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Q-Series with Applications to Binomial Coefficients, Integer Partitions and Sums of Squares

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Q-SERIES WITH APPLICATIONS TO BINOMIAL COEFFICIENTS, INTEGER PARTITIONS, AND SUMS OF SQUARES

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This thesis is submitted in partial fulfilment of the requirements for the degree of Master of Science in Mathematics

Under the Supervision of Dr. Mohamed El Bachraoui

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Declaration of Original Work

I, Amna Abdul Baset Saif Saif Al Suwaidi, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled *"Q-series with Applications to Binomial Coefficients, Integer Partitions, and Sums of Squares"*, hereby, solemnly declare that this thesis is my own original research work that has been done and prepared by me under the supervision of Dr. Mohamed El Bachraoui, in the College of Science at UAEU. This work has not previously been presented or published, or formed the basis for the award of any academic degree, diploma or a similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my thesis have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this thesis.

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Abstract

In this report we shall introduce q-series and we shall discuss some of their applications to the integer partitions, the sums of squares, and the binomial coefficients. We will present the basic theory of q-series including the most famous theorems and rules governing these objects such as the q-binomial theorem and the Jacobi's triple identity. We shall present the q-binomial coefficients which roughly speaking connect the binomial coefficients to q-series, we will give the most important results on q-binomial coefficients, and we shall provide some of our new results on the divisibility of binomial coefficients. Moreover, we shall give some well-known applications of q-series to sums of two squares and to integer partitions such as Ramanujan's modulo 5 congruence.

Keywords: q-series, q-binomial coefficients, binomial coefficients, integer partitions, sums of squares, q-analogues.

Title and Abstract (in Arabic)

متوالية $q-$ مع تطبيقات لمعاملات ذات الحدين، تقسيم الاعداد الصحيحة، و مجموع المربعات

الملخص

سنقدم النظريات الأساسية لمتوالية $q-$ بما في ذلك اهم النظريات المشهورة والقواعد التي تحكم هذه المواضيع مثل نظرية ذات الحدين $q-$ و متطابقة جاكوبي الثلاثية. وأيضا سنقدم معاملات $q-$ ذات الحدين التي لها علاقة بمعاملات ذات الحدين ومتوالية $q-$ ، و أهم النتائج على ذلك، و بعض من نتائجنا الجديدة على قابلية القسمة لمعاملات ذات الحدين. وعلاوة على ذلك، سنستعرض بعض التطبيقات المعروفة لمتوالية $q-$ مثل مجموع المربعين و تقسيم الاعداد الصحيحة مثل تطابق رامانوجان لمودولو 5 .

مفاهيم البحث الرئيسية: متوالية $q-$ ، معاملات $q-$ ذات الحدين، معاملات ذات الحدين، تقسيم الاعداد الصحيحة، مجموع المربعات، نظائر $q-$.

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Dedication

To my beloved parents and family

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Chapter 1: Introduction

The study of q-series has a long history which dates back into the famous mathematicians Euler and Gauss from the 18th century. The first result of Euler indicating his interest in q-series is referred to as Euler's pentagonal theorem and it states that

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty := \prod_{j=1}^{\infty}(1-q^j).$$

Euler was also first to notice the connection between q-series and the function $p(n)$ of integer partitions as he proved that

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \prod_{j=1}^{\infty}\frac{1}{(1-q^j)}.$$

As to Gauss, among his results in this direction we find

$$\sum_{n=0}^{\infty}(-1)^n q^{n^2} = \frac{(q;q)_\infty}{(-q;q)_\infty} = \prod_{j=1}^{\infty}\frac{1-q^j}{1+q^j},$$

which also has an interpretation in terms of integer partitions. Famous mathematicians from the 19th and 20th centuries who used q-series in their research include Jacobi, Riemann, Rogers, Jackson, and Ramanujan. Many of the work on q-series which has been provided by the mathematicians in the past was for the purpose of application to other branches of mathematics such as elliptic functions, differential equations, and combinatorics. The real study of q-series and hypergeometric series for their own as independent research topics started in the 20th century with very important contributions of Askey and Andrews along with their students and collaborators. Currently, q-series and hypergeometric series form a very active research area which attracts many mathematicians and researchers around the world. They have applications in many branches of mathematics and physics. In this report we shall introduce q-series and we shall discuss some of their applications to the integer partitions, the sums of squares, and the binomial coefficients. In chapter 1, we will present the basic theory of q-series including the most famous theorems and rules governing these objects such as the q-binomial theorem and the Jacobi's triple identity. To prepare the ground for the following chapters, we shall also briefly discuss integer partitions, the divisor functions, and the sums of squares. In chapter 2 we shall present the q-binomial coefficients which roughly speaking connect the binomial coefficients to q-series, we will give the most important results on q-binomial coefficients, and we shall provide some of our new results on the divisibility of binomial coefficients. Finally in chapter 3, we shall give some well-known applications of q-series to sums of two squares and to integer partitions such as Ramanujan's modulo 5 congruence.

Chapter 2: *Q*-Series, their properties, and some related arithmetic functions

In this chapter we will introduce the notion of q-series along with their properties. These properties include the q-analogue of the binomial theorem and the Jacobi's identities. We will also introduce briefly the functions which we are going to study through q-series in the following chapters. The results in this chapter are all known and can be found in the standard literature on q-series such as Andrews [1], Berndt [3], and Gasper and Rahman [7].

2.1 *Q*-series

Definition 2.1.1. For any fixed complex number q, any complex number a, and any non-negative integer n we let

$$(a;q)_0 = 1 \text{ and } (a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}) = \prod_{j=0}^{n-1}(1-aq^j).$$

Accordingly, we let

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j) = \lim_{n\to\infty}(a;q)_n.$$

If no confusion arises, we shall sometimes write $(a)_n$ and $(a)_\infty$ rather than $(a;q)_n$ and $(a;q)_\infty$ respectively.

A *q-series* is any series which involves expressions of the form $(a;q)_n$ and $(a;q)_\infty$.

Theorem 2.1.1. *Let q and z be complex numbers such that $|q|$ and $|z|<1$. Then*

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n=\frac{(az;q)_\infty}{(z;q)_\infty}. \tag{2.1}$$

Proof. Let

$$F(z):=\frac{(az;q)_\infty}{(z;q)_\infty}=\sum_{n=0}^{\infty}A_nz^n. \tag{2.2}$$

It is clear that $F(z)$converges uniformly on compact subset of $|z|<1$, and so it represents an analytic function on $|z|<1$. We have

$$\begin{aligned}(1-z)F(z)&=(1-z)\cdot\frac{(az;q)_\infty}{(z;q)_\infty}\\&=(1-z)\cdot\frac{(1-az)(1-azq)(1-azq^2)\cdots}{(1-z)(1-zq)(1-zq^2)\cdots}\\&=\frac{(1-az)(1-azq)(1-azq^2)\cdots}{(1-zq)(1-zq^2)(1-zq^3)\cdots}\\&=(1-az)\cdot\frac{(aqz;q)_\infty}{(qz;q)_\infty}\\&=(1-az)F(qz).\end{aligned}$$

Then, the following are equivalent

$$\begin{gathered}(1-z)\sum_{n=0}^{\infty}A_nz^n=(1-az)\sum_{n=0}^{\infty}A_nq^nz^n,\\ \sum_{n=0}^{\infty}A_nz^n-\sum_{n=0}^{\infty}A_nz^{n+1}=\sum_{n=0}^{\infty}A_nq^nz^n-\sum_{n=0}^{\infty}A_naq^nz^{n+1},\\ \sum_{n=0}^{\infty}(A_n-A_nq^n)z^n=\sum_{n=0}^{\infty}(A_n-A_naq^n)z^{n+1}.\end{gathered}$$

That is,

$$A_n - A_n q^n = A_{n-1} - A_{n-1} a q^{n-1},$$

or equivalently

$$A_n(1-q^n) = A_{n-1}(1-aq^{n-1}),$$

and so,

$$A_n = \frac{1-aq^{n-1}}{1-q^n} A_{n-1}, \; n \geq 1. \tag{2.3}$$

Iterating (2.3) and using the fact that $A_0 = 1$, we get

$$A_n = \frac{(a;q)_n}{(q;q)_n}, \;\; n \geq 0,$$

which substituted in (2.2) gives the result. □

Note that setting in the previous theorem $a = q^a$ and then taking the limits as $q \to 1$ one gets

$$\sum_{n=0}^{\infty} \frac{a(a+1)\cdots(a+n-1)}{n!} z^n = \frac{1}{(1-z)^a},$$

which is the classical binomial theorem. This is the reason why Theorem 2.1.1 is often referred to as *the q-binomial theorem*. The following result is due to Euler.

Corollary 2.1.2. *Let q and z be complex numbers such that $|q| < 1$.*

(a) If $|z| < 1$*, then*

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z)_\infty}. \tag{2.4}$$

(b) If $|z| < \infty$*, then*

$$\sum_{n=0}^{\infty} \frac{(-z)^n q^{\frac{n(n-1)}{2}}}{(q;q)_n} = (z)_\infty. \tag{2.5}$$

Proof. (a) Setting $a = 0$ in (2.1) we get,

$$\sum_{n=0}^{\infty} \frac{(0;q)_n}{(q;q)_n} z^n = \frac{(0 \cdot z;q)_\infty}{(z;q)_\infty},$$

Then,

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z)_\infty}.$$

(b) To get this part replace a by a/b and z by bz in (2.1).

$$\sum_{n=0}^{\infty} \frac{(a/b;q)_n}{(q;q)_n} (bz)^n = \frac{(az)_\infty}{(bz)_\infty}. \tag{2.6}$$

Notice that,

$$
\begin{aligned}
\lim_{b\to 0}(a/b;q)_n b^n &= \lim_{b\to 0}\left(1-\frac{a}{b}\right)\left(1-\frac{aq}{b}\right)\cdots\left(1-\frac{aq^{n-1}}{b}\right)b^n \\
&= \lim_{b\to 0}(b-a)(b-aq)\cdots\left(b-aq^{n-1}\right) \\
&= (-a)(-aq)\left(-aq^2\right)\cdots\left(-aq^{n-1}\right) \\
&= (-a)^n q^{1+2+\cdots+n-1} \\
&= (-a)^n q^{\frac{n(n-1)}{2}}.
\end{aligned}
$$

Now equality (2.5) follows by letting $b \to 0$ and setting $a = 1$. □

Definition 2.1.2. Ramanujan's general theta function $f(a,b)$ is defined by

$$
f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{2.7}
$$

We list some properties of the theta function in the following theorem.

Theorem 2.1.3.

(a) $f(a,b) = f(b,a)$,

(b) $f(1,a) = 2f(a,a^3)$,

(c) $f(-1,a) = 0$,

(d) $f(a,b) = a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n}), \quad n \in \mathbb{Z}$.

Proof. (a) This part follows by letting $m = -n$ in the series below:

$$\begin{aligned} f(a,b) &= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \\ &= \sum_{n=-\infty}^{\infty} b^{-n(-n+1)/2} a^{-n(-n-1)/2} \\ &= \sum_{m=-\infty}^{\infty} b^{m(m+1)/2} a^{m(m-1)/2} \\ &= f(b,a). \end{aligned}$$

(b) We have

$$\begin{aligned} f(1,a) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} &= \sum_{n=-\infty}^{\infty} a^{\frac{2n(2n+1)}{2}} + \sum_{n=-\infty}^{\infty} a^{\frac{(2n+1)(2n+2)}{2}} \\ &= \sum_{n=-\infty}^{\infty} a^{n(2n+1)} + \sum_{n=-\infty}^{\infty} a^{(n+1)(2n+1)} \end{aligned}$$

Changing the variable in the last summation with $m := n+1$, we find,

$$f(1,a) = \sum_{n=-\infty}^{\infty} a^{n(2n+1)} + \sum_{m=-\infty}^{\infty} a^{m(2m-1)}$$

Next expand $f(a,a^3)$ and $f(a^3,a)$,

$$f(a,a^3) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} a^{\frac{3n(n-1)}{2}} = \sum_{n=-\infty}^{\infty} a^{2n^2-n} = \sum_{n=-\infty}^{\infty} a^{n(2n-1)}$$

and,

$$f(a^3,a) = \sum_{n=-\infty}^{\infty} a^{\frac{3n(n+1)}{2}} a^{\frac{n(n-1)}{2}} = \sum_{n=-\infty}^{\infty} a^{2n^2+n} = \sum_{n=-\infty}^{\infty} a^{n(2n+1)}$$

Now we deduce as $f(a,b) = f(b,a)$ by part (a) that,

$$f(1,a) = f(a^3,a) + f(a,a^3) = 2f(a,a^3).$$

(c) To prove this part, we work with modulo 4,

$$\begin{aligned}
f(-1,a) = &\sum_{n=-\infty}^{\infty} (-1)^{\frac{n(n+1)}{2}} a^{\frac{n(n-1)}{2}} \\
&= \sum_{n=-\infty}^{\infty} (-1)^{\frac{4n(4n+1)}{2}} a^{\frac{4n(4n-1)}{2}} + \sum_{n=-\infty}^{\infty} (-1)^{\frac{(4n+1)(4n+2)}{2}} a^{\frac{(4n+1)(4n)}{2}} \\
&+ \sum_{n=-\infty}^{\infty} (-1)^{\frac{(4n+2)(4n+3)}{2}} a^{\frac{(4n+2)(4n+1)}{2}} + \sum_{n=-\infty}^{\infty} (-1)^{\frac{(4n+3)(4n+4)}{2}} a^{\frac{(4n+3)(4n+2)}{2}} \\
&= \sum_{n=-\infty}^{\infty} a^{2n(4n-1)} - \sum_{n=-\infty}^{\infty} a^{2n(4n+1)} \\
&- \sum_{n=-\infty}^{\infty} a^{(2n+1)(4n+1)} + \sum_{n=-\infty}^{\infty} a^{(2n+1)(4n+3)}
\end{aligned}$$

We have,

$$2(-n)(4(-n)+1) = (-2n)(-4n+1) = 8n^2 - 2n = 2n(4n-1),$$

and so,

$$\sum_{n=-\infty}^{\infty} a^{2n(4n-1)} - \sum_{n=-\infty}^{\infty} a^{2n(4n+1)} = 0.$$

Similarly,

$$\begin{aligned}
(2(-n-1)+1)(4(-n-1)+1) &= (-2(n+1)+1)(-4(n+1)+1) \\
&= 8(n^2+2n+1) - 6(n+1) + 1 = 8n^2 + 10n + 3 \\
&= (2n+1)(4n+3),
\end{aligned}$$

and so,

$$\sum_{n=-\infty}^{\infty} a^{(2n+1)(4n+3)} - \sum_{n=-\infty}^{\infty} a^{(2n+1)(4n+1)} = 0.$$

(d) Let n be an integer. Then

$$\begin{aligned} a^{n(n+1)/2} b^{n(n-1)/2} f\left(a(ab)^n, b(ab)^{-n}\right) &= a^{n(n+1)/2} b^{n(n-1)/2} \\ &\sum_{m=-\infty}^{\infty} a^{((m(m+1)(n+1)-nm(m-1))/2} b^{(m(m+1)n+m(m-1)(1-n))/2} \\ &= \sum_{m=-\infty}^{\infty} a^{(n+m)(n+m+1)/2} b^{(n+m)(n+m-1)/2}. \end{aligned}$$

Now replace $n+m$ by k in the previous series to obtain the desired identity.

□

The following result is often referred to as *Jacobi's Triple Product Identity*.

Theorem 2.1.4. *Let q and z be complex numbers such that $z \neq 0$ and $|q| < 1$. Then*

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq;q^2)_\infty (-q/z;q^2)_\infty (q^2;q^2)_\infty. \tag{2.8}$$

Proof. Replace q by q^2 and z by $-zq$ in equation (2.5) to deduce that

$$\begin{aligned} (-zq;q^2)_\infty &= \sum_{n=0}^{\infty} \frac{(zq)^n (q^2)^{\frac{n(n-1)}{2}}}{(q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2;q^2)_n} \\ &= \sum_{n=0}^{\infty} z^n q^{n^2} \frac{(q^{2n+2};q^2)_\infty}{(q^2;q^2)_\infty} \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} z^n q^{n^2} (q^{2n+2};q^2)_\infty \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n^2} (q^{2n+2};q^2)_\infty. \end{aligned}$$

We start the sum from $n=-\infty$ in the last step, since $(q^{2n+2};q^2)_\infty=0$, when n is negative integer. In $(q^{2n+2};q^2)_\infty$ apply (2.5) again with q replaced by q^2 and with $z=q^{2n+2}$. Then,

$$\begin{aligned}
(-zq;q^2)_\infty &= \frac{1}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty} z^n q^{n^2}\sum_{r=0}^{\infty}\frac{(-1)^r(q^{2n+2})^r(q^2)^{\frac{r(r-1)}{2}}}{(q^2;q^2)_r}\\
&= \frac{1}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty} z^n q^{n^2}\sum_{r=0}^{\infty}\frac{(-1)^r q^{(2n+2)r+r^2-r}}{(q^2;q^2)_r}\\
&= \frac{1}{(q^2;q^2)_\infty}\sum_{r=0}^{\infty}\frac{(-1)^r q^r}{(q^2;q^2)_r}\sum_{n=-\infty}^{\infty} z^n q^{n^2} q^{2nr+r^2}\\
&= \frac{1}{(q^2;q^2)_\infty}\sum_{r=0}^{\infty}\frac{(-1)^r q^r z^{-r}}{(q^2;q^2)_r}\sum_{n=-\infty}^{\infty} z^{n+r} q^{(n+r)^2}\\
&= \frac{1}{(q^2;q^2)_\infty}\sum_{r=0}^{\infty}\frac{(-q/z)^r}{(q^2;q^2)_r}\sum_{n=-\infty}^{\infty} z^n q^{n^2}\\
&= \frac{1}{(q^2;q^2)_\infty(-q/z;q^2)_\infty}\sum_{n=-\infty}^{\infty} z^n q^{n^2},
\end{aligned}$$

when the last identity follows by (2.4) with z replaced by $-q/z$ and q replaced by q^2. Then we obtain:

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq;q^2)_\infty(q^2;q^2)_\infty(-q/z;q^2)_\infty.$$

□

Note that in Ramanujan's notation (2.7), the Jacobi's triple product identity takes the shape

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.9}$$

To see this, put $a = zq$ and $b = \frac{q}{z}$. Then,

$$\begin{aligned}
f(a,b) &= \sum_{n=-\infty}^{\infty} (zq)^{\frac{n(n+1)}{2}} (\frac{q}{z})^{\frac{n(n-1)}{2}} \\
&= \sum_{n=-\infty}^{\infty} z^{\frac{n(n+1)}{2} - \frac{n(n-1)}{2}} \cdot q^{\frac{n(n+1)}{2} + \frac{n(n-1)}{2}} \\
&= \sum_{n=-\infty}^{\infty} z^n q^{n^2} \\
&= (-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty \\
&= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty \ .
\end{aligned}$$

The following three functions are among the most common examples of theta functions:

Definition 2.1.3.

(a) $\varphi(q) := f(q,q) := \sum_{n=-\infty}^{\infty} q^{n^2}$,

(b) $\psi(q) := f(q,q^3) := \sum_{n=0}^{\infty} q^{n(n+1)/2}$,

(c) $f(-q) := f(-q,-q^2) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}$.

Corollary 2.1.5. *We have*

(a) $\varphi(q) = (-q,q^2)_\infty^2(q^2;q^2)_\infty$,

(b) $\psi(q) = \frac{(q^2,q^2)_\infty}{(q,q^2)_\infty}$,

(c) $f(-q) = (q,q)_\infty$.

Proof. (a) Letting $a = b = q$ in (2.9) gives:

$$\begin{aligned}
\varphi(q) = f(q,q) &= (-q;q^2)_\infty(-q;q^2)_\infty(q^2;q^2)_\infty \\
&= (-q;q^2)_\infty^2(q^2;q^2)_\infty.
\end{aligned}$$

(b) We have

$$\psi(q) = f(q,q^3) = (-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty,$$

by (2.9) applied to $a = q$ and $b = q^3$.

Next,

$$\begin{aligned}(-q;q^4)_\infty(-q^3;q^4)_\infty &= (1+q)(1+q^{4+1})(1+q^{2*4+1})(1+q^{3*4+1})\cdots \\ &\quad (1+q^3)(1+q^{4+3})(1+q^{2*4+3})(1+q^{3*4+3})\cdots \\ &= (1+q)(1+q^{2+1})(1+q^{2*2+1})(1+q^{3*2+1})\cdots \\ &= (-q;q^2)_\infty.\end{aligned}$$

Also,

$$\begin{aligned}(q^4;q^4)_\infty &= (1-q^4)(1-q^8)(1-q^{12})(1-q^{16})\cdots \\ &= (1+q^2)(1+q^4)(1+q^6)(1+q^8)\cdots \\ &\quad (1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots \\ &= (-q^2;q^2)_\infty(q^2;q^2)_\infty.\end{aligned}$$

Now combining gives:

$$\psi(q) = (-q;q^2)_\infty(-q^2;q^2)_\infty(q^2;q^2)_\infty.$$

Further,

$$\begin{aligned}(-q;q^2)_\infty(-q^2;q^2)_\infty &= (1+q)(1+q^{2+1})(1+q^{2*2+1})(1+q^{3*2+1})\cdots\\ &\quad (1+q^2)(1+q^{2+2})(1+q^{2*2+2})(1+q^{3*2+2})\cdots\\ &= (1+q)(1+q^2)(1+q^{2+1})(1+q^{2+2})\cdots\\ &= (-q;q)_\infty.\end{aligned}$$

Also, we have

$$\begin{aligned}(-q;q)_\infty &= (1+q)(1+q^2)(1+q^3)\cdots\\ &= \frac{(1-q^2)(1-q^4)(1-q^6)\cdots}{(1-q)(1-q^2)(1-q^3)\cdots}\\ &= \frac{(q^2;q^2)_\infty}{(q;q)_\infty} = \frac{(q^2;q^2)_\infty}{(q^2;q^2)_\infty(q;q^2)_\infty}\\ &= \frac{1}{(q;q^2)_\infty}.\end{aligned}$$

Finally, we find

$$\psi(q) = (-q;q)_\infty(q^2;q^2)_\infty = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

(c) To get this part, set $a=-q$, $b=-q^2$ in (2.9). Then, from Definition 2.1.3 part (3),

$$\begin{aligned}f(-q) = f(-q,-q^2) &= (q;q^3)_\infty(q^2;q^3)_\infty(q^3;q^3)_\infty\\ &= (1-q)(1-q^4)(1-q^7)(1-q^{10})\cdots\\ &\quad (1-q^2)(1-q^5)(1-q^8)(1-q^{11})\cdots\\ &\quad (1-q^3)(1-q^6)(1-q^9)(1-q^{12})\cdots\\ &= (q;q)_\infty.\end{aligned}$$

□

The formula in the following theorem is often called *Jacobi's identity*.

Theorem 2.1.6. *We have*

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} = (q;q)_\infty^3 \tag{2.10}$$

Proof. In (2.8), replace z by z^2q to deduce that

$$\sum_{n=0}^{\infty} z^{2n}q^{n^2+n} = (-z^2q^2;q^2)_\infty(-1/z^2;q^2)_\infty(q^2;q^2)_\infty. \tag{2.11}$$

Divide both sides of (2.11) by $1+1/z^2$ to find that

$$\begin{aligned}\frac{\sum_{n=0}^{\infty} z^{2n+1}q^{n^2+n}}{z+1/z} &= \frac{(-z^2q^2;q^2)_\infty(-1/z^2;q^2)_\infty(q^2;q^2)_\infty}{1+1/z^2}\\ &= \frac{(-z^2q^2;q^2)_\infty(1+1/z^2)(1+q^2/z^2)(1+q^4/z^2)\cdots(q^2;q^2)_\infty}{1+1/z^2}\\ &= (-z^2q^2;q^2)_\infty(-q^2/z^2;q^2)_\infty(q^2;q^2)_\infty\end{aligned} \tag{2.12}$$

Observe that by theorem 2.1.3 part (c),

$$\begin{aligned}f(-1,-q^2) &= \sum_{n=-\infty}^{\infty}(-1)^{n(n-1)/2}(-q^2)^{n(n+1)/2}\\ &= \sum_{n=-\infty}^{\infty}(-1)^{n(n-1)/2}(-1)^{n(n+1)/2}q^{n^2+n}\\ &= \sum_{n=-\infty}^{\infty}(-1)^{n^2}q^{n^2+n}\\ &= \sum_{n=-\infty}^{\infty}(-1)^{n}q^{n^2+n} = 0.\end{aligned} \tag{2.13}$$

Thus, letting $z \to i$ in (2.12), using (2.13), and applying L'Hospital's rule, we find that

$$
\begin{aligned}
L.H.S: \lim_{z\to i} \frac{\sum_{n=0}^{\infty} z^{2n+1} q^{n^2+n}}{z+1/z} &= \lim_{z\to i} \frac{\sum_{n=0}^{\infty}(2n+1)(z)^{2n} q^{n^2+n}}{1-1/z^2} \\
&= \frac{\sum_{n=0}^{\infty}(2n+1)(i)^{2n} q^{n^2+n}}{1-1/(i)^2} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)}.
\end{aligned}
$$

$$
\begin{aligned}
R.H.S: \lim_{z\to i} = (-z^2q^2;q^2)_\infty(-q^2/z^2;q^2)_\infty(q^2;q^2)_\infty &= (-(i)^2q^2;q^2)_\infty(-q^2/(i)^2;q^2)_\infty(q^2;q^2)_\infty \\
&= (q^2;q^2)_\infty(q^2;q^2)_\infty(q^2;q^2)_\infty \\
&= (q^2;q^2)^3_\infty.
\end{aligned}
$$

So,

$$
\frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{n(n+1)} = (q^2;q^2)^3_\infty. \tag{2.14}
$$

Dividing the sum into two part:

$$
\frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{n(n+1)} = \frac{1}{2}\left(\sum_{n=-\infty}^{-1}(-1)^n(2n+1)q^{n(n+1)} + \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)}\right).
$$

In the prior sum, replace n by $-n-1$ and simplify, so the sum become:

$$
\frac{1}{2}\left(2\times\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)}\right) = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)}.
$$

Finally, we complete the proof by replacing q^2 by q in (2.14):

$$
\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} = (q;q)^3_\infty.
$$

□

2.2 Integer partitions, divisor functions, and sums of two squares

We now discuss briefly the arithmetic function which we would like to investigate later in Chapter 3 using the theory of q-series. For more details about these functions we refer for instance to Andrews [1], Berndt[3], and Williams [14].

Definition 2.2.1. Let n be a nonnegative integer. The integer partition function is the function $p(n)$ which counts the number of ways n can be written as a sum of positive integers where the order of the summands is not important. For example $p(4) = 5$ since $4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$. Note that by convention we have $p(0) = 1$. Euler in [5] stated that

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \prod_{j=1}^{\infty} \frac{1}{(1-q^j)}.$$

Noticing that the factor $(1+q^k+q^{k+k}+q^{k+k+k}+\cdots)$ represents the number of summands k in a partition of n Euler's proof proceeded as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} p(n)q^n &= (1+q^1+q^{1+1}+q^{1+1+1}\cdots)(1+q^2+q^{2+2}+q^{2+2+2}+\cdots) \\
&\quad (1+q^3+q^{3+3}+q^{3+3+3}+\cdots)\cdots \\
&= \prod_{j=1}^{\infty}(1+q^j+q^{j+j}+q^{j+j+j}+\cdots) \\
&= \prod_{j=1}^{\infty}(1+q^j+q^{2j}+q^{3j}+\cdots) \\
&= \prod_{j=1}^{\infty}\frac{1}{1-q^j} \\
&= \frac{1}{(q;q)_\infty}.
\end{aligned}$$

For a rigorous proof and other facts on integer partitions we refer to Andrews [1]. We introduce some divisor functions which will be needed in later applications.

Definition 2.2.2. Let n be a positive integer. The the function $\sigma(n)$ denotes the sum of the positive divisors of n. That is

$$\sigma(n) = \sum_{d|n} d.$$

It is well-known that the generating function for $\sigma(n)$ is

$$\sum_{n=1}^{\infty} \sigma(n)q^n = \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n},$$

which can be seen as follows:

$$\begin{aligned}\sum_{n=1}^{\infty} \sigma(n)q^n &= \sum_{n=1}^{\infty} q^n \sum_{d|n} d \\ &= \sum_{d=1}^{\infty} d \sum n: \ d \mid nq^n \\ &= \sum_{d=1}^{\infty} d \sum n = 1^{\infty} q^{dn} \\ &= \sum_{d=1}^{\infty} \frac{dq^d}{1-q^d}.\end{aligned}$$

Definition 2.2.3. For any positive integer let

$$d_{i,4}(n) = \sum_{d|n:\ d \equiv i (mod 4)} 1.$$

It is well-known that the generating functions for $d_{1,4}(n)$ and $d_{3,4}(n)$ are respectively given by

$$\sum_{n=1}^{\infty} d_{1,4}(n)q^n = \sum_{n=1}^{\infty} \frac{q^{4n-3}}{1-q^{4n-3}},$$

and

$$\sum_{n=1}^{\infty} d_{3,4}(n)q^n = \sum_{n=1}^{\infty} \frac{q^{4n-1}}{1-q^{4n-1}},$$

see for instance Berndt [3, Page 58].

Definition 2.2.4. If n is a nonnegative integer, let

$$r_2(n) = \#\{(x_1,x_2) \in \mathbb{Z}^2 : x_1^2 + x_2^2 = n\}.$$

So, $r_2(0) = 1$ and $r_2(1) = 4$ since $1 = (\pm 1)^2 + 0^2 = 0^2 + (\pm 1)^2$. Further, it easy to see that the generating function of $r_2(n)$ is given by

$$\sum_{n=0}^{\infty} r_2(n)q^n = \left(\sum_{n=0}^{\infty} q^{n^2}\right)^2. \tag{2.15}$$

Chapter 3: Some New Formulas Involving Binomial Coefficients

In this chapter we shall give a variety of new congruences for the binomial coefficients and we shall evaluate an alternating sum of binomial coefficients. While the results in section 1 are well-known, section 3 consists of some of our new work which can be found in [4]. As to section 2 we do not know whether the main result exists or not. The ideas of the proofs are in terms of the q-binomial coefficients for which we will devote section 1 of this chapter. We refer to Andrews [1] and Berndt [3] for more details about q-binomial coefficients.

3.1 The q-binomial coefficients

In this section we will introduce the q-binomial coefficients along with their important properties.

Definition 3.1.1. Let n be a nonnegative integer. The q-number and the q-factorial are respectively given by

$$[n] = \frac{1-q^n}{1-q},$$

$$[n]! = \frac{(q)_n}{(1-q)^n} = \prod_{j=1}^{n} [j].$$

Accordingly, if n and m are nonnegative integers such that $n \geq m$, then the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!} = \frac{(q)_n}{(q)_m (q)_{n-m}}.$$

Note that

$$\lim_{q\to 1}[n] = \frac{-n}{-1} = n,$$

$$\lim_{q\to 1}[n]! = \frac{-1}{-1}\cdot\frac{-2}{-1}\cdots\frac{-n}{-1} = n!,$$

and therefore

$$\lim_{q\to 1}\begin{bmatrix} n \\ m \end{bmatrix} = \binom{n}{m}.$$

For instance, from the previous identity we see that the q-binomial coefficients extend the usual binomial coefficients. Note also that $[n]!$ is a polynomial in q. However, it is not clear from the definition that the q-binomial coefficient $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]$ is a polynomial in q. Among other things, this fact will be proved below in Theorem 3.1.2. Besides, just as the binomial coefficients appear in many combinatorial applications the q-binomial coefficients have also a variety of combinatorial interpretations. We refer to Satnley [12] for more information on the matter. By way of example we mention here without proof the following interpretation which is related to integer partitions, refer to [1, Theorem 3.1].

Theorem 3.1.1. *For nonnegative integers M, N, and n let $p(N,M,n)$ denote the number of partitions of n into at most M parts each $\leq N$ and let $G(N,M;q)$ be the generating function for $p(N,M,n)$, that is,*

$$G(N,M;q) = \sum_{n=0}^{\infty} p(N,M,n)q^n.$$

Then $G(N,M;q) = \left[\begin{smallmatrix} M+N \\ M \end{smallmatrix}\right]$.

We now list some important facts on the q-binomial coefficients.

Theorem 3.1.2. *Let m and n be nonnegative integers such that $n \geq m$. Then* *(a)* $\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1.$

(b) $\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ n-m \end{bmatrix}.$

(c) $\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m \end{bmatrix} + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}.$

(d) $\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^{m} \begin{bmatrix} n-1 \\ m \end{bmatrix}.$

(e) $\begin{bmatrix} n \\ m \end{bmatrix}$ *is a polynomial in* q *of degree* $m(n-m)$*.*

Proof. (a) $\begin{bmatrix} n \\ 0 \end{bmatrix} = \frac{[n]!}{[0]![n]!} = \frac{1}{1 \cdot 1} = 1 = \begin{bmatrix} n \\ n \end{bmatrix}.$

(b) $\begin{bmatrix} n \\ n-m \end{bmatrix} = \frac{[n]!}{[n-m]![n-n+m]!} = \frac{[n]!}{[n-m]![m]!} = \begin{bmatrix} n \\ m \end{bmatrix}.$

(c)

$$\begin{aligned}
\begin{bmatrix} n \\ m \end{bmatrix} - \begin{bmatrix} n-1 \\ m \end{bmatrix} &= \frac{(q)_n}{(q)_m (q)_{n-m}} - \frac{(q)_{n-1}}{(q)_m (q)_{n-1-m}} \\
&= \frac{(q)_{n-1}}{(q)_m (q)_{n-m}} \left((1-q^n) - (1-q^{n-m}) \right) \\
&= \frac{(q)_{n-1}}{(q)_m (q)_{n-m}} \left(q^{n-m} - q^n \right) \\
&= \frac{(q)_{n-1}}{(q)_m (q)_{n-m}} q^{n-m} (1-q^m) \\
&= q^{n-m} \frac{(q)_{n-1}}{(q)_{m-1} (q)_{n-m}} \\
&= q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}.
\end{aligned}$$

(d) Replace m by $n-m$ to get

$$\begin{bmatrix} n \\ n-m \end{bmatrix} = \begin{bmatrix} n-1 \\ n-m-1 \end{bmatrix} + q^{n-m} \begin{bmatrix} n-1 \\ n-m \end{bmatrix},$$

which by part (b) is equivalent to:

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m \end{bmatrix} + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}.$$

(e) We proceed by induction on n. $n = 1$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1$, which is a polynomial of degree 1. Suppose now that the statement is true for $\begin{bmatrix} n \\ m \end{bmatrix}$. Then by virtue of part (d) we find

$$\begin{bmatrix} n+1 \\ m \end{bmatrix} = \begin{bmatrix} n \\ m-1 \end{bmatrix} + q^m \begin{bmatrix} n \\ m \end{bmatrix},$$

which is clearly a polynomial in q of degree $m(n+1-m)$.

□

Because of Theorem 3.1.2(e) the q-binomial coefficients are also referred to as *Gaussian polynomials*.

Theorem 3.1.3. *For all nonnegative integers m and n we have*

$$\sum_{j=0}^{n} q^j \begin{bmatrix} m+j \\ m \end{bmatrix} = \begin{bmatrix} n+m+1 \\ m+1 \end{bmatrix}.$$

Proof. We proceed by induction on n. If $n = 0$, we have

$$q^0 \begin{bmatrix} m+0 \\ m \end{bmatrix} = \begin{bmatrix} m+1 \\ m+1 \end{bmatrix}.$$

Suppose that the statement holds for n. Then with the help of Theorem 3.1.2(c) we find

$$\begin{aligned} \begin{bmatrix} n+m+2 \\ m+1 \end{bmatrix} &= \begin{bmatrix} n+m+1 \\ m+1 \end{bmatrix} + q^{n+1} \begin{bmatrix} n+m+1 \\ m \end{bmatrix} \\ &= \sum_{j=0}^{n} q^j \begin{bmatrix} m+j \\ m \end{bmatrix} + q^{n+1} \begin{bmatrix} n+m+1 \\ m \end{bmatrix} \\ &= \sum_{j=0}^{n+1} q^j \begin{bmatrix} m+j \\ m \end{bmatrix}. \end{aligned}$$

□

Corollary 3.1.4. *For all nonnegative integers m and n we have*

$$\sum_{j=0}^{n}\binom{m+j}{m}=\binom{n+m+1}{m+1}.$$

Proof. Letting $q \to 1$ in theorem 3.1.3 we obtain

$$\lim_{q\to 1}\sum_{j=0}^{n} q^{j}\begin{bmatrix} m+j \\ m \end{bmatrix}=\sum_{j=0}^{n}\binom{m+j}{m}=\lim_{q\to 1}\begin{bmatrix} n+m+1 \\ m+1 \end{bmatrix}=\binom{n+m+1}{m+1}.$$

□

Theorem 3.1.5. *If n is a nonnegative integer, then*

$$\sum_{j=0}^{n}(-1)^{j}\begin{bmatrix} n \\ j \end{bmatrix} z^{j} q^{\frac{j(j-1)}{2}}=(z;q)_{n}.$$

Proof. By the q-analogue of the binomial theorem, see Theorem 2.1.1,

$$\begin{aligned}\sum_{j=0}^{n}\frac{(q^{-n};q)_{j}q^{nj}}{(q;q)_{j}}z^{j} &= \sum_{j=0}^{\infty}\frac{(q^{-n};q)_{j}}{(q;q)_{j}}(q^{n}z)^{j} \\ &= \frac{(z;q)_{\infty}}{(q^{n}z;q)_{\infty}} \\ &= (z;q)_{n}\end{aligned}$$

Then for $|z| < 1$,

$$\begin{aligned}
(z;q)_n &= \sum_{j=0}^{\infty} \frac{(1-q^{-n})(1-q^{-n+1})\cdots(1-q^{-n+j-1})}{(q;q)_j} q^{nj} z^j \\
&= \sum_{j=0}^{\infty} \frac{q^{-n}(q^n-1)q^{-n+1}(q^{n-1}-1)\cdots q^{-n+j-1}(q^{n-j+1}-1)}{(q;q)_j} q^{nj} z^j \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j q^{\frac{j(j-1)}{2}} \Big((1-q^{n-(j-1)})\cdots(1-q^n)\Big) z^j}{(q;q)_j} \\
&= \sum_{j=0}^{n} (-1)^j q^{\frac{j(j-1)}{2}} z^j \frac{(q)_n}{(q)_j (q)_{n-j}} \\
&= \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} q^{\frac{j(j-1)}{2}} z^j.
\end{aligned}$$

□

Theorem 3.1.6 (The q-Chu-Vandermonde Sum)**.** *For all nonnegative integers m, n and h we have*

$$\sum_{k=0}^{h} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ h-k \end{bmatrix} q^{(n-k)(h-k)} = \begin{bmatrix} m+n \\ h \end{bmatrix}.$$

Proof. By Theorem 3.1.5 we have

$$\begin{aligned}
\sum_{h=0}^{m+n} (-1)^h \begin{bmatrix} m+n \\ h \end{bmatrix} z^h q^{\frac{h(h-1)}{2}} &= (z;q)_{m+n} \\
&= (z;q)_n (zq^n;q)_m \\
&= \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{k(k-1)}{2}} z^k \cdot \sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix} q^{nj+\frac{j(j-1)}{2}} z^j \\
&= \sum_{h=0}^{m+n} z^h \sum_{k=0}^{h} \left((-1)^k q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{h-k} q^{n(h-k)+\frac{(h-k)(h-k-1)}{2}} \begin{bmatrix} m \\ h-k \end{bmatrix} \right) \\
&= \sum_{h=0}^{m+n} z^h (-1)^h q^{\frac{h(h-1)}{2}} \sum_{k=0}^{h} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ h-k \end{bmatrix} q^{(n-k)(h-k)}.
\end{aligned}$$

Then the desired identity follows by comparing the coefficients of z. □

Corollary 3.1.7 (Chu-Vandermonde Sum)**.** *For all nonnegative integers m,n and h we have*

$$\sum_{k=0}^{h}\binom{n}{k}\binom{m}{h-k}=\binom{m+n}{h}.$$

Proof. Letting $q\to 1$ in Theorem 3.1.6 gives:

$$\lim_{q\to 1}\sum_{k=0}^{h}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\h-k\end{bmatrix}q^{(n-k)(h-k)}=\sum_{k=0}^{h}\binom{n}{k}\binom{m}{h-k}=\lim_{q\to 1}\begin{bmatrix}m+n\\h\end{bmatrix}=\binom{m+n}{h}.$$

□

3.2 An alternating sum of q-binomial coefficients

It is easy by logarithmic differentiation applied to the identity $(1-z)^n=\sum_{k=0}^{n}(-1)^{n-k}z^k$ to check that

$$\sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{n}{j}=\binom{n-1}{m-1}. \tag{3.1}$$

In this section we provide q-analogue for this formula.

Theorem 3.2.1. *If m and n are positive integers such that $m\le n$, then*

$$\begin{bmatrix}n\\m\end{bmatrix}=\frac{1}{m}\sum_{j=0}^{m-1}(-1)^{m-1-j}q^{\frac{j(j-1)-m(m-1)}{2}}\begin{bmatrix}n\\j\end{bmatrix}\frac{1-q^{(m-j)n}}{1-q^{m-j}}.$$

Proof. We assume throughout the proof that $|z|<1$ and $|q|<1$. Let $F(z)=(z;q)_n=$

$\prod_{j=0}^{n-1}(1-zq^j)$ and write

$$F(z) = \sum_{j=0}^{\infty} A(j) z^j.$$

Then

$$\begin{aligned}\log F(z) &= \sum_{j=0}^{n-1} \log(1-zq^j) = -\sum_{j=0}^{n-1}\sum_{m=1}^{\infty} \frac{q^{mj}}{m} z^m \\ &= -\sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{j=0}^{n-1} q^{mj} = -\sum_{m=1}^{\infty} \frac{1-q^{mn}}{1-q^m} \frac{z^m}{m}.\end{aligned}$$

Differentiating both sides with respect to z and then multiplying by z gives

$$\begin{aligned}zF'(z) &= -F(z) \sum_{m=1}^{\infty} \frac{1-q^{mn}}{1-q^m} z^m \\ &= -\left(\sum_{m=0}^{\infty} A(m) z^m\right)\left(\sum_{m=1}^{\infty} \frac{1-q^{mn}}{1-q^m} z^m\right),\end{aligned}$$

and so we get the recurrence relation

$$mA(m) = -\sum_{j=0}^{m-1} A(j) \frac{1-q^{(m-j)n}}{1-q^{m-j}}. \tag{3.2}$$

On the other hand, by Theorem 3.1.5

$$A(j) = (-1)^j q^{\frac{j(j-1)}{2}} \begin{bmatrix} n \\ j \end{bmatrix}.$$

Substituting this in the formula (3.2) we obtain

$$m(-1)^m q^{\frac{m(m-1)}{2}} \begin{bmatrix} n \\ m \end{bmatrix} = \sum_{j=0}^{m-1} (-1)^{j+1} q^{\frac{j(j-1)}{2}} \begin{bmatrix} n \\ j \end{bmatrix} \frac{1-q^{(m-j)n}}{1-q^{m-j}},$$

or equivalently,

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{1}{m} \sum_{j=0}^{m-1} (-1)^{m-1-j} q^{\frac{j(j-1)-m(m-1)}{2}} \begin{bmatrix} n \\ j \end{bmatrix} \frac{1-q^{(m-j)n}}{1-q^{m-j}}.$$

This completes the proof. □

Note that letting $q \to 1$ yields

$$\binom{n}{m} = \frac{n}{m} \sum_{j=1}^{m-1} (-1)^{m-1-j} \binom{n}{j},$$

and hence

$$\binom{n-1}{m-1} = \sum_{j=1}^{m-1} (-1)^{m-1-j} \binom{n}{j}$$

which is the formula (3.1).

3.3 Divisibility of binomial coefficients

Throughout the set of polynomials in q with integer coefficients will be written $\mathbb{Z}[q]$. The following fact is easily checked and we record it as a theorem for further reference.

Theorem 3.3.1. *Let n, a, and b be a positive integer such that $a \geq b$. If $\frac{1-q}{1-q^n} \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{Z}[q]$, then $\binom{a}{b}$ is divisible by n.*

Proof. Simply let $q \to 1$. □

We now give a result which enables us to generate new congruences involving binomial coefficients from old ones.

Theorem 3.3.2. *Let $a_1, \ldots, a_r$, $b_1, \ldots, b_r$, $c_1, \ldots, c_s$, $d_1, \ldots, d_s$ and n be positive integers such that $a_j \geq b_j$ for all $j = 1, \ldots, r$, $c_j \geq d_j$ for all $j = 1, \ldots, s$, and $\max\{a_j : \; j =$*

$1,\dots,r\} \geq \max\{c_j :\ j=1,\dots,s\}$. *Then*

$$\frac{1-q}{1-q^n}\frac{\prod_{j=1}^{r}\begin{bmatrix}a_j\\b_j\end{bmatrix}}{\prod_{j=1}^{s}\begin{bmatrix}c_j\\d_j\end{bmatrix}} \in \mathbb{Z}[q]$$

if and only if

$$\frac{1-q}{1-q^n}\frac{\prod_{j=1}^{r}\begin{bmatrix}a_j+k_jn\\b_j+l_jn\end{bmatrix}}{\prod_{j=1}^{s}\begin{bmatrix}c_j+u_jn\\d_j+v_jn\end{bmatrix}} \in \mathbb{Z}[q]$$

for all integers $k_1,\dots,k_j,\ l_1,\dots,l_j,\ u_1,\dots,u_j,\ v_1,\dots,v_j$ *such that* $a_j+k_jn \geq b_j+l_jn \geq 0$, $c_j+u_jn \geq d_j+v_jn \geq 0$ *for all* $j=1,\dots,r$, *and* $\max\{a_j+k_j :\ j=1,\dots,r\} \geq \max\{c_j+u_j :\ j=1,\dots,s\}$.

Proof. The implication from the right to the left is clear. Assume now that

$$A(q)=\frac{1-q}{1-q^n}\prod_{j=1}^{r}\frac{\begin{bmatrix}a_j\\b_j\end{bmatrix}}{\begin{bmatrix}c_j\\d_j\end{bmatrix}} \in \mathbb{Z}[q].$$

By the well-known identity

$$q^m-1=\prod_{d\mid m}\Phi_d(q),$$

where $\Phi_d(q)$ is the d-th cyclotomic polynomial in q, we obtain

$$A(q)=\prod_{d=2}^{\max\{a_1,\dots,a_r\}}\Phi_d(q)^{e_d},$$

where

$$e_d=-\chi(d\mid n)+\sum_{j=1}^{r}\left(\left\lfloor\frac{a_j}{d}\right\rfloor-\left\lfloor\frac{b_j}{d}\right\rfloor-\left\lfloor\frac{a_j-b_j}{d}\right\rfloor\right)-\sum_{j=1}^{s}\left(\left\lfloor\frac{c_j}{d}\right\rfloor-\left\lfloor\frac{d_j}{d}\right\rfloor-\left\lfloor\frac{c_j-d_j}{d}\right\rfloor\right),$$

with $\chi(S)=1$ if S is true and $\chi(S)=0$ if S is false. As $A(q)$ is a polynomial in q and $\Phi_d(q)$ is irreducible for any d we must have $e_d \geq 0$ for all $d=2,\dots,\max\{a_1,\dots,a_r\}$. As to $B(q)=\frac{1-q}{1-q^n}\prod_{j=1}^r \frac{\left[{a_j+k_jn \atop b_j+l_jn}\right]}{\left[{c_j+u_jn \atop d_j+v_jn}\right]}$, we have

$$B(q)=\prod_{d=2}^{\max\{a_1+k_1n,\dots,a_r+k_rn\}} \Phi_d(q)^{e_d},$$

where

$$\begin{aligned} e_d = -\chi(d \mid n) + \sum_{j=1}^r \left(\left\lfloor \frac{a_j+k_jn}{d} \right\rfloor - \left\lfloor \frac{b_j+l_jn}{d} \right\rfloor - \left\lfloor \frac{a_j-b_j+(k_j-l_j)n}{d} \right\rfloor \right) \\ - \sum_{j=1}^s \left(\left\lfloor \frac{c_j+u_jn}{d} \right\rfloor - \left\lfloor \frac{d_j+v_jn}{d} \right\rfloor - \left\lfloor \frac{c_j-d_j+(u_j-v_j)n}{d} \right\rfloor \right). \end{aligned}$$

Then clearly $e_d \geq 0$ unless $d \mid n$. But if $d \mid n$, then

$$\sum_{j=1}^r \left(\left\lfloor \frac{a_j+k_jn}{d} \right\rfloor - \left\lfloor \frac{b_j+l_jn}{d} \right\rfloor - \left\lfloor \frac{a_j-b_j+(k_j-l_j)n}{d} \right\rfloor \right) = \sum_{j=1}^r \left(\left\lfloor \frac{a_j}{d} \right\rfloor - \left\lfloor \frac{b_j}{d} \right\rfloor - \left\lfloor \frac{a_j-b_j}{d} \right\rfloor \right)$$

and

$$\sum_{j=1}^s \left(\left\lfloor \frac{c_j+u_jn}{d} \right\rfloor - \left\lfloor \frac{d_j+v_jn}{d} \right\rfloor - \left\lfloor \frac{c_j-d_j+(u_j-v_j)n}{d} \right\rfloor \right) = \sum_{j=1}^s \left(\left\lfloor \frac{c_j}{d} \right\rfloor - \left\lfloor \frac{d_j}{d} \right\rfloor - \left\lfloor \frac{c_j-d_j}{d} \right\rfloor \right)$$

and therefore $e_d \geq 0$ by assumption, implying that $B(q)$ is a polynomial in q. This completes the proof. □

In our first application of Theorem 3.3.2 we will need the well-known fact that the *q-Catalan number*

$$C_n(q)=\frac{1-q}{1-q^{n+1}}\begin{bmatrix}2n\\ n\end{bmatrix}$$

is a polynomial in q with nonnegative coefficients for all nonnegative integer n. See for instance [2].

Theorem 3.3.3. *(a) For any positive integer n we have*

$$2n(2n+1)\Big|\binom{6n}{2n-1}.$$

(b) For any positive integer n we have

$$3n(3n+1)\Big|\binom{12n}{6n-1}.$$

(c) For any nonnegative integer n we have

$$(3n+1)(3n+2)\Big|\binom{12n+4}{6n+1}.$$

Proof. (a) Combining Theorem 3.3.2 with the fact that $C_n(q) \in \mathbb{Z}[q]$ yields that

$$\frac{1-q}{1-q^{n+1}}\begin{bmatrix}2n+(n+1)\\ n\end{bmatrix} = \frac{1-q}{1-q^{n+1}}\begin{bmatrix}3n+1\\ n\end{bmatrix} \in \mathbb{Z}[q]$$

and therefore by Theorem 3.3.1 we find that

$$n(n+1)\Big|(3n+1)\binom{3n}{n-1}.$$

Observe now that if n is even, then $\gcd(n(n+1), 3n+1) = 1$ and thus

$$2n(2n+1)\Big|\binom{6n}{2n-1}.$$

(b) By a similar argument of part (a) we get

$$\frac{1-q}{1-q^{n+1}}\begin{bmatrix}2n+2(n+1)\\ n+n+1\end{bmatrix} = \frac{1-q}{1-q^{n+1}}\begin{bmatrix}4n+2\\ 2n+1\end{bmatrix} \in \mathbb{Z}[q],$$

which by Theorem 3.3.1 and the basic properties of the binomial coefficients yields

$$n(n+1)\Big|(4n+1)\binom{4n}{2n-1}.$$

But if $n \equiv 0 \pmod 3$, then $\gcd(n(n+1),(4n+1)) = 1$ and therefore we find

$$3n(3n+1)\Big|\binom{12n}{6n-1},$$

as desired. (c) This part follows in a completely the same way as part (b) by taking $n \equiv 1 \pmod 3$ instead of $n \equiv 0 \pmod 3$. □

We will now apply Theorem 3.3.2 to some results of Guo and Krattenthaler in [8].

Theorem 3.3.4. *(a) For any positive integer n we have*

$$3n(6n-1)\Big|\binom{6n}{3n-1}.$$

(b) For any positive integer n we have

$$(6n+1)(12n-1)\Big|\binom{12n-1}{6n-1}.$$

Proof. (a) Guo and Krattenthaler in [8, Theorem 3.1] proved that $\frac{1-q}{1-q^{6n-1}}\begin{bmatrix}12n\\ 3n\end{bmatrix} \in \mathbb{Z}[q]$ for all positive integer n. Then by Theorem 3.3.2 we find that

$$\frac{1-q}{1-q^{6n-1}}\begin{bmatrix}12n-(6n-1)\\ 3n\end{bmatrix}=\frac{1-q}{1-q^{6n-1}}\begin{bmatrix}6n+1\\ 3n\end{bmatrix}\in\mathbb{Z}[q],$$

from which it follows by Theorem 3.3.1 that

$$3n(6n-1)\Big|(6n+1)\binom{6n}{3n-1}.$$

As $\gcd(3n(6n-1),6n+1)=1$, the previous relation implies

$$3n(6n-1)\Big|\binom{6n}{3n-1}.$$

(b) Similarly we find

$$(6n-1)\Big|\binom{6n+1}{3n+1}=\frac{6n+1}{3n+1}\binom{6n}{3n}=\frac{2(6n+1)}{3n+1}\binom{6n-1}{3n-1},$$

which implies that

$$(3n+1)(6n-1)\Big|2(6n+1)\binom{6n-1}{3n-1}.$$

Now the desired result follows since $\gcd((3n+1)(6n-1),2(6n+1))=1$ for even n. □

Theorem 3.3.5. *(a) For any positive integer n we have*

$$4n(6n-1)\Big|\binom{6n}{4n-1}.$$

(b) For any positive integer n we have

$$(18n-1)(6n+1)\Big|\binom{18n-1}{6n-1}.$$

(c) For any nonnegative integer n we have

$$(18n+11)(6n+5)\Big|\binom{18n+5}{6n+3}.$$

Proof. (a) Guo and Krattenthaler [8, Theorem 3.1] showed that $\frac{1-q}{1-q^{6n-1}}\begin{bmatrix}12n\\4n\end{bmatrix} \in \mathbb{Z}[q]$ for all positive integer n. Then by Theorem 3.3.2 we have that

$$\frac{1-q}{1-q^{6n-1}}\begin{bmatrix}6n+1\\4n\end{bmatrix} \in \mathbb{Z}[q],$$

which by Theorem 3.3.1 implies

$$4n(6n-1)\Big|(6n+1)\binom{6n}{4n-1}.$$

Since $\gcd(4n(6n-1), 6n+1) = 1$ we get

$$4n(6n-1)\Big|\binom{6n}{4n-1}.$$

(b) Similarly using the basic fact that $\begin{bmatrix}12n\\4n\end{bmatrix} = \begin{bmatrix}12n\\8n\end{bmatrix}$, see Theorem 3.1.2, we have

$$(6n-1)\Big|\binom{12n-6n+1}{8n-6n+1} = \binom{6n+1}{2n+1} = \frac{3(6n+1)}{2n+1}\binom{6n-1}{2n-1}.$$

But $\gcd((2n+1)(6n-1), 3(6n+1)) = 1$ if $n \equiv 0 \pmod 3$ and hence

$$(18n-1)(6n+1) \Big| \binom{18n-1}{6n-1}.$$

(c) This part follows by the same argument of the previous part by taking $n \equiv 2 \pmod 3$ rather than $n \equiv 0 \pmod 3$. This concludes the proof. □

We will now apply Theorem 3.3.2 to a polynomial in $\mathbb{Z}[q]$ which is related to a result of Sun in [13]. We first need a lemma.

Lemma 3.3.6. *For all nonnegative integer n the function*

$$f(q) = \frac{1-q}{1-q^{2n+1}} \frac{\begin{bmatrix}6n\\3n\end{bmatrix}\begin{bmatrix}3n\\n\end{bmatrix}}{\begin{bmatrix}2n\\n\end{bmatrix}}$$

is a polynomial in q with integer coefficients.

Proof. It is easily seen that

$$f(q) = \prod_{d=2}^{6n} \Phi_d(q)^{e_d},$$

where as before $\Phi_d(q)$ is the d-th cyclotomic polynomial and

$$e_d = -\chi(d \mid 2n+1) + \left\lfloor \frac{6n}{d} \right\rfloor + \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{2n}{d} \right\rfloor - \left\lfloor \frac{2n}{d} \right\rfloor - \left\lfloor \frac{3n}{d} \right\rfloor,$$

with $\chi(S) = 1$ if S is true and $\chi(S) = 0$ if S is false. If $d \nmid 2n+1$, then $\lfloor \frac{2n}{d} \rfloor = \lfloor \frac{2n+1}{d} \rfloor$ and by Sun's inequality [13, Theorem 10] we have

$$\left\lfloor \frac{6n}{d} \right\rfloor + \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{2n}{d} \right\rfloor - \left\lfloor \frac{2n}{d} \right\rfloor - \left\lfloor \frac{3n}{d} \right\rfloor$$

$$= \left\lfloor \frac{6n}{d} \right\rfloor + \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{2n}{d} \right\rfloor - \left\lfloor \frac{2n+1}{d} \right\rfloor - \left\lfloor \frac{3n}{d} \right\rfloor \geq 0.$$

If $d \mid 2n+1$, then $\lfloor \frac{2n}{d} \rfloor = \lfloor \frac{2n+1}{d} \rfloor - 1$ and by Sun's inequality [13, Theorem 10] we have

$$\left\lfloor \frac{6n}{d} \right\rfloor + \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{2n}{d} \right\rfloor - \left\lfloor \frac{2n}{d} \right\rfloor - \left\lfloor \frac{3n}{d} \right\rfloor$$

$$= \left\lfloor \frac{6n}{d} \right\rfloor + \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{2n}{d} \right\rfloor - \left\lfloor \frac{2n+1}{d} \right\rfloor - \left\lfloor \frac{3n}{d} \right\rfloor + 1 \geq 1.$$

So in both cases $e_d \geq 0$ and therefore $f(q) \in \mathbb{Z}[q]$. □

Theorem 3.3.7. *For any nonnegative integer n we have*

$$(2n+1)\binom{6n+1}{3n+1} \Big| \binom{10n+1}{5n+1}\binom{5n+1}{3n+1}.$$

Proof. By Lemma 3.3.6 and Theorem 3.3.2 we have

$$\frac{1-q}{1-q^{2n+1}} \frac{\begin{bmatrix} 6n+2(2n+1) \\ 3n+2n+1 \end{bmatrix} \begin{bmatrix} 3n+2n+1 \\ n+2n+1 \end{bmatrix}}{\begin{bmatrix} 2n+2(2n+1) \\ n+2n+1 \end{bmatrix}} = \frac{1-q}{1-q^{2n+1}} \frac{\begin{bmatrix} 10n+2 \\ 5n+1 \end{bmatrix} \begin{bmatrix} 5n+1 \\ 3n+1 \end{bmatrix}}{\begin{bmatrix} 6n+2 \\ 3n+1 \end{bmatrix}} \in \mathbb{Z}[q].$$

Then

$$(2n+1)\binom{6n+2}{3n+1} \Big| \binom{10n+2}{5n+1}\binom{5n+1}{3n+1},$$

or equivalently,

$$2(2n+1)\binom{6n+1}{3n+1} \Big| 2\binom{10n+1}{5n+1}\binom{5n+1}{3n+1},$$

which gives the desired identity after dividing both sides by 2. □

Chapter 4: Applications of q-series to Sums of Squares and Partitions

The number of ways a nonnegative integer can be expressed as a sum of squares have been extensively studied in the past. For instance, Jacobi was the first to give such formulas for sums of two squares and sums of four squares. Many different methods have been found to derive Jacobi's formulas. In this chapter we shall see how q-series can be used to give Jacobi's formula on the representation of a nonnegative integer as a sum of two squares. Other elementary methods based on Liouville's sums can be found in Williams [14, Chapter 9]. We will provide a recurrence relation for the partition function $p(n)$ in terms of the divisor function $\sigma(n)$. Finally, we shall see an application of q-series to prove one of the most famous congruences for integer partitions which is due to Ramanujan.

4.1 Sums of two squares

In this section we will use the approach of Hirschhorn [9] to derive Jacobi's formula for the function $r_2(n)$ of the sum of two squares.

Theorem 4.1.1. *If n is a positive integer, then we have*

$$r_2(n) = 4(d_{1,4}(n) - d_{3,4}(n)).$$

Proof. Replace z by $-a^2q$ in the Jacobi's triple identity Theorem 2.1.4 to get:

$$\prod_{n=1}^{\infty}(1-a^2q^{2n})(1-a^{-2}q^{2n-2})(1-q^{2n}) = \sum_{n=-\infty}^{\infty}(-1)^n a^{2n} q^{n^2+n}.$$

Then replace q^2 by q:

$$\prod_{n=1}^{\infty}(1-a^2q^n)(1-a^{-2}q^{n-1})(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n a^{2n} q^{\frac{n^2+n}{2}}.$$

Taking one factor out, we get:

$$(1-a^{-2})\prod_{n=1}^{\infty}(1-a^2q^n)(1-a^{-2}q^n)(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n a^{2n} q^{\frac{n^2+n}{2}}.$$

Multiply be a:

$$(a-a^{-1})\prod_{n=1}^{\infty}(1-a^2q^n)(1-a^{-2}q^n)(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n a^{2n+1} q^{\frac{n^2+n}{2}}.$$

Now work with modulo 2 to obtain

$$\begin{aligned}
&(a-a^{-1})\prod_{n=1}^{\infty}(1-a^2q^n)(1-a^{-2}q^n)(1-q^n) \\
&\quad = \sum_{n=-\infty}^{\infty}(-1)^n a^{2n+1} q^{\frac{n^2+n}{2}} \\
&\quad = \sum_{n=-\infty}^{\infty} a^{4n+1} q^{2n^2+n} - \sum_{n=-\infty}^{\infty} a^{4n-1} q^{2n^2-n} \\
&\quad = a\sum_{n=-\infty}^{\infty} a^{4n} q^{2n^2+n} - a^{-1}\sum_{n=-\infty}^{\infty} a^{4n} q^{2n^2-n} \\
&\quad = a\sum_{n=-\infty}^{\infty} a^{4n} q^{n} (q^2)^{n^2} - a^{-1}\sum_{n=-\infty}^{\infty} a^{4n} q^{-n} (q^2)^{n^2} \\
&\quad = a\sum_{n=-\infty}^{\infty} (a^4q)^n (q^2)^{n^2} - a^{-1}\sum_{n=-\infty}^{\infty} (a^4q^{-1})^n (q^2)^{n^2} \\
&\quad = a\prod_{n=1}^{\infty}(1+a^4q^{4n-1})(1+a^{-4}q^{4n-3})(1-q^{4n}) \\
&\quad - a^{-1}\prod_{n=1}^{\infty}(1+a^4q^{4n-3})(1+a^{-4}q^{4n-1})(1-q^{4n}).
\end{aligned}$$

Differentiating the left hand side with respect to a, gives

$$(1+\frac{1}{a^2})(\prod_{n=1}^{\infty}(1-a^2q^n)(1-a^{-2}q^n)(1-q^n))+(a-a^{-1})(\prod_{n=1}^{\infty}(1-a^2q^n)(1-a^{-2}q^n)(1-q^n))',$$

which if $a=1$ becomes

$$2\prod_{n=1}^{\infty}(1-q^n)^3.$$

Next differentiating the right hand side with respect to a yields

$$\begin{aligned}
&\prod_{n=1}^{\infty}(1+a^4q^{4n-1})(1+a^{-4}q^{4n-3})(1-q^{4n})\\
&+a\prod_{n=1}^{\infty}(1+a^4q^{4n-1})(1+a^{-4}q^{4n-3})(1-q^{4n})\\
&\times\sum_{n=1}^{\infty}\frac{(4a^3q^{4n-1}(1+a^{-4}q^{4n-3})-4a^{-5}q^{4n-3}(1+a^4q^{4n-1}))}{(1+a^4q^{4n-1})(1+a^{-4}q^{4n-3})}\\
&+a^{-2}\prod_{n=1}^{\infty}(1+a^4q^{4n-3})(1+a^{-4}q^{4n-1})(1-q^{4n})\\
&-a^{-1}\prod_{n=1}^{\infty}(1+a^4q^{4n-3})(1+a^{-4}q^{4n-1})(1-q^{4n})\\
&\times\sum_{n=1}^{\infty}\frac{(4a^3q^{4n-3}(1+a^{-4}q^{4n-1})-4a^{-5}q^{4n-1}(1+a^4q^{4n-3}))}{(1+a^4q^{4n-3})(1+a^{-4}q^{4n-1})}
\end{aligned}$$

which if $a = 1$ becomes

$$\prod_{n=1}^{\infty}(1+q^{4n-1})(1+q^{4n-3})(1-q^{4n}) \times \left(1+4\sum_{n=1}^{\infty}\frac{q^{4n-1}(1+q^{4n-3})-q^{4n-3}(1+q^{4n-1})}{(1+q^{4n-1})(1+q^{4n-3})}\right)$$
$$+\prod_{n=1}^{\infty}(1+q^{4n-3})(1+q^{4n-1})(1-q^{4n}) \times \left(1-4\sum_{n=1}^{\infty}\frac{q^{4n-3}(1+q^{4n-1})-q^{4n-1}(1+q^{4n-3})}{(1+q^{4n-3})(1+q^{4n-1})}\right)$$
$$=\prod_{n=1}^{\infty}(1+q^{4n-1})(1+q^{4n-3})(1-q^{4n}) \times \left(1+4\sum_{n=1}^{\infty}\frac{q^{4n-1}}{1+q^{4n-1}}-\frac{q^{4n-3}}{1+q^{4n-3}}\right)$$
$$+\prod_{n=1}^{\infty}(1+q^{4n-3})(1+q^{4n-1})(1-q^{4n}) \times \left(1-4\sum_{n=1}^{\infty}\frac{q^{4n-3}}{1+q^{4n-3}}-\frac{q^{4n-1}}{1+q^{4n-1}}\right)$$
$$=2\prod_{n=1}^{\infty}(1+q^{4n-3})(1+q^{4n-1})(1-q^{4n}) \times \left(1-4\sum_{n=1}^{\infty}\frac{q^{4n-3}}{1+q^{4n-3}}-\frac{q^{4n-1}}{1+q^{4n-1}}\right)$$

Now combining these facts we get

$$\prod_{n=1}^{\infty}(1-q^{n})^{3}=\prod_{n=1}^{\infty}(1+q^{4n-3})(1+q^{4n-1})(1-q^{4n}) \times \left(1-4\sum_{n=1}^{\infty}\frac{q^{4n-3}}{1+q^{4n-3}}-\frac{q^{4n-1}}{1+q^{4n-1}}\right). \tag{4.1}$$

Next divide both sides of (4.1) by:

$$\prod_{n=1}^{\infty}(1+q^{n})^{2}(1-q^{n})=\prod_{n=1}^{\infty}(1+q^{n})(1-q^{2n}) \tag{4.2}$$

Then the left hand side becomes:

$$\prod_{n=1}^{\infty}\frac{(1-q^{n})^{3}}{(1+q^{n})^{2}(1-q^{n})}=\prod_{n=1}^{\infty}\left(\frac{1-q^{n}}{1+q^{n}}\right)^{2},$$

and the right hand side becomes:

$$\left(1-4\sum_{n=1}^{\infty}\frac{q^{4n-3}}{1+q^{4n-3}}-\frac{q^{4n-1}}{1+q^{4n-1}}\right),$$

as we can rewrite the equation (4.2) as following:

$$\begin{aligned}\prod_{n=1}^{\infty}(1+q^n)^2(1-q^n) &= \prod_{n=1}^{\infty}(1+q^{2n-1})(1+q^{2n})(1-q^{2n})\\ &= \prod_{n=1}^{\infty}(1+q^{2n-1})(1-q^{4n})\\ &= \prod_{n=1}^{\infty}(1+q^{4n-1})(1+q^{4n-3})(1-q^{4n}).\end{aligned}$$

So we have,

$$\prod_{n=1}^{\infty}\left(\frac{1-q^n}{1+q^n}\right)^2 = \left(1-4\sum_{n=1}^{\infty}\frac{q^{4n-3}}{1+q^{4n-3}}-\frac{q^{4n-1}}{1+q^{4n-1}}\right).$$

Now,

$$\begin{aligned}\prod_{n=1}^{\infty}\frac{1-q^n}{1+q^n} &= \prod_{n=1}^{\infty}\frac{(1-q^{2n-1})(1-q^{2n})}{(1+q^n)}\\ &= \prod_{n=1}^{\infty}(1-q^{2n-1})(1-q^n)\\ &= \prod_{n=1}^{\infty}(1-q^{2n-1})(1-q^{2n-1})(1-q^{2n})\\ &= \sum_{n=1}^{\infty}(-1)^n q^{n^2}.\end{aligned}$$

Then we have:

$$\left(\sum_{n=1}^{\infty}(-1)^n q^{n^2}\right)^2 = 1-4\sum_{n=1}^{\infty}\left(\frac{q^{4n-3}}{1+q^{4n-3}}-\frac{q^{4n-1}}{1+q^{4n-1}}\right).$$

Further, replace q by $-q$ in the previous identity to get

$$\begin{aligned}\left(\sum_{n=1}^{\infty} q^{n^2}\right)^2 &= 1-4\sum_{n=1}^{\infty}\left(\frac{-q^{4n-3}}{1-q^{4n-3}}+\frac{q^{4n-1}}{1-q^{4n-1}}\right)\\ &= 1+4\sum_{n=1}^{\infty}\left(\frac{q^{4n-3}}{1-q^{4n-3}}-\frac{q^{4n-1}}{1-q^{4n-1}}\right)\\ &= 1+4\sum_{n=1}^{\infty}(d_1(n)-d_3(n))q^n,\end{aligned}$$

which by virtue of (2.15) means that

$$\sum_{n=0}^{\infty} r_2(n)q^n = 1+4\sum_{n=1}^{\infty}\left(d_{1,4}(n)-d_{3,4}(n)\right)q^n.$$

Thus

$$r_2(n) = 4\left(d_{1,4}(n)-d_{3,4}(n)\right)$$

which is the desired formula. □

4.2 Application to integer partitions

We start by a recurrence relation for $p(n)$ in terms of the divisor function $\sigma(n) := \sum_{d|n} d$ which is due to Euler.

Theorem 4.2.1. *For each integer $n > 1$,*

$$np(n) = \sum_{j=0}^{n-1} p(j)\sigma(n-j). \tag{4.3}$$

Proof. Let,

$$F(q) = \sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}. \tag{4.4}$$

Put logarithm on both sides:

$$\begin{aligned}\log F(q) &= \log \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots} \\ &= -\log((1-q)(1-q^2)(1-q^3)\cdots) \\ &= -\sum_{n=1}^{\infty} \log(1-q^n) + C.\end{aligned}$$

Then differentiate both sides,

$$\frac{F'(q)}{F(q)} = -\sum_{n=1}^{\infty} \frac{-nq^{n-1}}{1-q^n} = \sum_{n=1}^{\infty} \frac{nq^{n-1}}{1-q^n}.$$

The denominator is a geometric series so we can write,

$$\frac{F'(q)}{F(q)} = \sum_{n=1}^{\infty} nq^{n-1} \sum_{m=0}^{\infty} q^{mn}.$$

Multiply both sides by q to get,

$$q\frac{F'(q)}{F(q)} = q\sum_{n=1}^{\infty} nq^{n-1} \sum_{m=0}^{\infty} q^{mn} = \sum_{n=1}^{\infty} nq^n \sum_{m=0}^{\infty} q^{mn}.$$

Then expand the summation,

$$
\begin{aligned}
q\frac{F'(q)}{F(q)} &= \sum_{n=1}^{\infty} nq^n \sum_{m=0}^{\infty} q^{mn} \\
&= q(1+q+q^2+\cdots)+2q^2(1+q^2+q^4+\cdots)+3q^3(1+q^3+\cdots) \\
&+4q^4(1+q^4+\cdots)+\cdots \\
&= (1)q+(1+2)q^2+(1+3)q^3(1+2+4)q^4+(1+5)q^5+\cdots \\
&= \sigma(1)q+\sigma(2)q^2+\sigma(3)q^3+\sigma(4)q^4+\sigma(5)q^5+\cdots \\
&= \sum_{n=1}^{\infty} \sigma(n)q^n.
\end{aligned}
$$

Differentiating both sides of equation (4.4), and multiplying by q, gives:

$$
qF'(q) = q\sum_{n=1}^{\infty} np(n)q^{n-1} = \sum_{n=1}^{\infty} np(n)q^n = \sum_{n=0}^{\infty} np(n)q^n.
$$

Then,

$$
q\frac{F'(q)}{F(q)} = \sum_{n=1}^{\infty} \sigma(n)q^n,
$$

or

$$
qF'(q) = F(q)\sum_{n=1}^{\infty} \sigma(n)q^n.
$$

Then,

$$
\sum_{n=0}^{\infty} np(n)q^n = F(q)\sum_{n=1}^{\infty} \sigma(n)q^n = \sum_{n=0}^{\infty} p(n)q^n \sum_{n=1}^{\infty} \sigma(n)q^n
$$

Finally we deduce that,

$$np(n) = \sum_{j=0}^{n-1} p(j)\sigma(n-j).$$

□

In the following result we present the method of Hirschhorn [11] to reproduce one of Ramanujna's famous congruence involving the partition function $p(n)$.

Theorem 4.2.2.

$$p(5n+4) \equiv 0 \pmod 5.$$

Proof. Remember that Jacobi's identity 2.1.6 states that

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}.$$

The coefficient $(-1)^n(2n+1)$ is congruent to $0, \pm 1$ or ± 2 modulo 5. Specifically, $(-1)^n(2n+1) \equiv 1$ if and only if $n \equiv 0$ or 9 (mod 10), $(-1)^n(2n+1) \equiv -1$ if and only if $n \equiv 4$ or 5 (mod 10), $(-1)^n(2n+1) \equiv 2$ if and only if $n \equiv 1$ or 8 (mod 10), $(-1)^n(2n+1) \equiv -2$ if and only if $n \equiv 3$ or 6 (mod 10), and $(-1)^n(2n+1) \equiv 0$ if and only if $n \equiv$2 or 7 (mod 10). For example we can check that $(-1)^n(2n+1) \equiv 1 \pmod 5$ if and only if $n \equiv 0$ or 9 (mod 10) as follows:

$$\begin{cases} n \equiv 0 & \pmod 2 \\ 2n+1 \equiv 1 & \pmod 5 \end{cases} \Rightarrow \begin{cases} n \equiv 0 & \pmod 2 \\ 2n \equiv 0 & \pmod 5 \end{cases} \Rightarrow \begin{cases} n \equiv 0 & \pmod 2 \\ n \equiv 0 & \pmod 5 \end{cases} \Rightarrow n \equiv 0 \text{ (mod 10)}$$

and,

$$\begin{cases} n \equiv 1 & \pmod 2 \\ 2n+1 \equiv -1 & \pmod 5 \end{cases} \Rightarrow \begin{cases} n \equiv 1 & \pmod 2 \\ 2n \equiv 3 & \pmod 5 \end{cases} \Rightarrow \begin{cases} n \equiv 1 & \pmod 2 \\ n \equiv 4 & \pmod 5 \end{cases} \Rightarrow n \equiv 9 \ (\bmod 10).$$

So, $(-1)^n(2n+1) \equiv 1$ iff $n \equiv 0$ or 9 (mod 10).

Thus we work with modulo 5 and get,

$$\begin{aligned}(q,q)_\infty^3 &\equiv \sum_{n=0}^{\infty} q^{10n(10n+1)/2} + \sum_{n=0}^{\infty} q^{(10n+9)(10n+10)/2} - \sum_{n=0}^{\infty} q^{(10n+4)(10n+5)/2} \\ &\quad - \sum_{n=0}^{\infty} q^{(10n+5)(10n+6)/2} + 2\sum_{n=0}^{\infty} q^{(10n+1)(10n+2)/2} + 2\sum_{n=0}^{\infty} q^{(10n+8)(10n+9)/2} \\ &\quad - 2\sum_{n=0}^{\infty} q^{(10n+3)(10n+4)/2} - 2\sum_{n=0}^{\infty} q^{(10n+6)(10n+7)/2} \\ &\equiv \sum_{n=0}^{\infty} q^{50n^2+5n} + \sum_{n=0}^{\infty} q^{50n^2+95n+45} - \sum_{n=0}^{\infty} q^{50n^2+45n+10} \\ &\quad - \sum_{n=0}^{\infty} q^{50n^2+55n+15} + 2\sum_{n=0}^{\infty} q^{50n^2+15n+1} + 2\sum_{n=0}^{\infty} q^{50n^2+85n+36} \\ &\quad - 2\sum_{n=0}^{\infty} q^{50n^2+35n+6} - 2\sum_{n=0}^{\infty} q^{50n^2+65n+21}.\end{aligned}$$

Then we have,

$$(q;q)_\infty^3 = X + 2qY,$$

where each of X, Y is a series in powers of q^5 since in the first four sums the powers of q are $\equiv 0 \pmod 5$, while in the latter four sums the powers of q are $\equiv 1 \pmod 5$. Also,

$$\begin{aligned}(q;q)_\infty^5 &= \prod_{n=1}^{\infty}(1-q^n)^5 = \prod_{n=1}^{\infty}(1-5q^n+10q^{2n}-10q^{3n}+5q^{4n}-q^{5n}) \\ &\equiv \prod_{n=1}^{\infty}(1-q^{5n}) \equiv (q^5;q^5)_\infty.\end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \frac{(q;q)_\infty^9}{(q;q)_\infty^{10}} = \frac{((q;q)_\infty^3)^3}{((q;q)_\infty^5)^2} \equiv \frac{((q;q)_\infty^3)^3}{(q^5;q^5)_\infty^2} \equiv \frac{(X+2qY)^3}{(q^5;q^5)_\infty^2}$$
$$\equiv \frac{X^3+6qX^2Y+12q^2XY^2+8q^3Y^3}{(q^5,q^5)_\infty^2}$$
$$\equiv \frac{X^3+qX^2Y+2q^2XY^2+3q^3Y^3}{(q^5,q^5)_\infty^2}.$$

We observe that the last expression on the right when expanded in q will not contain any fourth powers. In other words,

$$\sum_{n=1}^{\infty} p(5n+4)q^{5n+4} \equiv 0 \pmod{5}.$$

□

Similar proofs also exist for the following results of Ramanujan.

Theorem 4.2.3.

(a) $p(7n+5) \equiv 0 \pmod{7}$

(b) $p(11n+6) \equiv 0 \pmod{11}$

Bibliography

[1] G. E. Andrews, *The theory of partitions, Vol. 2*, Cambridge University Press, 1984.

[2] G. E. Andrews, *Catalan numbers, q-Catalan numbers and hypergeometric series*, J. Combin. Theory Ser. A 44 (1987), 267-273.

[3] B. C. Berndt, *Number theory in the spirit of Ramanujan, Vol. 34*, American Mathematical Society, 2006.

[4] M. El Bachraoui and A. Al Suwaidi, *On some new formulas involving the binoimal coefficients*, Submitted.

[5] L. Euler, *Introductio analysin infinitorium*, Lusanne 1 (1748), 253-275.

[6] J. A. Ewell, *A simple proof of Fermat's two-square theorem*, Amer. Math. Monthly 90 (1983), 635-637.

[7] G. Gasper and M. Rahman, *Basic Hypergeometric series*, Cambridge University Press, 2004.

[8] V.J.W. Guo and C. Krattenthaler, *Some divisibility properties of binomial and q-binomial coefficients*, J. Number Theory 135 (2014), 167-184.

[9] M. D. Hirschhorn, *A simple proof of Jacobi's two-square theorem*, Amer. Math. Monthly 92 (1985), 579-580.

[10] M. D. Hirschhorn, *A simple proof of Jacobi's four-square theorem*, Proc. Amer. Math. Soc. 101 (1987), 436-438.

[11] M. D. Hirschhorn, *Another short proof of Ramanujan's mod 5 partition congruence, and more*, Amer. Math. Monthly 106 (1999), 580-583.

[12] R. P. Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge University Press, 1999.

[13] Z.-W. Sun, *Products and sums divisible by central binomial coefficients*, Electron. J. Combin. 20(1) (2013), #P9.

[14] K. S. Williams, *Number Theory in the Spirit of Lioville, Vol. 76*, London Mathematical Society, 2011.