,\lambda) + \frac{\partial f}{\partial y'} z'(x,\lambda) \tag{2.33}$$

with initial conditions

$$z(a,\lambda) = 0, \qquad z'(a,\lambda) = 1, \tag{2.34}$$

where the differentiation in (2.33) is with respect to x.

The shooting method algorithm can be summarized as follow:

1. Choose an initial guess λ_k.

2. Solve the IVP (2.29)-(2.30) with $\lambda = \lambda_k$ to obtain the solution $y(x;\lambda_k)$.

3. Using the solution $y(x;\lambda_k)$, solve the IVP (2.33)-(2.34) to obtain the solution $z(x;\lambda_k)$.

4. Using $z(b,\lambda_k) = \dfrac{\partial y(b;\lambda_k)}{\partial \lambda}$, update λ_k to get λ_{k+1} from (2.32).

5. Stop when a stopping criteria is achieved, for example $|\lambda_{k+1} - \lambda_k| < \varepsilon$, for some desired accuracy ε.

The accuracy of the solution obtained by the shooting method as described above clearly depends on two things:

1. The method used to solve the initial value problems (2.29)-(2.30) and (2.33)-(2.34).

2. The accuracy of Newton's method.

2.3 Numerical Examples

In this section we apply the finite difference and the shooting methods described in the previous section to two examples and compare the results.

Example 2.3.1. Consider the second order boundary value problem

$$y'' = -e^{-2y}, \qquad 1 < x < 2, \tag{2.35}$$

$$y(1) = 0,\ y(1) = \ln(2). \tag{2.36}$$

It is easy to see that (2.35)–(2.36) has the exact solution $y_{exact}(x) = \ln(x)$.

The finite difference method:

Discretize the interval $[1,2]$ using nodes $x_i = 1+ih$, where $h = \frac{1}{n}$, we obtain the system

$$\begin{cases} y_2 - 2y_1 - h^2 e^{-2y_1} = 0, \\ y_3 - 2y_2 + y_1 - h^2 e^{-2y_2} = 0, \\ \qquad \vdots \\ y_{n-2} - 2y_{n-1} - h^2 e^{-2y_{n-1}} + \ln(2) = 0, \end{cases} \tag{2.37}$$

where $y_i \approx y(x_i)$. This system is nonlinear. Hence, we use the Multidimensional Newton's method, Let $\mathbf{Y} = [y_1\ y_2 \dots y_{n-1}]^T$. Let $\mathbf{F}(\mathbf{Y})$ be defined as in the system (2.37),

$$\mathbf{F}(\mathbf{Y}) = \begin{bmatrix} y_2 - 2y_1 - h^2 e^{-2y_1} \\ y_3 - 2y_2 + y_1 - h^2 e^{-2y_2} \\ y_4 - 2y_3 + y_2 - h^2 e^{-2y_3} \\ \vdots \\ y_{n-2} - 2y_{n-1} - h^2 e^{-2y_{n-1}} + \ln(2) \end{bmatrix}. \tag{2.38}$$

From Newton's method, we have the iterations

$$\mathbf{Y}_{k+1} = \mathbf{Y}_k - \mathbf{J}_k^{-1} \mathbf{F}(\mathbf{Y}_k) \tag{2.39}$$

where $\mathbf{J}$ is the Jacobian matrix of $\mathbf{F}$, given by

$$\mathbf{J}_{ij} = \frac{\partial \mathbf{F}_i}{\partial y_j}, \tag{2.40}$$

where $\mathbf{F}_i$ is the ith component of $\mathbf{F}(\mathbf{Y})$. From (2.38), the Jacobian $\mathbf{J}$ is the following triangular matrix

$$\mathbf{J} = \begin{bmatrix} \alpha_1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & \alpha_2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \alpha_3 & 1 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \\ & \cdots & 0 & 1 & \alpha_{n-2} & 1 \\ & & \cdots & 0 & 1 & \alpha_{n-1} \end{bmatrix} \tag{2.41}$$

where $\alpha_i = (-2+2h^2)e^{-2y_i}$.

The shooting method:

Starting with an initial guess λ_0, for each $k \geq 0$,

1. Solve the IVP
$$y'' = -e^{-2y},\ y(1) = 0,\ y'(1) = \lambda_k,$$
to obtain the solution $y(x;\lambda_k)$.

2. Using the solution $y(x;\lambda_k)$, solve the IVP
$$z'' = 2e^{-2y(x;\lambda_k)}z,\ z(1) = 0,\ z'(1) = 1,$$
to obtain the solution $z(x;\lambda_k)$.

3. Using $z(2,\lambda_k) = \dfrac{\partial y(2;\lambda_k)}{\partial \lambda}$, update λ_k to get λ_{k+1} according to
$$\lambda_{k+1} = \lambda_k - \frac{y(2,\lambda_k) - \ln(2)}{z(2,\lambda_k)}$$

4. Stop when $|\lambda_{k+1} - \lambda_k| < \varepsilon = 10^{-6}$.

We have implemented, both methods with $n = 10$ and $\varepsilon = 10^{-6}$ for the finite difference and $\lambda_0 = \ln(2)$ and $\varepsilon = 10^{-6}$ for the shooting method. The numerical solutions along with the exact solution for the finite difference and the shooting method method are displayed in Figure 2.1. The numerical solutions obtained by both methods are identical up to six decimal places as can be seen from Table 2.1 which displays the values of both numerical solutions and the exact solution at x_i as well as the absolute error. It can be seen that the absolute error is of the order if 10^{-6}.

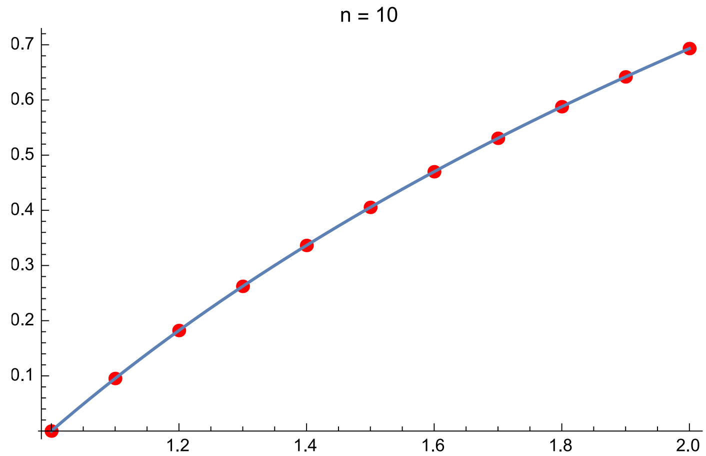

Figure 2.1: Numerical (red dots) and exact solution (continuous blue) of Example 2.3.1

x_i	y_i (FD and Shooting)	$y_{exact}(x_i)$	$\lvert Error \rvert$
1.	0	0.	0.
1.1	0.0952344	0.0953102	0.0000758247
1.2	0.182203	0.182322	0.000118563
1.3	0.262226	0.262364	0.000138723
1.4	0.336329	0.336472	0.000142949
1.5	0.40533	0.405465	0.000135575
1.6	0.469884	0.470004	0.000119499
1.7	0.530532	0.530628	0.0000967084
1.8	0.587718	0.587787	0.0000685864
1.9	0.641818	0.641854	0.0000361151
2.0	0.693147	0.693147	0.

Table 2.1: Numerical solutions of Example 2.3.1 using the finite difference and the shooting method

Chapter 3: The Homotopy Analysis Method

In this section, we consider in details a relatively new method for solving nonlinear boundary value problems, the homotopy analysis method (HAM). In Section 3.1, we give an overview of the method. In Section 3.2, we provide in details the derivation of the HAM algorithm for a general nonlinear operator. In Section 3.3, we make a note on an important parameter in the HAM. In Section 3.4, we reformulate the HAM for a particular nonlinear operator suitable for our application in Chapter 5.

3.1 Introduction

The homotopy analysis method (HAM) is an analytic method used to approximate the solutions of nonlinear problems. It was proposed by S. Liao [29] in 1992 (see also [30] and references therein). It is regarded as one of the most efficient analytic methods for solving nonlinear problems. The idea of the HAM is based on the concept of homotopy from topology. The HAM consists of a continuous deformation of an initial guess to the solution until the solution of the problem is reached. The attractiveness of the HAM is that the continuous deformation is done by solving *linear* sub-problems, which makes the method attractable.

Of course, there are many methods for solving nonlinear problems, but the HAM is favoured over other analytic methods because of many of its advantages. Some of its advantages is that it is independent of any changes in the parameters of the problem. It also provides freedom in choosing the initial guess solution and the linear operator, as we will see in the derivation. In its generality, it also encompasses other well known methods. For example, it has been proven in [32] that the Adomian decomposition method [33, 34] is a special case of the HAM. The HAM provides freedom in adding convergence control parameters in its derivation [31]. These control parameters can be used to speed up the convergence of the approximate solution.

3.2 Derivation of the HAM Iteration Formula

In this section, we provide a detailed derivation of the homotopy analysis method. Consider the nonlinear equation

$$\mathscr{A}(u(x)) = 0,\ x \in \Omega, \qquad \mathscr{B}(u(x)) = 0,\ x \in \partial\Omega \tag{3.1}$$

where $\mathscr{A}$ is a general nonlinear operator and $\mathscr{B}$ a boundary condition operator. We will restrict our derivation where $\mathscr{A}$ is a differential equation.

Define the homotopy

$$(1-p)[\mathscr{L}(u(x,p)) - \mathscr{L}(u_0(x))] = \hbar p \mathscr{A}(u(x,p)), \tag{3.2}$$

where the operator $\mathscr{L}$ is a linear differential operator satisfying $\mathscr{L}(0) = 0$, $u_0(x)$ is any initial approximation to the solution, called the base function, and p is the *embedding* parameter ($0 \leq p \leq 1$). The constant $\hbar$ is a non-zero auxiliary parameter in some literature referred to as convergence control parameter [31]. We will see later (Section 3.3) that it has to satisfy $-2 < \hbar < 0$.

Let $u(x,p)$ be the solution of (3.2), which depends on the homotopy parameter p (and of course on the parameter $\hbar$). Note that

* When $p = 0$, we have $\mathscr{L}(u(x,0)) - \mathscr{L}(u_0(x)) = 0 \Longrightarrow u(x,0) = u_0(x)$.
* When $p = 1$, we have $\hbar\mathscr{A}(u(x,1)) = 0$ and since $\hbar \neq 0 \Longrightarrow \mathscr{A}(u(x,1)) = 0$.

Therefore, the desired solution $u(x)$ of (3.1) is $u(x) = u(x,1)$. Let $u(x,p)$ be written as

$$u(x,p) = u_0(x) + \sum_{n=1}^{\infty} u_n(x) p^n \tag{3.3}$$

so that the sought solution $u(x) = u(x,1)$ is

$$u(x) = u_0(x) + \sum_{n=1}^{\infty} u_n(x), \tag{3.4}$$

with $u_0(x)$ solution of $\mathscr{L}(u(x)) = 0$ with $\mathscr{B}(u(x)) = 0$ and $u_n(x)$ are the solution components. Since the B.C. $\mathscr{B}(u(x)) = 0$ on $\partial\Omega$ is satisfied by $u_0(x)$, we impose that the components $u_n(x)$, $n \geq 1$, satisfy homogeneous B.Cs on $\partial\Omega$, that is for all $n \geq 1$, $u_n(x) = 0$ on $\partial\Omega$.

Next we derive an iterative algorithm to determine the components $u_n(x)$, $n \geq 1$. To this end, substitute (3.3) into (3.2) to obtain

$$
\begin{aligned}
& (1-p)\left[\mathscr{L}\left(\sum_{n=0}^{\infty} u_n(x)p^n\right) - \mathscr{L}(u_0(x))\right] = \hbar p \mathscr{A}(u(x,p)) \\
\Longrightarrow \quad & (1-p)\sum_{n=1}^{\infty} \mathscr{L}(u_n(x))p^n = \hbar p \mathscr{A}(u(x,p)) \\
\Longrightarrow \quad & \sum_{n=1}^{\infty} \mathscr{L}(u_n(x))p^n - \sum_{n=1}^{\infty} \mathscr{L}(u_n(x))p^{n+1} = \hbar p \mathscr{A}(u(x,p)).
\end{aligned} \tag{3.5}
$$

To find the component u_m, we differentiate (3.5) m times with respect to p, to obtain

$$
\begin{aligned}
& \sum_{n=1}^{\infty} \mathscr{L}(u_n)[n(n-1)\ldots(n-m+1)]p^{n-m} \\
& -\sum_{n=1}^{\infty} \mathscr{L}(u_n)[(n+1)n(n-1)\ldots(n-m+2)]p^{n+1-m} \\
& = \hbar \frac{d^m}{dp^m}(p\mathscr{A}(u(x,p))) = \hbar \sum_{k=0}^{m} \binom{m}{k} \frac{d^k}{dp^k}(p) \frac{d^{m-k}}{dp^{m-k}}(\mathscr{A}(u(x,p))) \\
& = \hbar \binom{m}{0} \frac{d^0}{dp^0}(p) \frac{d^m}{dp^m}(\mathscr{A}(u(x,p))) + \hbar \binom{m}{1} \frac{d}{dp}(p) \frac{d^{m-1}}{dp^{m-1}}(\mathscr{A}(u(x,p))) \\
& = \hbar p \frac{d^m}{dp^m}(\mathscr{A}(u(x,p))) + \hbar m \frac{d^{m-1}}{dp^{m-1}}(\mathscr{A}(u(x,p))).
\end{aligned}
$$

When we substitute $p = 0$, the first and sums on the left right side are zeros except when $n = m$ and $n = m-1$, respectively. The sum on the right hand side is zero except when $k = 1$. So we have

$$
m![\mathscr{L}(u_m) - \mathscr{L}(u_{m-1})] = \hbar m \frac{d^{m-1}}{dp^{m-1}}(\mathscr{A}(u(x,p)))\bigg|_{p=0}
$$

or

$$\mathscr{L}(u_m) = \mathscr{L}(u_{m-1}) + \frac{\hbar}{(m-1)!}\frac{d^{m-1}}{dp^{m-1}}(\mathscr{A}(u(x,p)))\bigg|_{p=0}, \tag{3.6}$$

which is the well-known HAM iteration formula to compute the components $u_n(x)$, $n \geq 1$, given $u_0(x)$ as the solution of $\mathscr{L}(u(x)) = 0$.

The iteration formula (3.6) can be written differently, by eliminating the term $\mathscr{L}(u_{m-1})$ on the right hand side, as stated in the following proposition.

Proposition 3.2.1. *The HAM iteration formula in (3.6) can be written as*

$$\mathscr{L}(u_m) = \hbar\left[\sum_{k=0}^{m-1}\frac{1}{k!}\frac{d^k}{dp^k}[\mathscr{A}(u(x,p)]\bigg|_{p=0}\right], \qquad m \geq 1. \tag{3.7}$$

Proof. Write down (3.6) for $k = 1$ to $k = m$, we have

$$\mathscr{L}(u_1) = \underbrace{\mathscr{L}(u_0)}_{=0} + \hbar\frac{1}{(0)!}\frac{d^0}{dp^0}(\mathscr{A}(u(x,p)))\bigg|_{p=0} \tag{3.8}$$

$$\mathscr{L}(u_2) = \mathscr{L}(u_1) + \hbar\frac{1}{(1)!}\frac{d^1}{dp^1}(\mathscr{A}(u(x,p)))\bigg|_{p=0} \tag{3.9}$$

$$\vdots \qquad\qquad \vdots$$

$$\mathscr{L}(u_{m-2}) = \mathscr{L}(u_{m-3}) + \hbar\frac{1}{(m-3)!}\frac{d^{m-3}}{dp^{m-3}}(\mathscr{A}(u(x,p)))\bigg|_{p=0} \tag{3.10}$$

$$\mathscr{L}(u_{m-1}) = \mathscr{L}(u_{m-2}) + \hbar\frac{1}{(m-2)!}\frac{d^{m-2}}{dp^{m-2}}(\mathscr{A}(u(x,p)))\bigg|_{p=0} \tag{3.11}$$

$$\mathscr{L}(u_m) = \mathscr{L}(u_{m-1}) + \hbar\frac{1}{(m-1)!}\frac{d^{m-1}}{dp^{m-1}}(\mathscr{A}(u(x,p)))\bigg|_{p=0} \tag{3.12}$$

Adding equations (3.8) to (3.12), we obtain

$$\mathscr{L}(u_m) = \hbar\left[\sum_{k=0}^{m-1}\frac{1}{k!}\frac{d^k}{dp^k}[\mathscr{A}(u(x,p)]\bigg|_{p=0}\right], \qquad m \geq 1. \tag{3.13}$$

which proves (3.7). □

Often the operator $\mathscr{A}$ can be written as $\mathscr{A} = \mathscr{L} + \mathscr{N}$ where $\mathscr{L}$ is the linear operator considered in the homotopy (3.2) and $\mathscr{N}$ is a nonlinear operator. In this case the HAM iteration formula (3.6) can be written as

$$\mathscr{L}(u_m) = (1+\hbar)\mathscr{L}(u_{m-1}) + \frac{\hbar}{(m-1)!}\frac{d^{m-1}}{dp^{m-1}}(\mathscr{N}(u(x,p)))\bigg|_{p=0}. \tag{3.14}$$

3.3 A Note On the Constant $\hbar$

It is has been reported in the literature [29] that the auxiliary parameter $\hbar$ has satisfy $-2 < \hbar < 0$. However, we could not find an explicit proof of this. Here, we provide such a proof. Suppose that we can decompose the general operator $\mathscr{A}$ as $\mathscr{A} = \mathscr{L} + \mathscr{N}$, where $\mathscr{L}$ is the linear operator considered in the homotopy (3.2) and $\mathscr{N}$ is a nonlinear operator. Let $A_j = \dfrac{\hbar}{j!}\dfrac{d^j}{dp^j}[\mathscr{N}(u(x,p)]\bigg|_{p=0}$. Then Equation (3.14) can be written as

$$\begin{aligned}
\mathscr{L}(u_m(x)) &= (1+\hbar)\mathscr{L}(u_{m-1}(x)) + A_{m-1} \\
&= (1+\hbar)^2\mathscr{L}(u_{m-2}(x)) + (1+\hbar)A_{m-2} + A_{m-1} \\
&= (1+\hbar)^3\mathscr{L}(u_{m-3}(x)) + (1+\hbar)^2A_{m-3} + (1+\hbar)A_{m-2} + A_{m-1} \\
&= (1+\hbar)^m\underbrace{\mathscr{L}(u_0(x))}_{=0} + (1+\hbar)^{m-1}A_0 + (1+\hbar)^{m-2}A_1 + \cdots + A_{m-1}.
\end{aligned}$$

or

$$\mathscr{L}(u_m(x)) = \sum_{i=0}^{m-1}(1+\hbar)^i A_{m-1-i}. \tag{3.15}$$

Now recall that the solution $u(x) = \sum_{m=0}^{\infty} u_m(x)$. Then, formally,

$$\mathscr{L}(u(x)) = \sum_{m=0}^{\infty} \mathscr{L}(u_m(x)) = \sum_{m=1}^{\infty} \sum_{i=0}^{m-1} (1+\hbar)^i A_{m-i-1} = \left(\sum_{i=0}^{\infty} (1+\hbar)^i \right) \left(\sum_{m=0}^{\infty} A_m \right) \tag{3.16}$$

We see from (3.16) that a necessary condition for the existence of a solution $u(x)$ is that

$$\sum_{i=0}^{\infty} (1+\hbar)^i < \infty \iff |1+\hbar| < 1 \iff -2 < \hbar < 0.$$

3.4 Reformulation of the HAM Iteration Formula Using Bell Polynomials

In this section, we shall reformulate the general HAM iteration formula (3.6) in terms of Bell polynomials [2] in the case where the nonlinear part of the operator $\mathscr{N}$ in the decomposition $\mathscr{A} = \mathscr{L} + \mathscr{N}$ is a function of u only. The Bell polynomials are defined below.

Definition 3.4.1. Let n and k be positive integers. The partial (or incomplete) Bell polynomial, denoted by $B_{n,k}$, is a multivariate polynomial in $(n-k+1)$ variables $x_1, x_2, \ldots, x_{n-k+1}$, defined by

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{j_1! j_2! \cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}} \tag{3.17}$$

where the sum is over indices $j_i \geq 1,\ i = 1, \ldots, n-k+1$, such that

$$\begin{aligned} &j_1 + j_2 + \cdots + j_{n-k+1} = k, \\ &j_1 + 2j_2 + 3j_3 + \cdots + (n-k+1) j_{n-k+1} = n. \end{aligned} \tag{3.18}$$

Examples of Bell polynomials are

$$B_{4,1}(x_1,x_2,x_3,x_4) = x_4$$

$$B_{4,2}(x_1,x_2,x_3) = 3x_2^2 + 4x_1x_3$$

$$B_{4,3}(x_1,x_2) = 6x_1^2x_2$$

$$B_{4,4}(x_1) = x_1^4$$

A key property of Bell polynomials is

$$B_{n,1}(x_1,\ldots,x_n) = x_n, \qquad B_{n,n}(x_1) = x_1^n. \tag{3.19}$$

We need the following proposition about the nth derivative of the composition of two functions

Proposition 3.4.1. *Given two functions $f(x)$ and $g(x)$, we have for $n \geq 1$,*

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{k=1}^{n} f^{(k)}(g(x)) \cdot B_{n,k}\left(g'(x), g''(x), \ldots, g^{(n-k+1)}(x)\right). \tag{3.20}$$

The proof of the above proposition can be found in the literature.

Now, from Definition 3.4.1 and Proposition 3.20, we have for $m \geq 2$,

$$\frac{d^{m-1}}{dp^{m-1}}(\mathcal{N}(u(x,p))) = \sum_{k=1}^{m-1} \mathcal{N}^{(k)}(u) B_{m-1,k}(\partial_p u, \partial_p^2 u, \ldots, \partial_p^{m-k} u),$$

where $\partial_p^i u(x,p) = \frac{\partial^i}{\partial p^i} u(x,p)$ and $\mathcal{N}^{(k)}(u(x,p)) = \frac{d^k}{dp^k}\mathcal{N}(u(x,p))$.

Since $\partial_p^i u(x,p)\big|_{p=0} = i! u_i(x)$, we have, for $m \geq 2$,

$$\begin{aligned} &\frac{d^{m-1}}{dp^{m-1}}(\mathcal{N}(u(x,p)))\bigg|_{p=0} \\ &= \sum_{k=1}^{m-1} \mathcal{N}^{(k)}(u_0(x)) B_{m-1,k}(1!u_1(x), 2!u_2(x), \ldots, (m-k)!u_{m-k}(x)), \end{aligned}$$

It follows that the HAM iteration (3.14) can be written as

$$\begin{aligned}
\mathscr{L}(u_m) &= (1+\hbar)\mathscr{L}(u_{m-1}) && (3.21)\\
&\quad + \frac{\hbar}{(m-1)!}\sum_{k=1}^{m-1}\mathscr{N}^{(k)}(u_0(x))B_{m-1,k}(1!u_1, 2!u_2, \ldots, (m-k)!u_{m-k}) \\
&= (1+\hbar)\mathscr{L}(u_{m-1}) + \frac{\hbar}{(m-1)!}\sum_{k=1}^{m-1}\mathscr{N}^{(k)}(u_0(x))B_{m-1,k}, && (3.22)
\end{aligned}$$

where, for easy of notation, we have suppressed the arguments of $B_{m-1,k}$. Note that a special case is when $\hbar = -1$. In this case, the different forms of the HAM iterations (3.14) and (3.22) simplify to

$$\begin{aligned}
\mathscr{L}(u_m(x)) &= \hbar\frac{1}{(m-1)!}\frac{d^{m-1}}{dp^{m-1}}[\mathscr{N}(u(x,p)]\bigg|_{p=0} \\
&= \frac{\hbar}{(m-1)!}\sum_{k=1}^{m-1}\mathscr{N}^{(k)}(u_0)B_{m-1,k}(1!u_1, 2!u_2, \ldots, (m-k)!u_{m-k}).
\end{aligned}$$

Chapter 4: Multi-layer Boundary Value Problem

In this chapter, we will consider multi-layer boundary value problems. We restrict our study to the second order case.

4.1 Description of the Problem

In this work, a multi-layer boundary value problem consists of the following boundary value problems:

$$y'' = f_i(x,y,y'), \qquad x \in [x_{i-1}, x_i], \;\; i = 1,2,\ldots,N, \tag{4.1}$$

subject to the the following boundary conditions

$$y(x_0) = \alpha, \qquad y(x_N) = \beta, \tag{4.2}$$

$$y(x_i^-) = y(x_i^+), \qquad 1 \le i \le N-1, \tag{4.3}$$

$$y'(x_i^-) = y'(x_i^+), \qquad 1 \le i \le N-1, \tag{4.4}$$

where $[x_{i-1}, x_i]$ is the ith subdomain of the overall domain $[a,b] = [x_0, x_N]$. The functions f_i, $1 \le i \le N$, are the functions that define the differential equation in the ith subdomain. The boundary conditions in (4.3) and (4.4) require the solution $y(x)$ to be continuous and smooth across the nodes x_i. Note that the nodes x_i need not be uniformly distributed in the interval $[a,b]$ as displayed in Figure 1.1.

We should mention that similar problems have been considered in the literature but with all functions $f_i = f$ are the same [20]. This really amounts to solving a regular boundary value problem

$$y'' = f(x,y,y'), \; a \le x \le b, \qquad y(a) = \alpha, \; y(b) = \beta,$$

using a domain decomposition, making sure that the solution is smooth across the subdomains interface regions. However, in many applications, such as in fluid flow though porous channels, as we will see in chapter 5, the governing equation have different from from layer to layer.

There are different approaches one can use to solve such a problem. In the next section, we shall consider three main methods.

4.2 Method of Solutions of Multi-Layer Boundary Value Problems

In this section, we outline three methods to solve problems of type (4.1)–(4.4). The simple is the finite difference method. The second method is the shooting method. The third method is the homotopy analysis method.

4.2.1 The Finite Difference Method

The finite difference method is a straight forward method for solving boundary value problems where the differential equation is discritized to obtain a system of algebraic equation for the approximate value of the solution at the discrete point x_i, as was discussed Subsection 2.2.1.

To adapt the finite difference method for a single layer boundary value problem to our problem, one has to take care of the internal boundary conditions (4.3)–(4.4), carefully. We should mention that [28] has used this approach. The idea in [28] was to uniformly discritize each subdomain $I_i = [x_{i-1}, x_i]$ using a local step size h_i. This gives the discrete points

$$x_k^{(i)} = x_{i-1} + kh_i, \qquad 1 \le i \le N,\ 0 \le k \le M_i,$$

where $x_0^{(1)} = x_0$, $x_{M_N}^{(N)} = x_N$, $x_{M_i}^{(i)} = x_0^{(i+1)} = x_i$. This way of discretization was adopted in oder to guarantee that the interface nodes, x_i, are part of the mesh points. Next, each equation in (4.1) is discretized at its local discrete points $x_k^{(i)}$, excluding the last local mesh points $x_{M_i}^{(i)}$. This gives the following rectangular system for the unknowns $y_k^{(i)} \approx y(x_k^{(i)})$,

$$y_{k+1}^{(i)} - 2y_k^{(i)} + y_{k-1}^{(i)} = h_i^2 f_i \left(x_k^{(i)}, \frac{y_{k+1}^{(i)} - y_{k-1}^{(i)}}{2h_i} \right), \qquad \begin{array}{l} i = 1,2,\ldots,N, \\ k = 1,2,\ldots,M_i - 1. \end{array} \tag{4.5}$$

which has $(M-N)$ equations and M unknowns, with $M = \sum_{i=1}^{N} M_i$. The system is then made square by eliminating $y_{M_i}^{(i)} \approx y(x_{M_i}^{(i)}) = y(x_i)$ by expressing it in terms of neighbouring points (see [28] for details). The result is a square system in the $(M-N)$ unknowns $y_k^{(i)}$, $i = 1,\ldots,N$, $k = 1,\ldots,M_i - 1$. The system was solve numerically using the multidimensional version Newton's method.

4.2.2 The Shooting Method

The shooting method for a single layer boundary value problem as described in subsection 2.2.2 has been extensively used to solve boundary value problems and has proven to be efficient. The adaptation of the shooting method to our multi-layer boundary value problem (4.1)–(4.4) consists of performing, in parallel, N shootings as illustrated in the Figure 4.1 below.

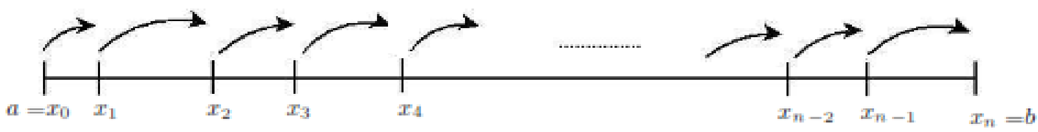

Figure 4.1: Shooting illustration

According to the suggested shooting strategy, we introduce $(2N-1)$ parameters $\lambda_i,\ i = 1,\dots,2N-1$, such that

$$y'(x_0) = \lambda_1, \tag{4.6}$$

$$y(x_i) = \lambda_{2i},\ y'(x_i) = \lambda_{2i+1},\ i = 1,\dots,N-1. \tag{4.7}$$

Then we solve, in parallel, the following set of initial-value problems:

1. In $[x_0,x_1]$ solve the IVP

$$y''(x) = f_1(x,y,y'), \qquad y(x_0) = \alpha,\ y'(x_0) = \lambda_1. \tag{4.8}$$

2. In the subintervals $[x_i,x_{i+1}]$, $i = 1,..,N-1$, solve the IVP

$$y''(x) = f_{i+1}(x,y,y'), \qquad y(x_i) = \lambda_{2i},\ y'(x_i) = \lambda_{2i+1}. \tag{4.9}$$

The parameters $\lambda_i,\ i = 1,\dots,2N-1$, are to be determined such that the boundary conditions (4.2) – (4.4) are satisfied. Let $y_1(x;\lambda_1)$, $y_{i+1}(x;\lambda_{2i},\lambda_{2i+1})$, $i = 1,...,N-1$, be the solutions to (4.8) and (4.9), respectively, where we have explicitly showed the dependence of the solutions y_i on the parameters λ_j. Imposing conditions (4.2)–(4.4), we obtain the set of equations

$$\begin{cases} y_1(x_1;\lambda_1) - y_2(x_1;\lambda_2,\lambda_3) = 0, \\ y_1'(x_1;\lambda_1) - y_2'(x_1;\lambda_2,\lambda_3) = 0, \\ y_{i+1}(x_{i+1};\lambda_{2i},\lambda_{2i+1}) - y_{i+2}(x_{i+1};\lambda_{2i+2},\lambda_{2i+3}) = 0, \quad 1 \le i \le N-2, \\ y_{i+1}'(x_{i+1};\lambda_{2i},\lambda_{2i+1}) - y_{i+2}'(x_{i+1};\lambda_{2i+2},\lambda_{2i+3}) = 0, \quad 1 \le i \le N-2, \\ y_N(x_N;\lambda_{2N-2},\lambda_{2N-1}) - \beta = 0. \end{cases} \tag{4.10}$$

Using the fact that $y_{i+1}(x_i;\lambda_{2i},\lambda_{2i+1}) = \lambda_{2i}$ and $y_{i+1}'(x_i;\lambda_{2i},\lambda_{2i+1}) = \lambda_{2i+1}$, the above system rewrites as

$$\begin{cases} y_1(x_1;\lambda_1) - \lambda_2 = 0, \\ y_1'(x_1;\lambda_1) - \lambda_3 = 0, \\ y_{i+1}(x_{i+1};\lambda_{2i},\lambda_{2i+1}) - \lambda_{2i+2} = 0, \qquad 1 \le i \le N-2, \\ y_{i+1}'(x_{i+1};\lambda_{2i},\lambda_{2i+1}) - \lambda_{2i+3} = 0, \qquad 1 \le i \le N-2, \\ y_N(x_N;\lambda_{2N-2},\lambda_{2N-1}) - \beta = 0. \end{cases} \tag{4.11}$$

System (4.11) can be regarded as a nonlinear homogeneous system in the unknown parameters $\lambda_i,\ 1 \le i \le 2N-1$. Introducing the notation $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2N-1})$, it can be written, in vector form, as

$$\mathbf{F}(\Lambda) = \mathbf{0},$$

where $\mathbf{F}(\Lambda) = [F_1(\Lambda), ..., F_{2N-1}(\Lambda)]^T : R^{2N-1} \longrightarrow R^{2N-1}$, is a vector valued function of the parameter vector Λ and $F_i(\cdot),\ i = 1, \ldots, 2N-1$, are scalar functions given by equations (4.11).

A popular method for solving the above system is the well-known multidimensional Newton's method. Let

$$\Lambda^{(k)} = (\lambda_1^{(k)}, \lambda_2^{(k)}, \ldots, \lambda_{2n-1}^{(k)})^T$$

be the values of the parameters at the k^{th} iteration. The classical multidimensional Newton's method calculates new values $\Lambda^{(k+1)}$ according to

$$\Lambda^{(k+1)} = \Lambda^{(k)} - \mathbf{J}^{-1}(\Lambda^{(k)})\mathbf{F}(\Lambda^{(k)}) \tag{4.12}$$

where $\mathbf{F}(\Lambda^{(k)}) = [F_1(\Lambda^{(k)}), \ldots, F_{2N-1}(\Lambda^{(k)})]^T$ and $\mathbf{J}^{-1}(\Lambda^{(k)})$ is the inverse of the $(2N-1) \times (2N-1)$ Jacobian matrix $\mathbf{J}$ evaluated at $\Lambda^{(k)}$. The entries of the Jacobian matrix are given by

$$\mathbf{J}_{ij} = \frac{\partial F_i}{\partial \Lambda_j} \tag{4.13}$$

The Jacobian matrix $\mathbf{J}$ turn out to be pentadiagonal and has the form:

$$\begin{aligned} J_{1,:} &= \begin{bmatrix} \frac{\partial y_1(x_1;\lambda_1)}{\partial \lambda_1} & -1 & 0 & \mathbf{0}_{2N-4} \end{bmatrix}, \\ J_{2,:} &= \begin{bmatrix} \frac{\partial y_1'(x_1;\lambda_1)}{\partial \lambda_1} & 0 & -1 & \mathbf{0}_{2N-4} \end{bmatrix}, \end{aligned}$$

for $i = 1, 2, \ldots, N-2$,

$$\begin{aligned} J_{2i+1,:} &= \begin{bmatrix} \mathbf{0}_{2i-1} & \frac{\partial y_{i+1}(x_{i+1};\lambda_{2i},\lambda_{2i+1})}{\partial \lambda_{2i}} & \frac{\partial y_{i+1}(x_{i+1};\lambda_{2i},\lambda_{2i+1})}{\partial \lambda_{2i+1}} & -1 & 0 & \mathbf{0}_{2N-2i-4} \end{bmatrix}, \\ J_{2i+2,:} &= \begin{bmatrix} \mathbf{0}_{2i-1} & \frac{\partial y_{i+1}'(x_{i+1};\lambda_{2i},\lambda_{2i+1})}{\partial \lambda_{2i}} & \frac{\partial y_{i+1}'(x_{i+1};\lambda_{2i},\lambda_{2i+1})}{\partial \lambda_{2i+1}} & 0 & -1 & \mathbf{0}_{2N-2i-4} \end{bmatrix}, \end{aligned}$$

and

$$J_{2N-1,:} = \begin{bmatrix} \mathbf{0}_{2N-3} & 1 & \frac{\partial y_N(x_N;\lambda_{2N-2},\lambda_{2N-1})}{\partial \lambda_{2N-1}} \end{bmatrix},$$

where the notation $J_{k,:}$ stands for the kth row of J and $\mathbf{0}_k$ stands for a row of k zeros.

The calculation of the entries of the Jacobian matrix requires the calculation of

$$\frac{\partial y_i(x_j)}{\partial \lambda_k} \quad \text{and} \quad \frac{\partial y_i'(x_j)}{\partial \lambda_k}$$

with the appropriate x_j and λ_k for a given y_i. Specifically, we require for $i = 1, 2, \ldots, N-2$,

$$\frac{\partial y_{i+1}(x_{i+1})}{\partial \lambda_k}, \ \frac{\partial y_{i+1}'(x_{i+1})}{\partial \lambda_k}, \ k = 2i, 2i+1, \tag{4.14}$$

$$\frac{\partial y_{i+2}(x_{i+1})}{\partial \lambda_k}, \ \frac{\partial y_{i+2}'(x_{i+1})}{\partial \lambda_k}, \ k = 2i+2, 2i+3, \tag{4.15}$$

and

$$\frac{\partial y_1^{(k)}(x_1)}{\partial \lambda_1}, \frac{\partial y_2^{(k)}(x_1)}{\partial \lambda_2}, \frac{\partial y_2^{(k)}(x_1)}{\partial \lambda_3}, k = 0, 1, \tag{4.16}$$

$$\frac{\partial y_N(x_N)}{\partial \lambda_{2N-2}}, \frac{\partial y_N(x_N)}{\partial \lambda_{2N-1}}. \tag{4.17}$$

The quantities in (4.14)–(4.17) are obtained from the solutions of an other set of initial value problems as explained below.

From equation (4.1), differentiating with respect to λ, we get

$$\frac{\partial y_j''}{\partial \lambda} = \frac{\partial f_j}{\partial y}\frac{\partial y_j}{\partial \lambda} + \frac{\partial f_j}{\partial y_j'}\frac{\partial y_j'}{\partial \lambda}.$$

Let $z_{j,k} = \dfrac{\partial y_j}{\partial \lambda_k}$. Assuming we can interchange the order of differentiation, we find that $z_{j,k}$ satisfies

$$z_{j,k}'' = \frac{\partial f_j(x, y_j, y_j')}{\partial y} z_{j,k} + \frac{\partial f_j(x, y_j, y_j')}{\partial y'} z_{j,k}'. \tag{4.18}$$

Therefore, we have the following I.V.Ps.

1. For $z_{1,1} = \frac{\partial y_1}{\partial \lambda_1}$ satisfies (4.18) for $x_0 \le x \le x_1$ with the I.Cs.

$$z_{1,1}(x_0) = 0,\ z_{1,1}'(x_0) = 1. \tag{4.19}$$

2. For $z_{j,2j-2} = \frac{\partial y_j}{\partial \lambda_{2j-2}}$ and $z_{j,2j-1} = \frac{\partial y_j}{\partial \lambda_{2j-1}}$, $j = 2, \ldots, N-1$, satisfy (4.18) with ICs

$$z_{j,2j-2}(x_{j-1}) = 1,\ z_{j,2j-2}'(x_{j-1}) = 0, \tag{4.20}$$

$$z_{j,2j-1}(x_{j-1}) = 0,\ z_{j,2j-1}'(x_{j-1}) = 1, \tag{4.21}$$

respectively.

3. For $j = N$, $z_{N,2N-2} = \frac{\partial y_N}{\partial \lambda_{2N-2}}$ and $z_{N,2N-1} = \frac{\partial y_N}{\partial \lambda_{2N-1}}$ satisfy (4.18) for $x_{N-1} \le x \le$

x_N with the I.Cs.

$$z_{N,2N-2}(x_{N-1}) = 1, \; z'_{N,2N-2}(x_{N-1}) = 0, \tag{4.22}$$

$$z_{N,2N-1}(x_{N-1}) = 0, \; z'_{n,2N-1}(x_{N-1}) = 1. \tag{4.23}$$

Once the solutions $z_{j,k}$ of the above IVPs (4.18) with (4.19)–(4.23) are obtained, the different Jacobian entries (4.14)–(4.17) are given by

$$\frac{\partial y_j(x_i)}{\partial \lambda_k} = z_{j,k}(x_i), \qquad \frac{\partial y'_j(x_i)}{\partial \lambda_k} = z'_{j,k}(x_i).$$

Finally, the shooting method algorithm to solve the multi-layer boundary value problem (4.1)–(4.4) can be summarized as follows.

1. At the k^{th} iteration, solve the IVPs (4.8)–(4.9) with parameters $\Lambda^{(k)}$.
2. Solve the IVPs (4.18) with (4.19)–(4.23) and construct the Jacobian matrix.
3. Use Newton's formula (4.12) to update the parameters Λ.
4. Repeat the process until a stopping criteria is satisfied. within a desired accuracy.

It is important to mention here that at each step solving the IVPs (4.8)–(4.9) can be done in parallel. Similarly, solving the IVPs (4.18) with (4.19)–(4.23) can also be done in parallel. This is an important advantage of shooting method for multi-layer boundary value problems. Let us consider an example to illustrate the shooting method for multi-layer boundary value problem.

Example 4.2.1. Consider the following "cooked up" three-layer problem

$$y'' = \begin{cases} -(y')^2 - \frac{20}{9}y' - \frac{100}{81}, & 0 < x < 1/2, \\ 2x(y' - \frac{14}{9})^2 + 3x(y - \frac{14}{9}x - \ln(3x) + 3)^2 - \frac{2}{x}, & 1/2 < x < 1, \\ -(y' - \frac{5}{9})^2 - (y - \frac{5}{9}x + 1) + \ln(3x), & 1 < x \leq 2, \end{cases} \tag{4.24}$$

subject to

$$
\begin{aligned}
&y(0) = 1/3,\ y(2) = \tfrac{1}{9} + \ln(6), \\
&y^{(i)}(1/2^-) = y^{(i)}(1/2^+),\ y^{(i)}(1^-) = y^{(i)}(1^+),\ i = 0, 1.
\end{aligned} \tag{4.25}
$$

It is easy to check that an exact solution to (4.24)–(4.25) is

$$
y(x) = \begin{cases} \ln(x+1) - \frac{10x}{9} + \frac{1}{3}, & 0 \le x < 1/2, \\ \frac{14x}{9} + \frac{1}{x} + \ln(3x) - 3, & 1/2 < x < 1, \\ \frac{5x}{9} + \ln(3x) - 1, & 1 < x < 2, \end{cases} \tag{4.26}
$$

We have applied the describe shooting algorithm with and zero initial guess for Λ. The various IVPs have been solved the **NDSolve** built in function in Mathematica. We have used a stopping criteria that $\|\Lambda^{(k+1)} - \Lambda^{(k)}\| < \varepsilon = 10^{-6}$. It took 6 iterations until the stopping criteria is satisfied. Figure 4.2 displays the solutions versus iterations and Figure 4.3 displays the absolute error between the approximate and exact solution. Table 4.1 displays the values of λ_i as well as $|y(x_i^-) - y(x_i^+)|$ and $|y'(x_i^-) - y'(x_i^+)|$ versus iterations.

Iter. / λ_i	1	2	3	4	5	6
λ_1	0.	-0.301427	-0.0968573	-0.110645	-0.111111	-0.111111
λ_2	0.	0.122723	0.190511	0.183409	0.183243	0.183243
λ_3	0	-0.521395	-0.431105	-0.444209	-0.444444	-0.444444
λ_4	0.	0.455019	0.661478	0.654395	0.654168	0.654168
λ_5	0	1.06598	1.49869	1.55436	1.55555	1.55556

Table 4.1: Values of λ_i vs iteration for Example 4.2.1

4.2.3 The Homotopy Analysis Method

In this subsection, we will show how we can use the homotopy analysis method as described in section 3.2 for a single layer boundary value problem to solve multi-

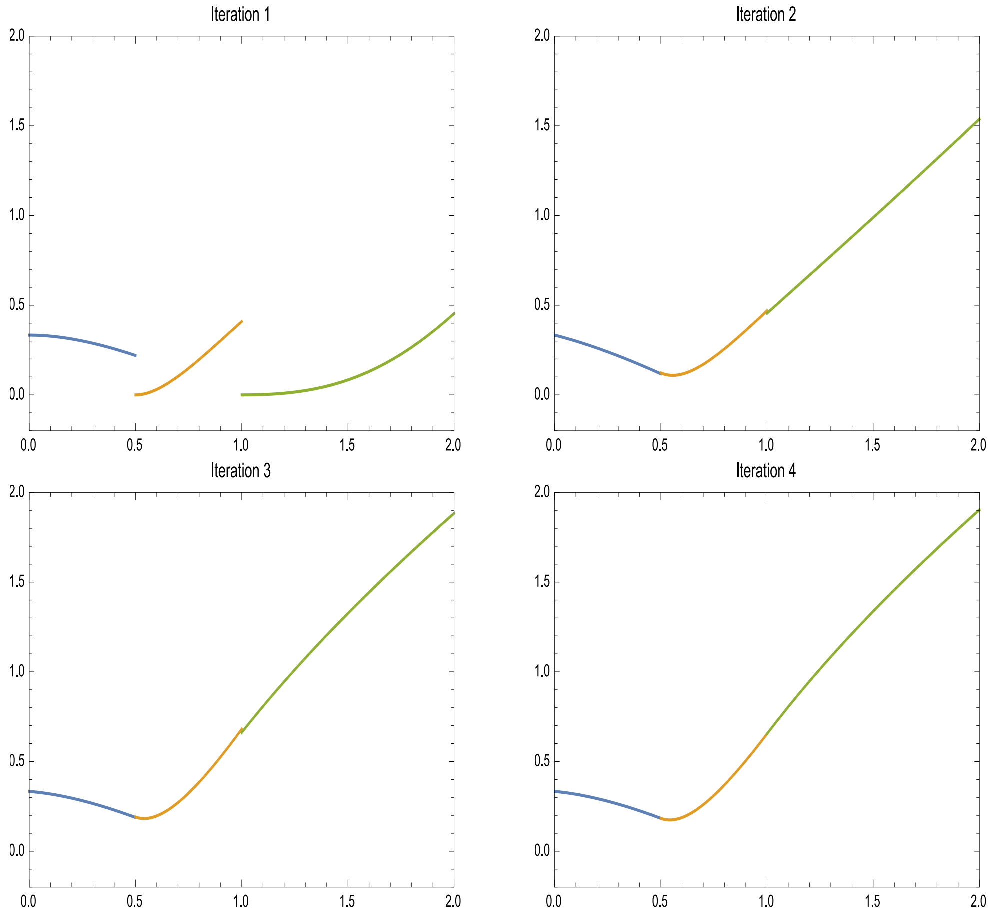

Figure 4.2: Numerical solutions vs iterations of Example 4.2.1

layer boundary of the form

$$y'' = f_i(x, y, y'), \qquad x \in [x_{i-1}, x_i], \ \ i = 1, 2, \dots, N, \tag{4.27}$$

subject to the boundary conditions

$$y(x_0) = \alpha, \qquad y(x_N) = \beta, \tag{4.28}$$

$$y(x_i^-) = y(x_i^+), \qquad 1 \le i \le N-1, \tag{4.29}$$

$$y'(x_i^-) = y'(x_i^+), \qquad 1 \le i \le N-1, \tag{4.30}$$

The main idea in adapting the HAM to multi-layer boundary value problem consists of

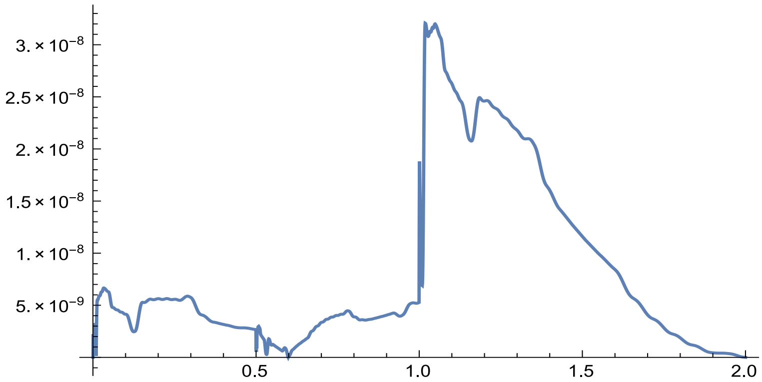

Figure 4.3: Absolute error between numerical solution and the exact solution of Example 4.2.1

1. Introducing $(N-1)$ parameters, λ_i, such that

$$y(x_i) = \lambda_i, \qquad 1 \leq i \leq N-1.$$

2. For each $i = 1, 2, \ldots, N$, use the HAM to solve independently the BVPs

$$y'' = f_i(x, y, y'), \qquad x \in [x_{i-1}, x_i],\ y(x_{i-1}) = \lambda_{i-1} \text{ and } y(x_i) = \lambda_i, \quad (4.31)$$

to get the local solutions $y_i(x; \lambda_{i-1}, \lambda_i)$ for each subinterval $[x_{i-1}, x_i]$. Here, $\lambda_0 = \alpha$ and $\lambda_N = \beta$. Note that we explicitly showed the dependence of y_i of λ_{i-1} and λ_i by writing $y_i(x; \lambda_{i-1}, \lambda_i)$.

3. Next satisfy the conditions $y'(x_i^-) = y'(x_i^+)$ for each $1 \leq i \leq N-1$, since $y(x_i^-) = y(x_i^+) = \lambda_i$ is already satisfied.

Step 3 produces a system of algebraic equations in the unknowns λ_i:

$$y_i'(x_i; \lambda_{i-1}, \lambda_i) = y_{i+1}'(x_i; \lambda_i, \lambda_{i+1}), \qquad i = 1, \ldots, N-1, \quad (4.32)$$

which can be solve by any suitable method.

We should mention that in our numerical simulations, we used the **NSolve** built-in function in Mathematica to solve (4.32) for λ_i, $i = 1, 2, \ldots, N-1$. We should note, here, that the **NSolve** built-in function in Mathematica gives all possible *real* solutions (we can always restrict **NSolve** to give only real solutions). The question now is: which set of solution is the right one, if more than one is found? One way is to choose the solution set that minimizes the residual

$$\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (y_i''(x) - f_i(x, y(x), y'(x))^2 dx.$$

In the next Chapter we will apply the HAM to solve for the velocity of the fluid flow through multi-layer porous media.

Chapter 5: Application to Fluid Flow Through Multi-layer Porous Media

In this Chapter, we consider an application of multi-layer boundary value problems to a physical problem. The physical problem is about fluid flow through multi-layer porous media. The purpose of the work in this chapter is to apply the homotopy analysis method to solve for the fluid velocity.

5.1 Introduction

The study of fluid flow through porous media has attracted many researchers [21]-[28]. This is due to its importance. Examples of such fluid flows are: flow of underground water through different subsurface layers and the flow of gas though different media. In multi-layer flows, the media consists of a number of porous layers, each with different characteristics. Each layer is characterized by its porosity. In Fig. 5.1 we display a sample 4 layer porous media for visualisation.

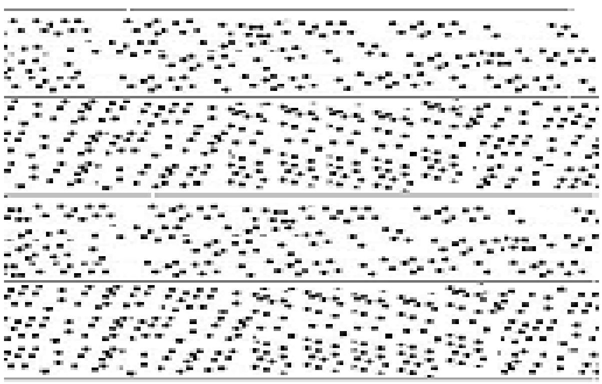

Figure 5.1: Multi-layer porous media

It is assume that the top and bottom layers are bounded by solid walls. The mathematical governing equations of the flow in each channel is given by either the

Darcy-Lapwood-Forchheimer-Brinkman (DFB) model [23],

$$\frac{d^2u}{dy^2} = ReC + \frac{u}{k} + \frac{ReC_d}{\sqrt{k}}u^2, \tag{5.1}$$

or by the Darcy-Lapwood-Brinkman (DLB) model,

$$\frac{d^2u}{dy^2} = ReC + \frac{u}{k}, \tag{5.2}$$

where $u(y),\ a \leq y \leq b$, is the velocity of the fluid and $y = a$ corresponds to the bottom solid and $y = b$ corresponds to the top solid wall. The various parameters in Eq. (5.1) (and in Eq. (5.2)) are physical parameters defined below [23]

1. $Re = \rho U_\infty L/\mu$ is the Reynolds number with ρ is the fluid density, U_∞ the free-stream characteristic velocity, μ is the fluid viscosity, and L is the channel characteristic length.

2. k is the permeability of the porous channel.

3. C_d is the drag coefficient.

4. $C < 0$ is a dimensionless pressure gradient.

For the sake of computations, we assume $a = -1$ and $b = 1$. The upper and lower layers are bounded by solid *impermeable* wall. This is what is known physically as the non-slip condition. This implies that $u = 0$ at the solid boundaries $y = -1$ and $y = 1$. At the interface regions y_i, between the different layers, the velocity $u(y)$ satisfies smoothness conditions, i.e.,

$$u(y_i^-) = u(y_i^+) \text{ and } u'(y_i^-) = u'(y_i^+). \tag{5.3}$$

Our goal in this chapter to investigate the use of the homotopy analysis method for multi-layer BVPs to solve for the velocity profile, $u(y)$, of the fluid. In the next section, we drive the homotopy solution to the problem for a general number N of channels. Then in section 5.3, we illustrate the method using a number of settings.

5.2 Derivation of the HAM Solution for Multi-layer Porous Media

Consider a multi-layer porous media consisting of N channels, $i = 1, 2, \ldots, N$. Each channel, i, situated at $y_{i-1} \leq y \leq y_i$, is characterised by its permeability k_i and is governed by the DFB model

$$\frac{d^2u}{dy^2} = ReC + \frac{u}{k_i} + \frac{ReC_d}{\sqrt{k_i}} u^2, \qquad y_{i-1} \leq y \leq y_i, \tag{5.4}$$

where $y = y_i$ is the interface between channel i and channel $i+1$. Note that $y_0 = -1$ and $y_N = 1$, which correspond to the bottom and top solid walls, respectively.

The nonslip-conditions at the outer walls ($y = -1$ for channel 1 and $y = 1$ for channel N) imply that $u(\pm 1) = 0$. The multi-layer BVP for the problem becomes

$$u'' = a_i + b_i u + c_i u^2, \qquad y_{i-1} \leq y \leq y_i, \qquad 1 \leq i \leq N, \tag{5.5}$$

with the BCs

$$\begin{aligned} &u(-1) = 0,\ u(1) = 0, \qquad \text{(Non-slip condition)} \\ &\left.\begin{aligned} u(y_i) &= u(y_{i+1}), \\ u'(y_i) &= u'(y_{i+1}), \end{aligned}\right\} 1 \leq i \leq N,\ \ \text{(Smoothness condition)} \end{aligned} \tag{5.6}$$

where we have used the notations $a_i = ReC,\ b_i = \frac{1}{k_i}$ and $c_i = \frac{ReC_d}{\sqrt{k_i}}$.

To derive the HAM solution to problem (5.5)– (5.6), first rewrite (5.5) in operator form as $\mathscr{A}_i(u(y)) = 0$, we see that

$$\mathscr{A}_i(u) = u'' - a_i - b_i u - c_i u^2 = 0 \tag{5.7}$$

As described in chapter 4, we introduce $N-1$ parameters λ_i to represent the velocity at the interface y_i, i.e., $u(y_i)=\lambda_i$. Next, we solve (5.7), using HAM, for each channel i, using the appropriate BCs. For channel 1, we solve (5.7) with BCs $u(-1)=0$ and $u(y_1)=\lambda_1$. For channel N, we solve (5.7) with BCs $u(y_{N-1})=\lambda_{N-1}$ and $u(1)=0$, and for other channels, $2\le i\le N-1$, we solve (5.7) with BCs $u(y_{i-1})=\lambda_{i-1}$ and $u(y_i)=\lambda_i$.

For each channel, we choose the linear operator

$$\mathscr{L}_i(u)=u''-b_iu. \tag{5.8}$$

and hence the nonlinear operators $\mathscr{N}_i(u)=-a_i-c_iu^2$.

According to the HAM method, the solution of channel i, denoted here by $u^{(i)}(y)$, is given by

$$u^{(i)}(y)=u_0^{(i)}(y)+\sum_{m=1}^{\infty}u_m^{(i)}(y)$$

where $u_0^{(i)}(y)$ is the solution of $\mathscr{L}_i(u)=0$ with BCs $u_0(y_{i-1})=\lambda_{i-1}$ and $u_0(y_i)=\lambda_i$. It is easy to see that

$$u_0^{(i)}(y)=C_{i,1}e^{\sqrt{b_i}\,y}+C_{i,2}e^{-\sqrt{b_i}\,y},$$

with

$$\begin{aligned}
C_{i,1} &= \frac{\lambda_i e^{\sqrt{b_i}y_i}-\lambda_{i-1}e^{\sqrt{b_i}y_{i-1}}}{e^{2\sqrt{b_i}y_i}-e^{2\sqrt{b_i}y_{i-1}}},\\
C_{i,2} &= \frac{e^{\sqrt{b_i}y_i+\sqrt{b_i}y_{i-1}}\left(\lambda_i e^{\sqrt{b_i}y_{i-1}}-\lambda_{i-1}e^{\sqrt{b_i}y_i}\right)}{e^{2\sqrt{b_i}y_{i-1}}-e^{2\sqrt{b_i}y_i}}.
\end{aligned}$$

The other solution components $u_m^{(i)}(y)$ are given by

$$\mathscr{L}_i(u_m^{(i)}(y))=(1+\hbar)\mathscr{L}_i(u_{m-1}^{(i)}(y))+\frac{\hbar}{(m-1)!}\frac{d^{m-1}}{dp^{m-1}}[\mathscr{N}_i(u^{(i)}(y,p)]\big|_{p=0},\ m\ge 1, \tag{5.9}$$

with homogeneous BCs, $u_m^{(i)}(y_{i-1})=u_m^{(i)}(y_i)=0$. In the following, for ease of notations, we will suppress the argument y from $u_m^{(i)}(y)$ and denote it $u_m^{(i)}$.

Since each $\mathscr{N}_i$ in (5.9) is a function of u only, we can rewrite (5.9), for $m \geq 2$, using Bell polynomials (see Proposition 3.20 and Eq. (3.22)) as

$$\begin{aligned}\mathscr{L}_i(u_m^{(i)}) &= (1+\hbar)\mathscr{L}_i(u_{m-1}^{(i)}) \\ &\quad + \frac{\hbar}{(m-1)!}\sum_{k=1}^{m-1}\mathscr{N}_i^{(k)}(u_0^{(i)})B_{m-1,k}(u_1^{(i)}, 2!u_1^{(i)}, \ldots, (m-k)!u_{m-k}^{(i)}), \quad m \geq 2,\end{aligned} \tag{5.10}$$

In our case, the nonlinear operators $\mathscr{N}_i(u) = -a_i - c_i u^2$ are quadratic in u so that $\mathscr{N}_i'(u) = -2c_i u$, $\mathscr{N}_i''(u) = -2c_i$, and $\mathscr{N}_i^{(k)} = 0$ for $k \geq 3$. This simplies the HAM iteration formula (5.10) as follows:

$$\mathscr{L}_i(u_1^{(i)}) = (1+\hbar)\underbrace{\mathscr{L}_i(u_0^{(i)})}_{=0} + \hbar\mathscr{N}_i(u_0^{(i)}) = -\hbar(a_i + c_i(u_0^{(i)})^2) \tag{5.11}$$

$$\begin{aligned}\mathscr{L}_i(u_2^{(i)}) &= (1+\hbar)\mathscr{L}_i(u_1^{(i)}) + \hbar\mathscr{N}_i'(u_0^{(i)})B_{1,1}(u_1^{(i)}) \\ &= (1+\hbar)\mathscr{L}_i(u_1^{(i)}) - 2\hbar c_i u_0^{(i)} u_1^{(i)}\end{aligned} \tag{5.12}$$

and for $m \geq 3$,

$$\begin{aligned}\mathscr{L}_i(u_m^{(i)}) &= (1+\hbar)\mathscr{L}_i(u_{m-1}^{(i)}) + \hbar\mathscr{N}_i'(u_0^{(i)})B_{m-1,1} + \hbar\mathscr{N}_i''(u_0^{(i)})B_{m-1,2} \\ &= (1+\hbar)\mathscr{L}_i(u_{m-1}^{(i)}) - 2c_i\hbar u_0^{(i)} u_{m-1}^{(i)} + -2c_i\hbar B_{m-1,2}.\end{aligned} \tag{5.13}$$

In our simulations, we used (5.11)–(5.13).

Once all $u_m^{(i)}$ have been solve for up to an approximation order M, each solution $u^{(i)}$ is approximated by

$$u^{(i)}(y) \approx u^{(i,M)}(y) = u_0^{(i)}(y) + \sum_{m=1}^{M} u_m^{(i)}(y). \tag{5.14}$$

Note that each $u^{(i,M)}(y)$ has the introduced parameters λ_{i-1} and λ_i which have to be solved for such that the smoothness conditions across y_i are satisfied, that is,

$$\frac{d}{dy}u^{(i,M)}(y_i) = \frac{d}{dy}u^{(i+1,M)}(y_i), \qquad i = 1, \ldots, N-1. \tag{5.15}$$

This gives a system of nonlinear algebraic equations of size $N-1$ in the unknown parameters $\lambda_i,\ i=1,\dots,N-1$. As mentioned in Chapter 4, we have relied on the Mathematica built-in function **NSolve** to find all possible solutions. We should re-mention here that Mathematica **NSolve** gives a range of possible positive solutions for all of which the solution satisfy the boundary condition (5.6).

To pick the suitable solution set for $\{\lambda_i,\ i=1,\dots,N-1\}$, we used a residual measure. The residual of each solution $u^{(i,M)}(y;\lambda_{i-1},\lambda_i)$ is taken as

$$R_i = \int_{y_{i-1}}^{y_i} \left(\mathscr{A}_i[u^{(i,M)}(y;\lambda_{i-1},\lambda_i)]\right)^2 dy. \tag{5.16}$$

The optimal solution set $\{\lambda_i^*,\ i=1,\dots,N-1\}$ is then chosen such that

$$\sum_{i=1}^{N} R_i^2 \qquad \text{is the smallest.} \tag{5.17}$$

5.3 Examples

In this section, we apply the homotopy analysis method as described in the previous section to a two-, three-, four-, and five-channel porous media problems.

5.3.1 Two-Channel Problem

As a first example, we consider the two-channel problem, where both top and bottom channels are modelled by the DFB. The permeability of the top channel is set to $k_t = 1$. We consider 4 cases where the permeability of the bottom channel is $k_b = 0.01,\ 0.1,\ 0.4$, and 1. The following parameter values have been used:

$$Re = 10,\ C = -1,\ C_d = 0.055.$$

The velocity profiles of the fluid flow for all cases is depicted in Figure 5.2 using order 8 HAM. The velocity, $u(0)$, and shear stress, $u'(0)$, for each case is displayed in Table 5.1. Table 5.2 displays the residuals of the order HAM approximate solution.

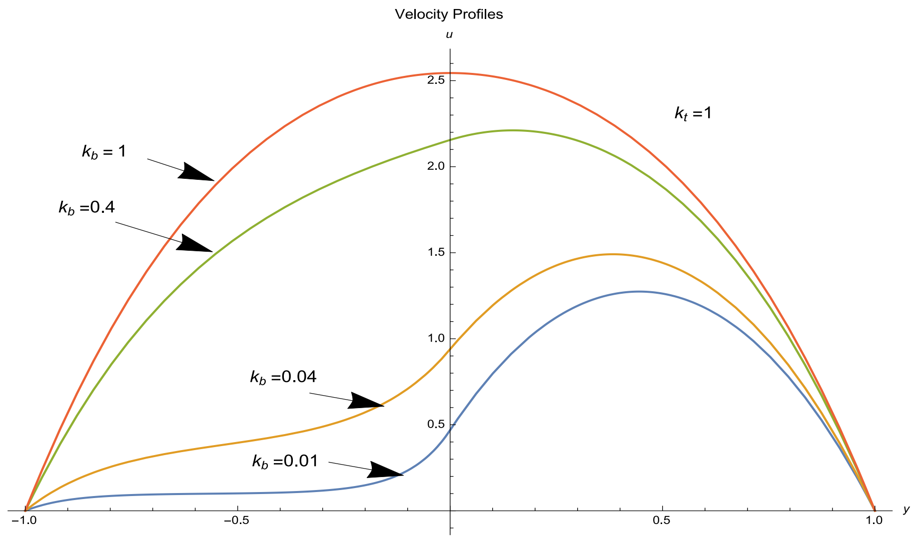

Figure 5.2: Velocity profile of the two channel problem, $k_b = 0.01$, 0.04, 0.4, 1 and $k_t = 1$ with order 8 HAM

k_b	HAM order 4		HAM order 5		HAM order 6	
	$u(0)$	$u'(0)$	$u(0)$	$u'(0)$	$u(0)$	$u'(0)$
0.01	0.467699	3.72726	0.469067	3.74119	0.468905	3.73953
0.04	0.937248	2.9574	0.939343	2.96784	0.938816	2.96513
0.4	2.15714	0.760447	2.15822	0.761982	2.15528	0.762155
1	2.54715	9.05×10^{-13}	2.54697	-5.0×10^{-10}	2.5444	-3.62×10^{-10}

Table 5.1: Values of $u(0)$ and $u'(0)$ vs. HAM order for the two channel example

k_b	HAM Residue		
	Order 4	Order 5	Order 6
0.01	1.52×10^{-3}	9.52×10^{-5}	1.98×10^{-5}
0.04	9.69×10^{-4}	1.71×10^{-4}	8.2×10^{-6}
0.4	2.10×10^{-4}	5.49×10^{-4}	3.05×10^{-5}
1	1.68×10^{-4}	2.45×10^{-4}	1.71×10^{-5}

Table 5.2: Residue of the numerical solution vs. HAM order for the two channel example

It can be seen from the above results of this preliminary example that the HAM is accurate in resolving the velocity of the fluid as well as the shear stress at the interface, namely $u(0)$ and $u'(0)$. From Table 5.1 we see that $u(0)$ and $u'(0)$ are converging as the order increases. From Table 5.2 we see that the residue is decreasing as the order increases which means that the approximate solution is accurate. Just one remark in this example is when $k_b = 0.1$, it seems that the residue is not decreasing as fast as the other three cases.

5.3.2 Four Channel-Problem

We now consider an example of four channels with the following permeability values:

$$k_1 = 0.01,\ k_2 = 0.16,\ k_3 = 0.49,\ k_4 = 1. \tag{5.18}$$

We used the same parameter values as in the two channel case $R_e = 10,\ C = -1$ and $C_d = 0.055$. The velocity profile of the fluid is shown in Figure 5.3. The red dots correspond to the interface points between the channels.

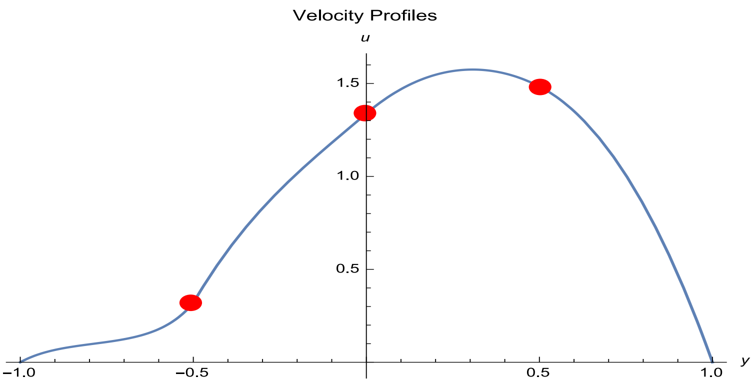

Figure 5.3: Velocity profile of the four channel problem with $k_1 = 0.01, k_2 = 0.16, k_3 = 0.49, k_4 = 1$ with order 5 HAM

The values of the fluid velocity, $u(y_i)$, and shear stress,$u'(y_i)$, at the interface points, $y_i = -1/2,\ 0,\ 1/2$, are displayed in Table 5.3 for HAM order M=3, 4 and 5. The residue of the approximate solution for HAM order $M = 3,4,5$, are also displayed in Table 5.3. It can be seen from the residue values in Table 5.3 that the HAM can produce an accurate approximate solution even for low order. In our example for order $M = 5$ the residue is of the order 10^{-7}.

	$u(-0.5)$	$u'(-0.5)$	$u(0)$	$u'(0)$	$u(0.5)$	$u'(0.5)$	Residue
$M=3$	0.319876	2.23683	1.34	1.59203	1.48568	-0.957193	7.9×10^{-4}
$M=4$	0.319661	2.23615	1.33801	1.59161	1.48359	-0.957381	2.4×10^{-5}
$M=5$	0.319704	2.23655	1.3384	1.59168	1.48379	-0.958273	1.4×10^{-7}

Table 5.3: Values of $u(y_i)$ and $u'(y_i)$ for $y_i=-1/2,0,1/2$, vs. HAM order for the four channel example

Chapter 6: Conclusion

In this thesis, we examined a number of numerical techniques to solve boundary value problems. We presented mainly three techniques: the finite difference technique, the shooting method, and the homotopy analysis method (HAM). We considered in details the homotopy analysis method. We also, consider multi-layer boundary value problems because of their applicability in fluid flow through porous channels. We applied the HAM to solve two cases of fluid flow. The results were accurate which show the efficiency and accuracy of the HAM. We should mention that in some cases, one has to go for larger number of iteration in the HAM to get an accurate solution. In our examples, we considered channels with constant permeability k. A future research problem is to consider channels with variable permeability.

References

[1] R. L. Burden and J. D. Faires, *Numerical Analysis*, Brooks/Cole, Cengage Learning, (2011).

[2] E.T. Bell, *Exponential polynomials*, Ann. Math. 35 (1934), pp. 258–277.

[3] P. Hartman, *Ordinary differential equations*, Wiley, New York, (1965).

[4] E.A. Al-Said and M.A. Noor, *Quartic spline method for solving fourth order obstacle boundary value problems*, J. Comput. Appl. Math. 143 (2002), pp. 107–116.

[5] A.M. Wazwaz, *The numerical solution of special fourth-order boundary value problems by the modified decomposition method*, Int. J. comput. Math., 79 (2002), pp. 345–356.

[6] D. Jiang, H. Liu and X. Xu, *Nonresonant singular fourth-order boundary value problems*, Appl. Math. Lett., 18 (2005), pp. 69–75.

[7] E. T. Swartz and R. O. Pohl, *Thermal resistance at interfaces*, Appl. Phys. Lett., 51 (1987).

[8] M. Mikhailov and M. Ozisik, *Unified analysis and solutions of heat and mass diffusion*, Dover Publications, New York, (1994).

[9] L. Jackson and K. Schrader,*Comparison theorems for nonlinear differential equations*, Journal of differential equations, 3 (1967), pp. 248–255 .

[10] K. Schrader,*Boundary-value problems for second-order ordinary differential equations*, Journal of differential equations, 3 (1967), pp. 403–413

[11] K. Schrader,*Solutions of second order ordinary differential equations*, Journal of differential equations, 4 (1968), pp. 510–518.

[12] P. Bailey, L. Shampine and P. Waltman, *The first and second boundary value problems for nonlinear second order differential equations*, Journal of differential equations, 2 (1966), pp. 399–411.

[13] H. B. Keller, *Numerical Methods for Two Point BoundaryValue Problems*, Blaisdell Publishing Company, Waltham, Massachusetts, (1968).

[14] K. W. Schrader, *Existence Theorems for Second Order Boundary Value Problems*, Journal of differential equations, 5 (1969), pp. 572–584.

[15] A. Lasota andZ. Opial, *On the existence and uniqueness of solutions of a boundary value problem for an ordinary, second order differential equation*, Colloq. Math., 18 (1967), pp. l–5.

[16] D. Barr and P. Miletta,*A Necessary and Sufficient Condition for Uniqueness of Solutions to Two Point Boundary Value Problems*, Pacific Journal of Mathematics, 57(2), (1975).

[17] R. Aitbayev. *Existence and Uniqueness for a Two-Point Interface Boundary Value Problem*, 242 (2013), pp. 1–12.

[18] P. B. Bailey and L. F. Shampine,*Existence from Uniqueness for Two Point Boundary Value Problems*, J. Mathematical Analysis and Applications, 25 (1969), pp. 569–574.

[19] R.P. Agarwal, *Boundary Value Problems for High Ordinary Differential Equations*, World Scientific, Singapore, (1986).

[20] H. Pasic, *Multipoint Boundary-Value Solution of Two-Point Boundary-Value Problems*, J. Optimization Theory and Application, 100, No. 2 (1999), pp. 397–416.

[21] Z. Chen, G. Huan and Y. Ma, *Computational methods for multiphase flows in porous media*, Soc. Indust. Appl. Math., (2006).

[22] F.M. Allan and M.H. Hamdan, *Fluid mechanics of the interface region between two porous layers*, Appl. Math. Comput., 128 (2002), pp. 37–43.

[23] R.A. Ford, and M.H. Hamdan, *Coupled Parallel Flow through Composite Porous Layers*, Applied Mathematics and Computation, 97 (1998), pp. 261–271.

[24] G.S. Beavers and D.D. Joseph, *Boundary conditions at a naturally permeable wall*, J. Fluid Mech., 30 (1967), pp. 197–207.

[25] F. Allan, M. Hajji and M. Anwar, *The characteristics of fluid flow through multilayer porous media*, J. Applied Mechanics,76 (2009).

[26] K. Vafai and S. Kim, *Fluid Mechanics of the Interface Region between a Porous Medium and a Fluid Layer- An Exact Solution*, International Journal of Heat and Fluid Flow, 11 (1990), pp. 254–256.

[27] F.M. Allan and M.H. Hamdan,*Arbitrary Finite Difference Schemes for Coupled Parallel Flow over Porous Fluids*, 5th WSEAS Int. Conf. on FLUID MECHANICS, 8 (2008), pp. 25–27.

[28] M.A. Hajji, *A numerical scheme for multi-point special boundary-value problems and application to fluid flow through porous layers*, Applied Mathematics and Computation, 217(12) (2009), pp. 5632–5642.

[29] S. Liao, *Homotopy Analysis Method in Nonlinear Differential Equations*, Higher Education Press, Beijing and Springer-Verlag Berlin Heidelberg, (2012), pp. 15–25.

[30] S. Liao, *Notes on the homotopy analysis method: Some definition and theorems*, Commun. Nonlinear Sci. Numer. Simulat., 14(4) (2010), pp. 983–997.

[31] S. Liao, *An optimal homotopy-analysis approach for strongly nonlinear differential equations*, Commun. Nonlinear Sci. Numer. Simulat., 15 (2010), pp. 2003–2016.

[32] F.M. Allan, *Derivation of the Adomian decomposition method using the homotopy analysis method*, Applied Mathematics and Compututation, 190 (2007), pp. 6–14.

[33] G. Adomian, *Nonlinear stochastic differential equations*, J. Math. Anal. Applic., 55 (1976), pp. 441–452.

[34] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic Publishers, Boston (1994).

[35] H.B. Keller,*Existence Theory for Two Point Boundary Value Problems*, Communicated By H. A. Antosiewicz,18 (1966).