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UNIVERSAL CONSTRAINTS OF KLEINIAN GROUPS AND HYPERBOLIC GEOMETRY

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Declaration of Original Work

I, Hala Alaqad, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this dissertation entitled *"Universal Constraints of Kleinian Groups and Hyperbolic Geometry"*, hereby, solemnly declare that this thesis is my own original research work that has been done and prepared by me under the supervision of Dr. Jianhua Gong, in the College of Science at UAEU. This work has not previously been presented or published, or formed the basis for the award of any academic degree, diploma or a similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my dissertation have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this dissertation.

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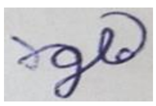
   Signature  Date July 7, 2020

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Abstract

Recent advances in geometry have shown the wide application of hyperbolic geometry not only in Mathematics but also in real-world applications. As in two dimensions, it is now clear that most three-dimensional objects (configuration spaces and manifolds) are modeled on hyperbolic geometry. This point of view explains a great many things from large-scale cosmological phenomena, such as the shape of the universe, right down to the symmetries of groups and geometric objects, and various physical theories. Kleinian groups are basically discrete groups of isometries associated with tessellations of hyperbolic space. They form the fundamental groups of hyperbolic manifolds. Over the last few decades, the theory of Kleinian groups has flourished because of its intimate connections with low-dimensional topology and geometry, especially with three-manifold theory.

The universal constraints for Kleinian groups in part arise from a novel description of the moduli spaces of discrete groups and generalize known universal constraints for Fuchsian groups - discrete subgroups of isometries of the hyperbolic plane. These generalizations will underpin a new understanding of the geometry and topology of hyperbolic three-manifolds and their associated singular spaces, hyperbolic three-orbifolds.

The novel approach in this dissertation is to use a fundamental result concerning spaces of finitely generated Kleinian groups: they are closed in the topology of algebraic convergence. Indeed, this is also true in higher dimensions when fairly minor additional and necessary conditions are imposed – for instance, giving a uniform bound on the torsion in a sequence, or asking that the limit set is in geometric position. In fact, this property holds more generally for groups of isometries of negatively curved metrics because of the Margulis-Gromov lemma. In particular, new polynomial trace identities in the Lie group $\mathrm{SL}(2, \mathbb{C})$ are discovered to expose various quantifiable inequalities (including Jørgensen's inequality) in a more general setting for Kleinian groups and the geometry of associated three-manifolds. This approach offers further substantive advances to

address the quite complicated analytic and topological properties of hyperbolic orbifolds, thereby advancing the solutions of important unsolved problems.

Keywords: Kleinian groups, moduli spaces, quantifiable inequalities, triple of complex parameters, universal constraints, hyperbolic geometry.

Title and Abstract (in Arabic)

القيود الشاملة لزمر كلاينين و الهندسة الزائدية

الملخص

أظهرت التطورات الحديثة في الهندسة العديد من التطبيقات للهندسة الزائدية ليست في الرياضيات فحسب لكن أيضا في التطبيقات الحياتية. كما هو الحال في الأبعاد الثنائية، فقد أصبح من الواضح الآن أن معظم الأشياء ثلاثية الأبعاد (مساحات التكوين والمشعبات) يمكن نمذجتها في الهندسة الزائدية. فهذا الأمر يشرح العديد من الظواهر الكونية واسعة النطاق. فعلى سبيل المثال شكل الكون، وصولًا إلى تماثل المجموعات والأشياء الهندسية والنظريات الفيزيائية المختلفة. زمر كلاينين يمكن وصفها ببساطة على إنها زمر منفصلة من القياسات المترافقة مع جزيئات الفضاء الزائدي، وتشكل الزمر الأساسية من المشعبات الزائدية. فعلى مدى العقود القليلة الماضية، ازدهرت نظرية زمر كلاينين بسبب روابطها الوثيقة مع التبولوجيا ذات الأبعاد المنخفضة والهندسة، وخاصة مع نظرية المشعب الثلاثي.

تنشأ القيود العالمية لزمر الكلاينين من وصف جديد للفراغات المعيارية للمجموعات المنفصلة ومن تعميم القيود الشاملة لزمر الفيوجين والتي هي عبارة عن زمر جزئية منفصلة من قياسات المستوى الزائدي. هذه التعميمات ستدعم الفهم الجديد للهندسة وتبولوجيا المشعبات الثلاثة الزائدية والمساحات المفردة المرتبطة بها، والمزدوجات المدارية الزائدية الثلاثية.

الطريقة الجديدة في هذه الأطروحة هي استخدام نتيجة أساسية تتعلق بفراغات زمر الكلاينين التي تم إنشاؤها بشكل نهائي: فهي مغلقة في توبولوجيا التقارب الجبرية. هذا الأمر صحيح أيضا في الأبعاد العليا عندما يتم فرض شروط إضافية، فعلى سبيل المثال إعطاء حد موحد للالتواء في متسلسلة، أو طلب أن يكون نهاية المجموعة في وضع هندسي. في الحقيقة، هذه الخاصية تتحقق بشكل أعم لزمرمن القياسات ذات المقاييس المنحنية سلبًا بسبب نتيجة مارجوليس-غروموف زسينهوس. على وجه الخصوص، أنشأنا هويات تتبع جديدة متعددة الحدود في زمر لي شبة البسيطة لكشف متباينات مختلفة قابلة للقياس الكمي (بما في ذلك متباينة Jørgensen) في وضع أكثرعمومية لزمر الكلاينين والهندسة من ثلاثة مشعبات مرتبطة. يقدم هذا النهج تقدمًا جوهريًا إضافيًا لمعالجة الخصائص التحليلية التبولوجية المعقدة تمامًا للمجموعات المدارية الزائدية، وبالتالي تطوير حلول للمشكلات المهمة التي لم يتم حلها.

مفاهيم البحث الرئيسية: زمر الكلاينين ، فراغات معيارية ، متباينات كمية، عوامل ثلاثية معقدة، قيود شاملة ، هندسة زائدية.

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Dedication

To my beloved parents and son

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Chapter 1: Introduction

Euclidean geometry is too limited to explain many of the various phenomena in the real world. Recent advances in geometry have shown the wide application of hyperbolic geometry not only in Mathematics but also in real-world applications. As in two dimensions, it is now clear that most three-dimensional objects (configuration spaces and manifolds) are modeled on hyperbolic geometry. This point of view explains a great many things from large-scale cosmological phenomena, such as the shape of the universe, right down to the symmetries of groups and geometric objects, and various physical theories.

Kleinian groups were introduced by Poincaré in the 1880's as subgroups of the Möbius group Möb($\overline{\mathbb{C}}$) acting discontinuously on some open domain of the Riemann sphere $\overline{\mathbb{C}}$. Nowadays the term "Kleinian group" is being often used for a discrete subgroup of hyperbolic isometries [29]. A Kleinian group is adopted as a non-elementary discrete group in this dissertation. Thus, Kleinian groups are basically discrete groups of hyperbolic isometries associated with tessellations of hyperbolic space. They form the fundamental groups of hyperbolic manifolds. Over the last few decades, the theory of Kleinian groups has flourished because of its intimate connections with low-dimensional topology and geometry, especially with 3-manifold theory [53, 54].

The identification of precise inequalities for discrete groups of Möbius transformations started with Jørgensen's famous inequality [35] from 1976, after earlier results of Shimizu from 1963 [51] and Leutbecher from 1967 [37] which gave estimates in the important special case when a generator is parabolic. Jørgensen's inequality is the first important universal constraint in studying the geometry of Kleinian groups. It is natural and interesting to generalize Jørgensen's inequality, there are many papers concerning such generalizations, for example, those published by Brooks and Matelski [5], Gilman [32], Rosenberger [50], and Tan [52]. The universal constraints for Kleinian groups generalize the known universal constraints for Fuchsian groups - discrete subgroups of isometries of hyperbolic plane [1]. The fundamental result concerning spaces of finitely generated Kleinian groups is that they are closed in the topology of algebraic convergence (see Jørgensen Theorem 3.2.13). Indeed this is also true in higher dimensions when fairly minor additional conditions are imposed, for instance, giving a uniform bound on the torsion in a sequence, or asking that the limit set be in geometric position [39]. In fact, this property holds more generally for groups of isometries of negatively curved metrics because of the Margulis-Gromov lemma [4, 18] which gives an estimate of the norm of iterated commutators [40].

The purposes of this dissertation are to expose various non-trivial quantifiable inequalities (including Jørgensen's inequality) in a more general setting for Kleinian groups and the geometry of associated 3-manifolds and to identify various sharp inequalities building on earlier work of Martin and his collaborators [39, 20, 21]. This approach offers further substantive advances to address the quite complicated analytic and topological properties of hyperbolic orbifolds, thereby advancing the solutions of important unsolved problems. In particular, these generalizations will underpin a new understanding of the geometry and topology of hyperbolic 3-manifolds and hyperbolic 3-orbifolds [9, 26, 27, 28].

The work of the dessertation in part arise from a novel description of the moduli spaces of discrete groups. One can describe the space of two-generator Kleinian groups $\langle f, g\rangle$ as a subspace of the three complex dimensional space $\mathbb{C}^3$ via the mapping

$$\langle f, g\rangle \longmapsto \left(\gamma\left(f, g\right), \beta\left(f\right), \beta(g)\right).$$

Indeed, every two-generator Kleinian group $\langle f, g\rangle$ can be determined uniquely up to conjugacy by a triple of complex parameters $\left(\gamma\left(f, g\right), \beta\left(f\right), \beta(g)\right)$ (Theorem 2.3.2).

A number of important tools are characterized, such as the finite order of elliptic elements by the explicit formula (Theorem 2.3.5), the classification of hyperbolic isometries by the conjugation and by the number of fixed points (Theorems 2.1.17 and 2.2.7), the clarification about the key concepts with multiple definitions in the literature (e.g., elementary groups in Theorem 2.4.6, discontinuity in Theorem 3.1.14, and the limit set in Lemma 3.1.6), the feature of elementary groups (Theorem 2.4.6), and the list of possible parameters for discrete elementary groups given by Tables 1, 2, and 3. These characterizations play key roles in enhancing the theory of Kleinian groups in Chapter 3 and in establishing the universal constraints for Kleinian groups in Chapter 4.

The novel approach here to establish the universal constraints for Kleinian groups is to use the closedness of the following subspaces, that is an essential tool for the scheme of establishing the quantifiable inequalities. The dissertation extends that the subspace $\mathcal{D}$ of triples of parameters for Kleinian groups:

$$\mathcal{D} = \big\{(\gamma, \beta, \beta') \in \mathbb{C}^3 : (\gamma, \beta, \beta') \text{ are the parameters of a Kleinian group } \langle f,\, g\rangle\big\}$$

is closed in $\mathbb{C}^3$ in the usual topology (Theorem 3.2.15), and that the subspace $\mathcal{D}_2$ of the first two parameters for Kleinian groups

$$\mathcal{D}_2 = \{(\gamma,\, \beta) : \text{for some } \beta' \text{ such that } (\gamma, \beta, \beta') \text{ are the parameters of a Kleinian group}\}.$$

is a closed in two complex dimensional space $\mathbb{C}^2$ in the usual topology (Theorem 3.3.4) by considering two projections: one is from $\mathcal{D}$ to $\mathcal{D}_2$ and the other is from $\mathcal{D}$ to the subspace on the slice $z_3 = -4$ in $\mathbb{C}^3$.

This dissertation discovers infinitely many polynomial trace identities in the Lie group SL(2, $\mathbb{C}$) that are useful for establishing various quantifiable inequalities for Kleinian groups and obtaining geometric information about Kleinian groups. These trace polynomials can be expressed simply in terms of the Chebyshev polynomial (Theorem 4.1.1 and Theorem 4.1.2). The Chebyshev polynomials were developed by Chebyshev in the mid-19th century for a completely different purpose and that they form an orthogonal system of polynomials which makes them of great use in Numerical Analysis and Approximation Theory that are very different fields from that of the current field Geometric Analysis.

The scheme of establishing the following sorts of quantifiable inequalities is implemented for two-generator Kleinian groups $\langle f, g\rangle$:

$$|\gamma(f, g) - \gamma_0| + |\beta(f) - \beta_0| \geq r,$$

where $\gamma_0 = \gamma(\phi, \psi)$ and $\beta_0 = \beta(\phi)$ are the parameters for a discrete elementary two-generator group $\langle \phi, \psi\rangle$ (Theorems 4.3.4, 4.3.5, 4.3.7, 4.3.8 and 4.3.10). However, the challenge here is how to find the various greatest lower bounds and to choose suitable trace polynomials.

An important application of the quantifiable inequalities shall be developed is in an explicit description of certain moduli spaces of Kleinian groups. The first such exploration of these moduli spaces appears to be that of Lyndon and Ullman [38] who used the Shimizu-Leutbecher inequality to describe the space of Kleinian groups generated by two parabolic elements, depending on how you look at it this is the moduli space of the punctured torus, or the four times punctured sphere. These early investigations led Riley to his important description of what is now known as the “Riley slice". Recent, the moduli space of Kleinian groups generated by elliptic elements of order 2 and 3 has been illustrated by the Figure 1.1 (see [41, 57]). Outside the bounded region (topologically a disk) the group is free on these generators, inside the groups are represented by isolated points and are rigid groups. Inside circles represent inequalities for groups with these generators. These descriptions of moduli spaces are successfully used in completing the solution to Siegel’s famous problem on hyperbolic lattices in three dimensions [48, 42, 44] and also in the identification of the finitely many two-generator arithmetic Kleinian groups with elliptic or parabolic generators.

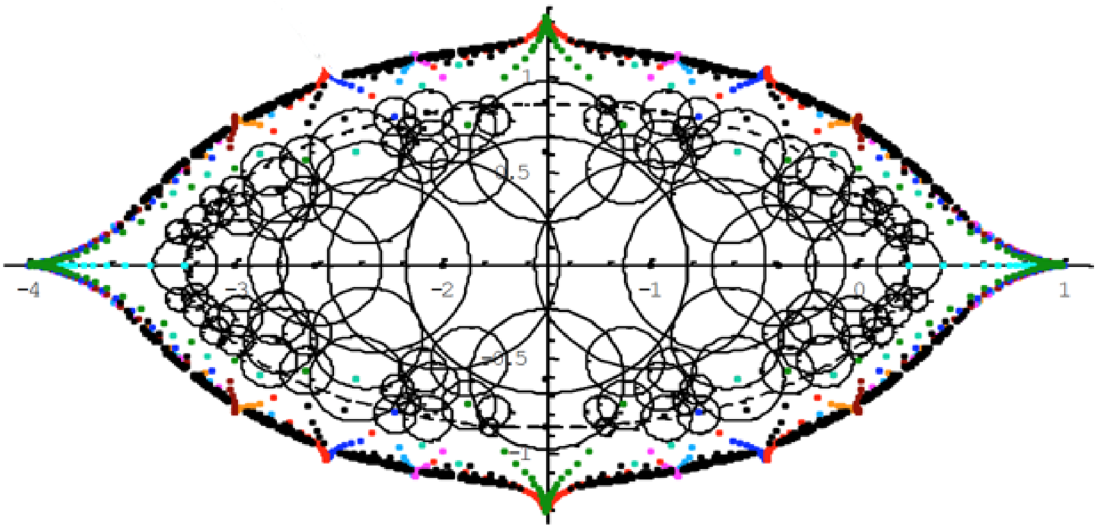

Figure 1.1: The (2, 3) commutator plane

Using these inequalities it is possible give quantifiable universal constraints on the geometry and topology of hyperbolic n-manifolds. Especially in the case of 3-manifolds and orbifolds, these inequalities underpin various computer searches for extremal hyperbolic manifolds, orbifolds, and groups. For instance,

- Bounds on the thick and the decomposition of hyperbolic 3-manifolds [3].
- That the projective general linear group $PGL(2, \mathcal{O}(\sqrt{-3}))$ is the smallest co-volume non-compact hyperbolic lattice [47] and also is the smallest co-volume lattice with singular set of degree $p \geq 6$ [19].
- The proof that homotopy hyperbolic 3-manifolds are hyperbolic [11, 14].
- Estimates on the first Betti number of closed hyperbolic 3-manifolds [7]
- That the orientable cusped hyperbolic 3-manifolds of minimum volume is the figure 8-knot complement and its sister [8].
- That the Fomenko-Weeks manifold as minimal volume hyperbolic 3-manifold [12, 13].

- That the $\mathbb{Z}_2$-extension of the 3-5-3 Coxeter group is the smallest co-volume hyperbolic lattice [22, 31, 48].

- The Margulis constants associate with discrete groups [23, 24].

In fact there are many more of these sorts of things, and higher dimensional versions of these quantifiable inequalities allow some estimates for hyperbolic n-manifolds, see for instance [39, 34, 55].

Chapter 2: Parameters of Groups of Isometries

It is reasonable that one can associate two kinds of structures with a set. A topological group is both an algebraic group and a topological space and the two structures are related. Precisely, a group G is a *topological group* if the following two maps are continuous: $G \times G \longrightarrow G$ by $(x, y) \longmapsto xy$ and $G \longrightarrow G$ by $x \longmapsto x^{-1}$. Two topological groups can be naturally identified if there exists a group isomorphism between them that is a homeomorphism as well. Though out the dissertation, a topological space is always non-empty and a neighborhood refers to an open neighborhood. Let $\mathbb{C}$ be the complex plane, and let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the extended complex plane, it is also known as the *Riemann sphere* .

There are at least three different ways of thinking about groups in this dissertation: as subgroups of the hyperbolic isometry group $\mathrm{Isom}^+(\mathbb{H}^3)$ on the Poincaré upper half-space model, as subgroups of Möbius group $\mathrm{M\ddot{o}b}^+\left(\overline{\mathbb{C}}\right)$, and as subgroups of the projective special linear group $\mathrm{PSL}(2, \mathbb{C})$. Each of these groups has its own topology, however the topological isomorphism tells that the concept of discreteness is the same (see Theorem 2.2.3 in Section 2.2). The following three elements will be identified throughout this dissertation: a Möbius transformation in $\mathrm{M\ddot{o}b}^+(\overline{\mathbb{C}})$, a hyperbolic isometry in $\mathrm{Isom}^+(\mathbb{H}^3)$, and a matrix in $\mathrm{PSL}(2, \mathbb{C})$. Thus, one can interact between Complex Analysis, Abstract Algebra, Hyperbolic Geometry, and Topology.

An advanced tool for studying two-generator Kleinian groups is the triple of complex parameters for each two-generator group that is introduced in Definition 2.3.1. Several important tools are characterized, such as the finite order of elliptic elements by the explicit formula (Theorem 2.3.5), the classification of hyperbolic isometries by the conjugation and by the number of fixed points (Theorems 2.1.17 and 2.2.7), the clarification about different definitions of elementary groups in the literature (Theorem 2.4.6), the feature of elementary groups (Theorem 2.4.6), and the list of parameters for discrete elementary groups given by Tables 1, 2, and 3. These characterizations play key roles in enhancing the theory of Kleinian groups in Chapter 3 and in establishing the universal constraints for Kleinian groups in Chapter 4.

2.1 Representations of Möbius groups

One of important tools is conjugation. There are three classifications of the special Möbius group Möb$^+(\overline{\mathbb{C}})$ according to the conjugation in Theorem 2.1.17 (including the standard representation by matrices) and the conjugate invariants such as the trace in Definition 2.1.6 and the number of fixed points that is prepared in Lemma 2.1.14 and will be finalized in Theorem 2.2.7 in the next section. The image of an invariant set (including the set of fixed points in $\overline{\mathbb{C}}$) under a conjugation is described in Theorem 2.1.9.

Definition 2.1.1 *A linear fractional transformation of the following form is called a* Möbius transformation *on the Riemann sphere* $\overline{\mathbb{C}}$:

$$f(z) = \frac{az+b}{cz+d}, \;\; a,b,c,d \in \mathbb{C} \textit{ and } ad-bc \neq 0.$$

Denote Möb$(\overline{\mathbb{C}})$ *the set of all Möbius transformations on* $\overline{\mathbb{C}}$:

$$\text{Möb}(\overline{\mathbb{C}}) = \left\{ \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{C} \textit{ and } ad-bc \neq 0 \right\}.$$

Clearly, Möb$(\overline{\mathbb{C}})$ is a group under composition, Möb$(\overline{\mathbb{C}})$ is simply said a *Möbius group* on $\overline{\mathbb{C}}$.

Furthermore, Möb($\overline{\mathbb{C}}$) is a topological group under composition with the topology induced by the following metric:

$$d(f,g) = \mathrm{Sup}_{z\in\overline{\mathbb{C}}}\ (q(f(z),g(z)),$$

where q denotes the chordal distance in the Riemann sphere $\overline{\mathbb{C}}$:

$$q(z_1,z_2) = \begin{cases} \frac{2|z_1-z_2|}{(1+|z_1|^2)^{\frac{1}{2}}(1+|z_2|^2)^{\frac{1}{2}}}, & \text{if } z_1, z_2 \in \mathbb{C}; \\ \frac{2}{(1+|z_1|^2)^{\frac{1}{2}}}, & \text{if } z_1 \in \mathbb{C} \text{ and } z_2 = \infty. \end{cases}.$$

Each Möbius transformation

$$f(z) = \frac{az+b}{cz+d} \in \text{Möb}\left(\overline{\mathbb{C}}\right)$$

is associated with a matrix

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where, f is the associated matrix of each element $f(z)$ in Möb $\left(\overline{\mathbb{C}}\right)$.

Note that each element of Möb($\overline{\mathbb{C}}$) will keep the same if one multiplies both numerator and denominator of $\frac{az+b}{cz+d}$ by any non-zero number, but the corresponding matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ may be changed. One can take $\sqrt{ad-bc}$ to normalize Möb($\overline{\mathbb{C}}$) by introducing the following subgroup of Möb($\overline{\mathbb{C}}$) :

$$\text{Möb}^+(\overline{\mathbb{C}}) = \left\{ f \in \text{Möb}(\overline{\mathbb{C}}) : f(z) = \frac{az+b}{cz+d},\ ad-bc = 1 \right\}.$$

Obviously, Möb$^+$($\overline{\mathbb{C}}$) is a topological subgroup of Möb($\overline{\mathbb{C}}$).

Consider the *special linear group* $\mathrm{SL}(2,\mathbb{C})$ of 2×2 matrices over $\mathbb{C}$ under multiplication:

$$\mathrm{SL}(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d, \in \mathbb{C} \text{ and } ad - bc = 1 \right\}.$$

Then consider the *projective special linear group* $\mathrm{PSL}(2,\mathbb{C})$ of the following quotient group:

$$\begin{aligned} \mathrm{PSL}(2,\mathbb{C}) &= \mathrm{SL}(2,\mathbb{C}) / \{\pm Id\} \\ &= \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d, \in \mathbb{C} \text{ and } ad - bc = 1 \right\}, \end{aligned}$$

where Id is the 2×2 identity matrix. Both matrices $\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be regarded as being the same as they lead to the same Möbius transformation on the Riemann sphere $\overline{\mathbb{C}}$.

As a quotient group, $\mathrm{PSL}(2,\mathbb{C})$ preserves the group structure of the group $\mathrm{SL}(2,\mathbb{C})$. Clearly, $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{PSL}(2,\mathbb{C})$ are topological groups under multiplication with the topology induced by the following norm:

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

Each Möbius transformation of $\text{Möb}^+(\overline{\mathbb{C}})$ will be identified with the 2×2 matrix of $\mathrm{PSL}(2,\mathbb{C})$ because of the following well known theorem.

Theorem 2.1.2 *The topological groups* $\text{Möb}^+(\overline{\mathbb{C}})$ *and* $\mathrm{PSL}(2,\mathbb{C})$ *are topologically isomorphic:*

$$\text{Möb}^+(\overline{\mathbb{C}}) \cong \mathrm{PSL}(2,\mathbb{C}).$$

Definition 2.1.3 *Consider a topological group G. Suppose that Γ and Γ' are subgroups of G, Γ' is said to be* conjugate *to Γ in G if there exists $h \in G$ such that*

$$\Gamma' = h \circ \Gamma \circ h^{-1} = \left\{hgh^{-1} : g \in \Gamma\right\},$$

denoted by $\Gamma' \sim \Gamma$ and h is called a conjugator *. In particular, two elements g and g' of G are conjugate in G if there exists $h \in G$ such that $g' = h \circ g \circ h^{-1}$, denoted by $g' \sim g$.*

For example, let Γ and Γ' be two subgroups of $\text{Möb}(\overline{\mathbb{C}})$, if there is $\phi \in \text{Möb}(\overline{\mathbb{C}})$ such that $\Gamma' = \phi \circ \Gamma \circ \phi^{-1}$, then Γ' is conjugate to Γ in $\text{Möb}(\overline{\mathbb{C}})$. Clearly, the following three statements are equivalent:

(a) f is self conjugate by h.

(b) f and h commute.

(c) The commutator $[f, h] = I$.

Let f, g, f', g' and h be elements of a topological group G, then two subgroups Γ and Γ', which are generated by f, g and f', g', respectively, are conjugate by h if and only if the generators f and f', g and g' are conjugate by h, respectively.

Furthermore, it is natural to expect the conjugation is an *equivalence relation* on the subgroups of a topological group in the following proposition, that is the conjugation provides a *partition* for a topological group.

Proposition 2.1.4 *Suppose that G is a topological group acting on a topological space. Then the following are true.*

(a) *Every subgroup Γ of $G : \Gamma \sim \Gamma$.*

(b) *Every pair of two subgroups $\Gamma$and Γ' of $G : \Gamma \sim \Gamma'$ implies $\Gamma' \sim \Gamma$.*

(c) *Every triple of subgroups* Γ, Γ', Γ'' *of* G : $\Gamma \sim \Gamma'$ *and* $\Gamma' \sim \Gamma''$ *implies* $\Gamma \sim \Gamma''$.

Proof. The part (a) and (b) are trivial.

(c) Since $\Gamma \sim \Gamma'$ and $\Gamma' \sim \Gamma''$, there exist $\phi_1, \phi_2 \in G$ such that $\Gamma = \phi_1 \circ \Gamma' \circ \phi_1^{-1}$ and $\Gamma' = \phi_2 \circ \Gamma'' \circ \phi_2^{-1}$. It follows that

$$\Gamma = \phi_1 \circ \left(\phi_2 \circ \Gamma'' \circ \phi_2^{-1}\right) \circ \phi_1^{-1} = (\phi_1 \circ \phi_2) \circ \Gamma'' \circ (\phi_1 \circ \phi_2)^{-1}.$$

Thus there exists $\phi = \phi_1 \circ \phi_2 \in G$ such that $\Gamma = \phi \circ \Gamma'' \circ \phi^{-1}$, i.e., $\Gamma \sim \Gamma''$. □

Most interest things in the geometry of Möbius groups are conjugate invariants. This dissertation will pay the attention to the information "up to conjugacy". Notice that the trace is the first conjugate invariant of matrices and therefore on $\mathrm{SL}(2, \mathbb{C})$, but the trace is not well defined in $\mathrm{PSL}(2, \mathbb{C})$. Fortunately it is well defined up to sign in $\mathrm{PSL}(2, \mathbb{C})$. Thus, one can define $\mathrm{tr}^2(f)$ for each $f \in \mathrm{Möb}^+(\overline{\mathbb{C}})$ by the square of the trace of $\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C})$:

$$\mathrm{tr}^2(f) = (a + d)^2.$$

Remark 2.1.5 *In Linear Algebra, it is known that if two matrices are conjugate then they have the same trace but the converse need not be held. However, Beardon showed in ([2], Theorem* 4.3.1.*) that two non-identity elements* f *and* g *of* $\mathrm{Möb}^+(\overline{\mathbb{C}})$ *are conjugate if and only if* $\mathrm{tr}^2(f) = \mathrm{tr}^2(g)$.

Now using the square of the trace define three types of non-identity elements in Möbius group $\mathrm{Möb}^+(\overline{\mathbb{C}})$.

Definition 2.1.6 *A non-identity element* f *of* $\mathrm{Möb}^+(\overline{\mathbb{C}})$ *is said to be*

(a) elliptic if $\mathrm{tr}^2(f) \in [0, 4)$,

(b) parabolic if $\mathrm{tr}^2(f) = 4$,

(c) loxodromic if $\mathrm{tr}^2(f) \in \mathbb{C} - [0,4]$.

The second conjugate invariant is the number of fixed points. Here is the definition of fixed points of homeomorphisms.

Definition 2.1.7 *A subset* S *of a topological space* X *is said to be* invariant *under a self-homeomorphism* f *if* $f(S) = S$. *A point* x *in* X *is called a* fixed point *of* f *if the singleton* $S = \{x\}$ *is an invariant set under* f. *In the case of* $X = \overline{\mathbb{C}}$ *and* $f \in \mathrm{M\ddot{o}b}(\overline{\mathbb{C}})$, *the set of fixed points of* f *in* $\overline{\mathbb{C}}$ *is denoted by* $\mathrm{Fix}(f)$:

$$\mathrm{Fix}(f) = \left\{z \in \overline{\mathbb{C}} : f(z) = z\right\}.$$

Moreover, denote the set of fixed points of a group G *by* $\mathrm{Fix}(G)$:

$$\mathrm{Fix}(G) = \left\{z \in \overline{\mathbb{C}} : f(z) = z \textit{ for all } f \in G\right\}.$$

Proposition 2.1.8 *Suppose that* G *is a topological group acting on a topological space* X, x *is a point of* X, *and* g *is an element of* G. *If* g *fixes* x *then the cyclic group* $\langle g \rangle$ *fixes* x *and hence* g^n *fixes* x *for* $n \in \mathbb{Z}$.

Proof. Since $g \in G$ fixes x, g^{-1} fixes x. Thus, g^n fixes x for each $n \in \mathbb{Z}$ because that:

$$g^n(x) = g \circ \cdots \circ g(x) = x, \text{ for } n = 0, 1, 2, \cdots$$
$$g^n(x) = g^{-1} \circ \cdots \circ g^{-1}(x) = x, \text{ for } n = -1, -2, \cdots$$

It follows that the cyclic group $\langle g \rangle = \{g^n : n \in \mathbb{Z}\}$ fixes x. $\square$

Theorem 2.1.9 *Suppose that G is a topological group acting on a topological space X. Let g and h be two distinct elements of G. Let S be a subset of X.*

Proposition 2.1.10 (a) *If S is invariant under g, then the image $h(S)$ is invariant under the conjugacy $h \circ g \circ h^{-1}$.*

(b) *If S is the set of fixed points of g, then $h(S)$ is the set of fixed points of $h \circ g \circ h^{-1}$.*

Proof. (a) Since S is invariant under g, $h(S) = S$, and hence $h \circ g \circ h^{-1}(h(S)) = h \circ g(S) = h(S)$, i.e., $h(S)$ is invariant under the conjugacy $h \circ g \circ h^{-1}$.

(b) Let $f = h \circ g \circ h^{-1}$, and let F be the set of the fixed points of $h \circ g \circ h^{-1}$. It needs to show $h(S) = \mathrm{F}$.

First to show $h(S) \subseteq \mathrm{F}$: Assume x is a fixed point of g, i.e., $g(x) = x$. Then $h(x)$ is a fixed point of f because that $f(h(x)) = h \circ g \circ h^{-1}(h(x)) = h(g(x)) = h(x)$.

Second to show $\mathrm{F} \subseteq h(S)$: Since f is conjugate to g by h, g is conjugate to f by h^{-1}. So by the first part, $h^{-1}(\mathrm{F}) \subseteq S$. And then perform h on both sides: $h \circ h^{-1}(\mathrm{F}) \subseteq h(\mathrm{F})$ gives $\mathrm{F} \subseteq h(S)$. □

In particular, the previous proposition gives the following corollary.

Corollary 2.1.11 *Suppose that two Möbius transformations f and g in* $\text{Möb}^+(\overline{\mathbb{C}})$ *are conjugate by h in* $\text{Möb}(\overline{\mathbb{C}})$, *then $f = h \circ g \circ h^{-1}$ and*

$$\mathrm{Fix}(f) = \mathrm{Fix}\left(h \circ g \circ h^{-1}\right) = h(\mathrm{Fix}(g)).$$

Traditionally, a map f is conformal in a domain D of $\mathbb{C}$ if f is analytic and $f'(z) \neq 0$ for $z \in D$. It is an orientation preserving mapping, i.e., preserving size and orientation of the angles. Since an injective analytic function f of D implies that $f'(z) \neq 0$ for $z \in D$, every conformal mapping here is a traditional conformal mapping and hence an orientation preserving mapping. By contrast, a traditional conformal mapping needs not to be a conformal mapping given by the following definition, for example, $f(z) = e^z$.

Definition 2.1.12 *Let D and D' be two domains in $\overline{\mathbb{C}}$. A map $f : D \to D'$ is called a* conformal mapping *if it is bijective and analytic. If $D = D'$, the set of conformal mappings is denoted by* $\mathrm{Conf}\,(D)$. *Clearly,* $\mathrm{Conf}\,(D)$ *is a group under composition, simply say* $\mathrm{Conf}\,(D)$ *is the* conformal group *on D.*

In the cases $D = D' = \mathbb{C}$ or $\overline{\mathbb{C}}$, the detailed proof of a known fact is showed in the following theorem.

Theorem 2.1.13 *(a) A map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with $f\,(\infty) = \infty$ is a conformal mapping (i.e., the restriction $f|_{\mathbb{C}}$ is a self conformal mapping on $\mathbb{C}$) if and only if $f(z) = az + b$, for some $a \neq 0, b \in \mathbb{C}$. Denote the group of self conformal mappings on $\mathbb{C}$ by* $\mathrm{Conf}\,(\mathbb{C})$,

$$\mathrm{Conf}\,(\mathbb{C}) = \{az + b : a \neq 0, b \in \mathbb{C}\}.$$

(b) A map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a conformal mapping if and only if $f \in \text{Möb}(\overline{\mathbb{C}})$. That is the group $\mathrm{Conf}\left(\overline{\mathbb{C}}\right)$ *coincides with the Möbius group* $\text{Möb}\left(\overline{\mathbb{C}}\right)$:

$$\mathrm{Conf}\left(\overline{\mathbb{C}}\right) = \text{Möb}\left(\overline{\mathbb{C}}\right).$$

Proof. (a) It is obvious that $f(z) = az + b$ with $a \neq 0$ is conformal on $\mathbb{C}$ and $f(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$ is conformal on $\overline{\mathbb{C}}$. Now show the necessity. Since the conformal mapping f is analytic in $\mathbb{C}$, the Taylor series expansion of $f(z)$ is

$$f(z) = \Sigma_{n=0}^{\infty} a_n z^n, \text{ for } z \in \mathbb{C}.$$

Since f cannot be a constant, there is $n \in \mathbb{N}$ such that $a_n \neq 0$, and then the isolated singularity of f at ∞ is an essential singularity or a pole. Set

$$g(z) = f(\frac{1}{z}) = \Sigma_{n=0}^{\infty} \frac{a_n}{z^n},$$

then g is analytic in $\mathbb{C} - \{0\}$. Recall the Casorati-Weierstrass theorem: If g is analytic in a deleted open disk U of z_0 and has an essential singularity at z_0, then $g(U)$ is dense in $\mathbb{C}$.

One can assume that g has an essential singularity at $z_0 = 0:$ let U_0 and U_1 be the open disks with radius $\frac{1}{3}$ centred at $z_0 = 0$ and $z_1 = 1$, respectively, then $U_0 \cap U_1 = \varnothing$. By Casorati-Weierstrass theorem, $g(U_0)$ is dense in $\mathbb{C}$ and hence $g(U_0) \cap g(U_1) \neq \varnothing$, but $U_0 \cap U_1 = \varnothing$. Let $w_0 \in g(U_0) \cap g(U_1)$, then there exist $z' \in U_0$ and $z'' \in U_1$, such that $f\left(\frac{1}{z'}\right) = f\left(\frac{1}{z''}\right) = w_0$ for $z = \frac{1}{z'}$ and $\frac{1}{z''} \in \mathbb{C}$. It follows that $f(z)$ is not injective in $\mathbb{C}$. It is the contradiction to the bijection of f in $\mathbb{C}$. Thus, $z_0 = 0$ is not an essential singularity of g and hence g has the unique pole of some order m at $z_0 = 0$. It follows that $g(z) = f(\frac{1}{z}) = \Sigma_{n=0}^{m} \frac{a_n}{z^n}$, so $a_m \neq 0$ for some $m \geq 1$ and $a_n = 0$ for all $n \geq m+1$. Thus, f is the polynomial of degree $m \geq 1:$

$$f(z) = a_0 + a_1 z + ... + a_m z^m$$

and hence f has exactly m roots, say $z_1, z_2, \cdots, z_m$ such that $f(z_1) = f(z_2) = \cdots = f(z_m) = 0$. On the other hand side, f is bijective, then $m = 1$. Therefore,

$$f(z) = a_0 + a_1 z.$$

(b) Suppose $f(\infty) = \infty$. Since f is conformal in $\overline{\mathbb{C}}$, the restriction $f|_{\mathbb{C}}$ is a self conformal mappings on $\mathbb{C}$.

By the previous part (a), $f|_{\mathbb{C}} = az + b$ for some $a \neq 0, b \in \mathbb{C}$, and hence $f = az + b$ for some $a \neq 0, b \in \mathbb{C}$.

Suppose $f(\infty) \neq \infty$, then $f(\infty) = w_0 \in \mathbb{C}$. Set

$$g(z) = \frac{1}{f(z) - w_0}.$$

Then $g(\infty) = \infty$ and the restriction $g|_{\mathbb{C}}$ is a self conformal mappings on $\mathbb{C}$. It follows from (a) that

$$g(z) = cz + d$$

for some $c \neq 0, d \in \mathbb{C}$. Thus,

$$f(z) = w_0 + \frac{1}{cz+d} = \frac{az+b}{cz+d} \in \text{Möb}(\overline{\mathbb{C}}).$$

where $a = w_0 c$ and $b = w_0 d + 1 \in \mathbb{C}$. $\square$

The relationship between the discriminant and the trace motivates the following lemma about the characterization of $\text{Möb}^+(\overline{\mathbb{C}})$ by the number of fixed points in $\overline{\mathbb{C}}$.

Lemma 2.1.14 *Let f be a non-identity element in* $\text{Möb}^+(\overline{\mathbb{C}})$*, then*

(a) f is parabolic if and only if it fixes exactly one point in $\overline{\mathbb{C}}$.

(b) f is either elliptic or loxodromic if and only if it fixes exactly two distinct points in $\overline{\mathbb{C}}$.

Proof. Let f be a non-identity element in $\text{Möb}^+(\overline{\mathbb{C}})$ and hence $f = \frac{az+b}{cz+d}$, for $a, b, c, d, \in \mathbb{C}$ and $ad - bc = 1$.

The fixed points in $\overline{\mathbb{C}}$ can be found by solving the equation:

$$\frac{az+b}{cz+d} = z.$$

Equivalently, solving the quadratic equation gives

$$cz^2 + (d - a)z - b = 0.$$

Notice the following discriminant,

$$\begin{aligned}\triangle &= (d-a)^2 + 4bc \\ &= (d-a)^2 + 4\,(ad-1) \\ &= (d+a)^2 - 4 \\ &= \mathrm{tr}^2(f) \ - 4.\end{aligned}$$

Thus, f has exactly one fixed point in $\overline{\mathbb{C}}$ if and only if $\mathrm{tr}^2(f) = 4$ and two distinct fixed points in $\overline{\mathbb{C}}$ if and only if $\mathrm{tr}^2(f) \neq 4$. Now it is straight forward to complete the proof by Definition 2.1.6. □

Moreover, the standard form of conjugation can be presented in the following lemma.

Lemma 2.1.15 *Let f be a non-identity element in* $\text{Möb}^+(\overline{\mathbb{C}})$*. Then:*

(a) *If f is parabolic then f is conjugate in* $\text{Möb}(\overline{\mathbb{C}})$ *to* $p(z) = z + 1$.

(b) *If f is either elliptic or loxodromic then f is conjugate in* $\text{Möb}(\overline{\mathbb{C}})$ *to* $h(z) = re^{i\theta}z$*, for* $\theta \in (0, 2\pi)$*. Furthermore,*

$$\mathrm{tr}^2(h) = \left(r + r^{-1}\right)\cos\theta + 2 + i\left(r - r^{-1}\right)\sin\theta.$$

Proof. (a) Since f is parabolic, by Lemma 2.1.14, f has a unique fixed point z_0 in $\overline{\mathbb{C}}$. Take $z_1 \neq z_0 \in \overline{\mathbb{C}}$ and let $z_2 = f(z_1)$ then $z_2 \neq z_1$ and $z_2 \neq z_0$. It is well known that $\text{Möb}(\overline{\mathbb{C}})$ is transitive on the set of triples of distinct points in

$\overline{\mathbb{C}}$. Thus there is a unique $g \in \text{Möb}(\overline{\mathbb{C}})$ such that

$$g(z_0) = \infty,\ g(z_1) = 0,\ g(z_2) = 1.$$

By Corollary 2.1.11, gfg^{-1} fixes the unique point $g(z_0) = \infty$. It follows from Theorem 2.1.13 that

$$gfg^{-1}(z) = az + b \text{ for } a \neq 0, b \in \mathbb{C}.$$

Moreover,

$$gfg^{-1}(0) = gf(z_1) = g(z_2) = 1.$$

Thus, $gfg^{-1}(0) = 1$ gives $b = 1$. Since gfg^{-1} has the unique point ∞, so $a = 1$. Otherwise, gfg^{-1} has another fixed point $z = \frac{1}{a-1}$. Now $gfg^{-1} = p$ and hence f is conjugate in $\text{Möb}(\overline{\mathbb{C}})$ to $p(z) = z + 1$.

(b) Since f is either elliptic or loxodromic, by Lemma 2.1.14, f has two distinct fixed points, say z_0 and z_1 in $\overline{\mathbb{C}}$. Let $z_2 \in \overline{\mathbb{C}} - \{z_0, z_1\}$. Applying for the transitivity of $\text{Möb}(\overline{\mathbb{C}})$ on the set of triples of distinct points in $\overline{\mathbb{C}}$, there is a unique $g \in \text{Möb}(\overline{\mathbb{C}})$ such that

$$g(z_0) = \infty,\ g(z_1) = 0,\ g(z_2) = 1.$$

By Corollary 2.1.11, gfg^{-1} fixes two points $g(z_0) = \infty$ and $g(z_1) = 0$. Applying for Theorem 2.1.13 for the fixed point ∞ of gfg^{-1}, $gfg^{-1} = az + b$ for some $a = re^{i\theta}, b \in \mathbb{C}$. On the other hand, the fixed point 0 of gfg^{-1} gives $b = 0$. One may assume $\theta \in [0, 2\pi]$. Since f is a non-identity element in $\text{Möb}^+(\overline{\mathbb{C}})$, $a \neq 1$ and hence $\theta \in (0, 2\pi)$. Otherwise, $\theta = 0, 2\pi$ give $a = r\cos\theta + ir\sin\theta = 1$. It follows that

$$gfg^{-1} = re^{i\theta}z = h(z),\ \text{for } \theta \in (0, 2\pi).$$

Thus, f is conjugate in Möb$(\overline{\mathbb{C}})$ to $h(z)$. Furthermore, the associated matrix and the trace are the following:

$$\begin{pmatrix} r^{\frac{1}{2}} e^{i\theta/2} & 0 \\ 0 & r^{-\frac{1}{2}} e^{-i\theta/2} \end{pmatrix},$$

$$\begin{aligned} \operatorname{tr}^2(h) &= \left(r^{\frac{1}{2}} e^{i\theta/2} + r^{-\frac{1}{2}} e^{-i\theta/2}\right)^2 \\ &= re^{i\theta} + r^{-1}e^{-i\theta} + 2 \\ &= \left(r + r^{-1}\right)\cos\theta + 2 + i\left(r - r^{-1}\right)\sin\theta. \end{aligned}$$

□

Now summarize the matrix representation associated with fixing or interchanging 0 and ∞, it is a useful tool in the dissertation.

Lemma 2.1.16 *Let f be an non-trivial element in* Möb$(\overline{\mathbb{C}})$.

(a) If f fixes 0 then $f(z) = \frac{az}{cz+d}$ for $a, c, d \in \mathbb{C}, ad \neq 0,$ and $(a, c) \neq (d, 0)$.

The associated matrix is

$$f = \begin{pmatrix} \lambda & 0 \\ \mu & \frac{1}{\lambda} \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C})$$

for some $\lambda \neq 0, \mu \in \mathbb{C}$ and $(\lambda, \mu) \neq (\pm 1, 0)$.

(b) If f fixes ∞ then $f(z) = az + b$ for $a \neq 0, b \in \mathbb{C}$ and $(a, b) \neq (1, 0)$.

The associated matrix is

$$f = \begin{pmatrix} \lambda & \mu \\ 0 & \frac{1}{\lambda} \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C})$$

for some $\lambda \neq 0, \mu \in \mathbb{C}$ *and* $(\lambda, \mu) \neq (\pm 1, 0)$.

(c) *If* f *fixes* 0 *and* ∞ *then* $f(z) = az$ *for* $a \neq 0, 1 \in \mathbb{C}$. *The associated matrix is*

$$f = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C})$$

for some $\lambda \neq 0, \pm 1 \in \mathbb{C}$.

(d) *If* f *interchanges* 0 *and* ∞ *then* $f(z) = \frac{b}{z}$ *for* $b \neq 0 \in \mathbb{C}$. *The associated matrix is*

$$f = \begin{pmatrix} 0 & -\lambda \\ \frac{1}{\lambda} & 0 \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C})$$

for some $\lambda \neq 0 \in \mathbb{C}$.

Proof. First of all, since $f \in \text{Möb}(\overline{\mathbb{C}})$, One can assume $f = \frac{a'z+b'}{c'z+d'}$ for $a', b', c', d' \in \mathbb{C}$ and $a'd' - b'c' \neq 0$. If $b' = c' = 0$ then $a' \neq d'$.

(a) Since f fixes 0, $f(0) = \frac{b'}{d'} = 0$ and hence $b' = 0$ and $d' \neq 0$, which give $f = \frac{az}{cz+d}$, where $a = a', c = c', d = d' \neq 0 \in \mathbb{C}$. Since $b' = 0$, $a'd' - b'c' = a'd' = ad \neq 0$ and $(a, c) \neq (d, 0)$. Moreover, the normalization provides the matrix representative, $f = \begin{pmatrix} \lambda & 0 \\ \mu & \frac{1}{\lambda} \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C})$, where $\lambda = \sqrt{\frac{a}{d}} \neq 0$, $\mu = \frac{c}{\sqrt{ad}} \in \mathbb{C}$, and $(\lambda, \mu) \neq (\pm 1, 0)$ as $(a, c) \neq (d, 0)$.

(b) Since f fixes ∞, by Theorem 2.1.13, $f(z) = az + b$, where $a = a' \neq 0, b = b' \in \mathbb{C}$. Since $c = c' = 0$ and $d = d' = 1, (a, b) \neq (1, 0)$. After normalization, the matrix representative is $f = \begin{pmatrix} \lambda & \mu \\ 0 & \frac{1}{\lambda} \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C})$, where $\lambda = \sqrt{a} \neq 0$, $\mu = \frac{b}{\sqrt{a}} \in \mathbb{C}$, and $(\lambda, \mu) \neq (\pm 1, 0)$ as $(a, b) \neq (1, 0)$.

(c) Obviously, it is the consequences of the previous parts (a) and (b).

(d) Since f interchanges 0 and ∞,

$$f(0) = \frac{b'}{d'} = \infty, \; f(\infty) = \frac{a'}{c'} = 0.$$

It follows that $d' = 0, a' = 0$, $b' \neq 0$, and $c' \neq 0$, which gives $f(z) = \frac{b}{z}$, where $b = \frac{b'}{c'} \neq 0 \in \mathbb{C}$. By normalizing, the matrix representative becomes $f = \begin{pmatrix} 0 & -\lambda \\ \frac{1}{\lambda} & 0 \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C})$, where $\lambda = i\sqrt{b} \neq 0 \in \mathbb{C}$. $\square$

Finally, the elements of $\mathrm{M\ddot{o}b}^+(\overline{\mathbb{C}})$ are classified by the conjugations in the following theorem.

Theorem 2.1.17 *Let f be a non-identity element of $\mathrm{M\ddot{o}b}^+(\overline{\mathbb{C}})$, then*

(a) f is parabolic if and only if it is conjugate in $\mathrm{M\ddot{o}b}(\overline{\mathbb{C}})$ to the translation $p(z) = z + 1$. The associated matrix is

$$p = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C}).$$

(b) f is loxodromic if and only if it is conjugate in $\mathrm{M\ddot{o}b}(\overline{\mathbb{C}})$ to a dilation $h(z) = re^{i\theta}z$, for $\theta \in (0, 2\pi)$ and $r \neq 1$. Let $\lambda^2 = re^{i\theta}$, then the associated matrix is

$$h = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C}), \ \textit{for} \ |\lambda| \neq 0, 1$$

(c) f is elliptic if and only if it is conjugate in $\mathrm{M\ddot{o}b}(\overline{\mathbb{C}})$ to a rotation about the origin O through an angle $\theta : e(z) = e^{i\theta}z$, for $\theta \in (0, 2\pi)$. Let $\lambda^2 = e^{i\theta}$, then the associated matrix is

$$e = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C}) \ \textit{where} \ |\lambda| = 1, \lambda \neq \pm 1.$$

Proof. First of all, applying for Remark 2.1.5, two non-identity elements f and g of $\mathrm{M\ddot{o}b}^+(\overline{\mathbb{C}})$ are conjugate if and only if $\mathrm{tr}^2(f) = \mathrm{tr}^2(g)$.

(a) The necessity is the part (a) of Lemma 2.1.15. Conversely, suppose that f is conjugate in Möb($\overline{\mathbb{C}}$) to $p(z) = z + 1$. Thus,

$$\mathrm{tr}^2(f) = \mathrm{tr}^2(p) = 4.$$

Thus, f is parabolic by Definition 2.1.6. The proof of (a) is completed.

(b) Suppose that f is loxodromic, by Lemma 2.1.15, f is conjugate in Möb($\overline{\mathbb{C}}$) to $h(z) = re^{i\theta}z$ for $\theta \in (0, 2\pi)$, and the trace is given by

$$\mathrm{tr}^2(h) = \left(r + r^{-1}\right)\cos\theta + i\left(r - r^{-1}\right)\sin\theta + 2.$$

By Definition 2.1.6, the loxodromic element f gives $\mathrm{tr}^2(f) \in \mathbb{C} - [0, 4]$ and hence $r \neq 1$. Otherwise, if $r = 1$ then

$$\mathrm{tr}^2(h) = 2\cos\theta + 2 = 4\cos^2\left(\frac{\theta}{2}\right).$$

Since $\theta \in (0, 2\pi)$, $0 \leq \mathrm{tr}^2(\mathrm{h}) < 4$ and hence $\mathrm{tr}^2(\mathrm{f}) \in [0, 4) \subseteq [0, 4]$, it is contradict to $\mathrm{tr}^2(f) \in \mathbb{C} - [0, 4]$. Thus, f is conjugate in Möb($\overline{\mathbb{C}}$) to $h(z) = re^{i\theta}z$, for $\theta \in (0, 2\pi)$ and $r \neq 1$.

Conversely, suppose that f is conjugate in Möb($\overline{\mathbb{C}}$) to $h(z) = re^{i\theta}z$, for $\theta \in (0, 2\pi)$ and $r \neq 1$. By Lemma 2.1.15, the trace is given by

$$\mathrm{tr}^2(h) = \left(r + r^{-1}\right)\cos\theta + 2 + i\left(r - r^{-1}\right)\sin\theta.$$

If $\mathrm{tr}^2(h) \in \mathbb{R}$ then either $r - r^{-1} = 0$ or $\sin\theta = 0$. Since $r - r^{-1} = 0$ gives $r = 1$ that contradicts $r \neq 1$, so $\sin\theta = 0$. It follows that $\theta = \pi$ and hence

$$\begin{aligned}\mathrm{tr}^2(h) &= -r - r^{-1} + 2\\ &= -\left(\sqrt{r} - \sqrt{r^{-1}}\right)^2 \leq 0.\end{aligned}$$

If $-\left(\sqrt{r}-\sqrt{r^{-1}}\right)^2 = 0$, then $r = 1$ that contradicts to $r \neq 1$, so $\mathrm{tr}^2(h) < 0$ and hence

$$\mathrm{tr}^2(h) \in \mathbb{C} - [0, \infty) \subset \mathbb{C} - [0, 4] \,.$$

Therefore, f is loxodromic by Definition 2.1.6. The proof of (b) is completed.

(c) Suppose that f is elliptic, then f is neither parabolic nor loxodromic. By Lemma 2.1.15 and the necessity of the previous (a) and (b), f is conjugate in $\text{Möb}(\overline{\mathbb{C}})$ to $e(z) = e^{i\theta}z$, for $\theta \in (0, 2\pi)$.

Conversely, suppose that f is conjugate in $\text{Möb}(\overline{\mathbb{C}})$ to $e(z) = e^{i\theta}z$, for $\theta \in (0, 2\pi)$, again by Lemma 2.1.15,

$$\mathrm{tr}^2(e) = 2\cos\theta + 2 = 4\cos^2\left(\frac{\theta}{2}\right).$$

Since $\theta \in (0, 2\pi), 0 \leq \mathrm{tr}^2(e) < 4$ and hence $0 \leq \mathrm{tr}^2(f) < 4$. Thus, f is elliptic by Definition 2.1.6. The proof of (c) is completed. □

Now the convergence group is introduced in the following definition (see [25, 33]).

Definition 2.1.18 *A group G of self-homeomorphisms of $\overline{\mathbb{C}}$ is said to be a* convergence group *if it has the* convergence property*: Each infinite subfamily of G contains an infinite sequence of distinct elements $\{f_j\}$ such that one of the following is true.*

(a) *There exists a self-homeomorphism f of $\overline{\mathbb{C}}$ such that*

$$\lim_{j\to\infty} f_j = f \text{ and } \lim_{j\to\infty} f_j^{-1} = f^{-1}$$

uniformly in $\overline{\mathbb{C}}$.

(*b*) *There exists points* x_o, y_o *in* $\overline{\mathbb{C}}$ *such that*

$$\lim_{j\to\infty} f_j = y_o \text{ and } \lim_{j\to\infty} f_j^{-1} = x_o$$

uniformly on all compact subsets of $\overline{\mathbb{C}}\backslash\{x_o\}$ *and* $\overline{\mathbb{C}}\backslash\{y_o\}$, *respectively.*

Recall an important lemma at the end of this section so that the results based on discrete convergence group by Martin [25] can be cited later.

Lemma 2.1.19 *Every subgroup of* $\text{Möb}(\overline{\mathbb{C}})$ *is a convergence group.*

2.2 Isometry groups on hyperbolic 3-space

There are many useful models of hyperbolic space such as the upper half-space model, the open ball model, the hemisphere model, the Klein model, and the hyperboloid model. Each model has its own metric, lines, isometries, and so on. For convenience, it is taken the upper half-space model $\mathbb{H}^3$ known as the Poincaré upper half-space model for hyperbolic 3-space throughout this dissertation, henceforth the boundary of $\mathbb{H}^3$ is the Riemann sphere $\overline{\mathbb{C}}$ and denote $\mathbb{H}^3 \cup \overline{\mathbb{C}}$ by $\overline{\mathbb{H}^3}$.

It is useful that the elliptic and loxodromic elements of $\text{Isom}^+(\mathbb{H}^3)$ can be represented by their axes that are introduced in Definition 2.2.5. The images of fixed points in $\overline{\mathbb{H}^3}$ and the axes under a conjugation are described, and then complete the classification of the elements in $\text{Isom}^+(\mathbb{H}^3)$ by the fixed points in Theorem 2.2.7. One of important tools used in the dissertation is given by the fundamental Theorem 2.2.3 so that one can interact between Complex Analysis, Abstract Algebra, Hyperbolic Geometry, and Topology.

Definition 2.2.1 *The* upper half-space model *of hyperbolic* 3*-space is defined by*

$$\mathbb{H}^3 = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0 \right\}$$

equipped with an (infinitesimal) hyperbolic metric *:*

$$ds = \frac{|dx|}{x_3} = \frac{\sqrt{dx_1^2 + dx_2^2 + dx_3^2}}{x_3}.$$

Both Euclidean straight lines perpendicular to the x_1x_2-plane and vertical Euclidean semicircles centred on the x_1x_2-plane are called hyperbolic lines *in* $\mathbb{H}^3$*.*

As a result, if γ is a smooth arc with the parametric equation in $\mathbb{H}^3$:

$$(x_1, x_2, x_3) = (x_1(t), x_2(t), x_3(t)), a \leq t \leq b,$$

then the hyperbolic length of γ, denoted by $L_H(\gamma)$, is given following:

$$L_H(\gamma) = \int_\gamma \frac{\sqrt{d^2x_1 + d^2x_2 + d^2x_3}}{x_3} = \int_a^b \frac{\sqrt{(x_1'(t))^2 + (x_2'(t))^2 + (x_3'(t))^2}dt}{x_3(t)}.$$

There are two typical planes interested here. First, consider any smooth curve γ on the horizontal plane $x_3 = c$ with $c > 0$. Since $dx_3 = 0$, the hyperbolic length becomes

$$L_H(\gamma) = \frac{1}{c} \int_\gamma \sqrt{d^2x_1 + d^2x_2}.$$

In the case $c = 1$, the the hyperbolic length coincides with its Euclidean length, and hence the hyperbolic geometry on the plane $x_3 = 1$ can be regarded as the Euclidean geometry there.

Second, consider any smooth curve γ on the vertical upper half-plane

$x_1 = a \in \mathbb{R}$ with $x_3 > 0$. Since $dx_1 = 0$, the hyperbolic length becomes

$$L_H(\gamma) = \int_\gamma \frac{\sqrt{d^2x_2 + d^2x_3}}{x_3},$$

which is clearly identical to the hyperbolic length of γ in the upper half-plane model $\mathbb{H}^2$ for hyperbolic plane. In other words, the restriction of three-dimensional hyperbolic length on $\mathbb{H}^3$ to a vertical upper half-plane can be regarded as the two-dimensional hyperbolic length on $\mathbb{H}^2$. Further, it is not difficult to see the following lemma.

Lemma 2.2.2 *Let $P(a,b,c_1)$ and $Q(a,b,c_2)$ be two points in $\mathbb{H}^3$ with $0 < c_1 \leq c_2$, let q be a vertical Euclidean semi-circle in $\mathbb{H}^3$ with center $C\,(a,b,0)$ and radius r, and let S and T be points of q such that the radii CS and CT make angles α and β $(\alpha \leq \beta)$ with the projection of q onto x_1x_2 plane, respectively. Then the hyperbolic lengths of the geodesics segments are*

$$L_H(PQ) = \ln\frac{c_2}{c_1} \quad \textit{and} \quad L_H(ST) = \ln\frac{\csc\beta - \cot\beta}{\csc\alpha - \cot\alpha}.$$

Let $\mathrm{Isom}^+(\mathbb{H}^3)$ be the set of the orientation preserving hyperbolic isometries of $\mathbb{H}^3$, then it is a topological group under composition with topology induced by the hyperbolic metric. It is well known that each element of $\mathrm{Isom}^+(\mathbb{H}^3)$ preserves the set of the hyperbolic lines. Together with Theorem 2.1.2, now the following fundamental theorem is reached.

Theorem 2.2.3 *The hyperbolic isometry group $\mathrm{Isom}^+(\mathbb{H}^3)$, the Möbius group $\mathrm{M\ddot{o}b}^+\left(\overline{\mathbb{C}}\right)$, and the projective special linear group $\mathrm{PSL}(2,\mathbb{C})$ are topologically isomorphic:*

$$\mathrm{M\ddot{o}b}^+(\overline{\mathbb{C}}) \cong \mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3).$$

In fact, the plane $x_3 = 0$ can be viewed as the complex plane $\mathbb{C}$ by $(x_1, x_2, 0) \longleftrightarrow z = x_1 + x_2 i$, and hence every Möbius transformation $f = \frac{az+b}{cz+d} \in$ Möb$^+(\overline{\mathbb{C}})$ has an extension to an isometry from $\mathbb{H}^3$ to $\mathbb{H}^3$ by using the Poincarè extension. In fact, each $f \in$ Möb$^+(\overline{\mathbb{C}})$ is composition of translations $f_1 = z + b$, dilations $f_2 = \lambda z$, and inversions $f_3 = \frac{1}{z}$, for some b and $\lambda \in \mathbb{C}$. Furthermore, these three forms of mappings can be extended to $\mathbb{H}^3$ as follows:

a "horizontal" translation $\widetilde{f_1} : (z, x_3) \mapsto (z + b, x_3)$,
a "horizontal" rotation or dilation $\widetilde{f_2} : (z, x_3) \mapsto (\lambda z, x_3)$, and
an inversion $\widetilde{f_3} : (z, x_3) \mapsto \left(\frac{\overline{z}}{|z|^2 + x_3^2}, \frac{x_3}{|z|^2 + x_3^2} \right)$.

Conversely, every orientation preserving isometry g of $\mathbb{H}^3$ extends to the boundary $\overline{\mathbb{C}}$, $\widetilde{g} : \overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$, and then the restriction $\widetilde{g}|_{\overline{\mathbb{C}}}$ is a conformal mapping of $\overline{\mathbb{C}}$, and hence, by Theorem 2.1.13, it is a Möbius transformation, so $\widetilde{g}|_{\overline{\mathbb{C}}} \in$ Möb$^+(\overline{\mathbb{C}})$.

Now turn the attention to the fixed points in $\overline{\mathbb{H}^3}$. Let f be a hyperbolic isometry in Isom$^+(\mathbb{H}^3)$. Denote $\mathrm{Fix}_{\overline{\mathbb{H}^3}}(f)$ by the set of fixed points of f in $\overline{\mathbb{H}^3}$. It is clear that

$$\mathrm{Fix}_{\overline{\mathbb{H}^3}}(f) \cap \overline{\mathbb{C}} = \mathrm{Fix}(f).$$

As a special case of Proposition 2.1.9, the following corollary is obtained directly.

Corollary 2.2.4 *Let g be an element of* Isom$^+(\mathbb{H}^3)$*, and let h be the natural extension of an element of* Möb$(\overline{\mathbb{C}})$*. Then*

$$\mathrm{Fix}_{\overline{\mathbb{H}^3}}\left(h \circ g \circ h^{-1}\right) = h\left(\mathrm{Fix}_{\mathbb{H}^3}(g)\right).$$

Next, introducing the axes to represent elliptic and loxodromic elements in Isom$^+(\mathbb{H}^3)$ as they have two fixed points in $\overline{\mathbb{C}}$. Notice that each parabolic element has a unique fixed point in $\overline{\mathbb{H}^3}$ that is indeed in $\overline{\mathbb{C}}$. Thus there are not any axes for parabolic elements in $\mathrm{Isom}^+(\mathbb{H}^3)$.

Definition 2.2.5 *Let f be an elliptic or loxodromic element of* $\mathrm{Isom}^+(\mathbb{H}^3)$. *The hyperbolic line in $\mathbb{H}^3$ joining the end points that are the fixed points of f on the boundary $\overline{\mathbb{C}}$ is called the* axis *of f, denoted by $axis(f)$.*

The following Figure 2.1 illustrates two axes.

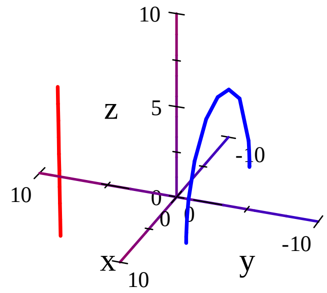

Figure 2.1: The two axes of isometries

It is clear that

$$\mathrm{axis}(f) \cap \overline{\mathbb{C}} = \mathrm{Fix}\,(f)\,. \tag{2.1}$$

Lemma 2.2.6 *Let g be an element of* $\mathrm{Isom}^+(\mathbb{H}^3)$, *and let h be the natural extension of an element of* $\mathrm{M\ddot{o}b}(\overline{\mathbb{C}})$. *Then*

(*a*) $\mathrm{axis}(g)$ *is invariant under g :*

$$g(\mathrm{axis}(g)) = \mathrm{axis}(g).$$

(*b*) *the axis of a conjugation is given by*

$$\mathrm{axis}(h \circ g \circ h^{-1}) = h(\mathrm{axis}(g)).$$

Proof. According to Definition 2.2.5, axis(f) is determined by the fixed point set Fix(f).

(a) As a result, axis (g) is determined by Fix(g) that is the set of fixed points of g, and hence $g(\text{axis}(g))$ and axis(g) have the same ending points, so $g(\text{axis}(g)) = \text{axis}(g)$.

(b) Applying for Corollary 2.1.11, Fix $(f) = h\,(\text{Fix}(g))$ and hence the ending points of the axis $h\,(\text{axis}(g))$ coincide with Fix $(h \circ g \circ h^{-1})$. Therefore, $\text{axis}(h \circ g \circ h^{-1}) = h(\text{axis}(g))$. □

Now it is ready to classify the elements of Möb$^+(\overline{\mathbb{C}})$ in the following theorem by using the fixed points in $\mathbb{H}^3$.

Theorem 2.2.7 *Suppose that f is a non-identity element of* Möb$^+(\overline{\mathbb{C}})$, *then*

(a) *f is parabolic if and only if f has a single fixed point in $\overline{\mathbb{C}}$ (and no fixed points in $\mathbb{H}^3$).*

(b) *f is elliptic if and only if f has two fixed points in $\overline{\mathbb{C}}$ and* axis (f) *is the fixed point set.*

(c) *f is loxodromic if and only if f has two fixed points in $\overline{\mathbb{C}}$ and no fixed points in $\mathbb{H}^3$ (*axis (f) *could be an invariant set under f).*

Proof. (a) It is that the part (a) of Lemma 2.1.14 states. Further, applying for Theorem 2.1.17, f is conjugate to a translation $p = z + 1$ and hence the natural extension, recall $p \in \text{Isom}^+(\mathbb{H}^3)$ has no fixed points in $\mathbb{H}^3$.

(b) Applying for Theorem 2.1.17, f is elliptic if and only if it is conjugate to a rotation $e = e^{i\theta}z$ with $0 < \theta < 2\pi$. It is clear that

$$\text{Fix}\,(e) = \{0, \infty\} \subseteq \overline{\mathbb{C}} \text{ and } \text{Fix}_{\overline{\mathbb{H}^3}}\,(e) = x_3\text{-axis} = \text{axis}\,(e)\,.$$

Choose a Möbius transformation $g \in \text{Möb}(\overline{\mathbb{C}})$ such that $f = geg^{-1}$. Further, applying for Corollary 2.2.4 and Lemma 2.2.6,

$$\mathrm{Fix}_{\overline{\mathbb{H}^3}}(f) = \mathrm{Fix}_{\overline{\mathbb{H}^3}}\left(geg^{-1}\right) = g\left(\mathrm{Fix}_{\overline{\mathbb{H}^3}}(e)\right) = g\left(\mathrm{axis}\,(e)\right) = \mathrm{axis}\,(f)\,.$$

Thus, axis (f) is the fixed point set of f in $\mathbb{H}^3$. Conversely, suppose that $\mathrm{Fix}_{\overline{\mathbb{H}^3}}(f) = \mathrm{axis}\,(f)$. One may assume that f has two fixed points 0 and ∞ in $\overline{\mathbb{C}}$ by a conjugation if necessary, then axis $(f) = x_3$-axis as whose end points are 0 and ∞. Thus, it is a rotation $f = e^{i\theta}z$ with $0 < \theta < 2\pi$.

(c) Applying for Theorem 2.1.17, f is loxodromic if and only if it is conjugate by g in Möb$(\overline{\mathbb{C}})$ to a dilation $h(z) = re^{i\theta}z$, for $\theta \in (0, 2\pi)$ and $r \neq 1$, i.e., $f = ghg^{-1}$. It is clear that Fix $(h) = \{0, \infty\} \subseteq \overline{\mathbb{C}}$ and the natural extension has no fixed points in $\mathbb{H}^3$, i.e., $\mathrm{Fix}_{\overline{\mathbb{H}^3}}(h) = \varnothing$. By Corollary 2.2.4 and Lemma 2.2.6,

$$\mathrm{Fix}_{\overline{\mathbb{H}^3}}(f) = \mathrm{Fix}_{\overline{\mathbb{H}^3}}\left(ghg^{-1}\right) = g\left(\mathrm{Fix}_{\overline{\mathbb{H}^3}}(h)\right) = g\left(\varnothing\right) = \varnothing.$$

Thus, if f is loxodromic then it has two fixed points in $\overline{\mathbb{C}}$ and no fixed points in $\mathbb{H}^3$. Conversely, if f has two fixed points in $\overline{\mathbb{C}}$ and no fixed points in $\mathbb{H}^3$. Thus, f is loxodromic as it is neither parabolic nor elliptic by the parts (a) and (b). $\square$

2.3 Parameters of two-generator groups

An advanced tool for studying Kleinian groups is the triple of complex parameters that determine uniquely up to conjugacy the two-generator group (Theorem 2.3.2). This dissertation characterizes the finite order of elliptic elements by the explicit formula (Theorem 2.3.5) and use the elementary method showing a number of trace identities in $\mathrm{SL}(2, \mathbb{C})$ including Fricke identity (Propositions 2.3.8 and 2.3.9). At the end of this section, some impressive geometric quantities are introduced, such as translation length, holonomy, and complex hyperbolic distance.

Definition 2.3.1 *Consider the group* $\langle f, g\rangle$ *generated by a pair of elements* f *and* g *in* $\mathrm{Isom}^+(\mathbb{H}^3)$, *and the multiplicative commutator* $[f,g] = fgf^{-1}g^{-1}$. *The following three complex numbers*

$$\gamma(f, g) = \mathrm{tr}([f,g]) - 2,\ \beta(f) = \mathrm{tr}^2(f) - 4,\ \beta(g) = \mathrm{tr}^2(g) - 4$$

are called parameters *for the two-generator group* $\langle f, g\rangle$.

Notice that two groups $\mathrm{Isom}^+(\mathbb{H}^3)$ and $\mathrm{PSL}(2,\mathbb{C})$ are isomorphic, thus for each f in $\mathrm{Isom}^+(\mathbb{H}^3)$ there is a unique representative matrix A in $\mathrm{PSL}(2,\mathbb{C})$, and hence $\mathrm{tr}(f) = \mathrm{tr}(A)$. So the triple of parameters $(\gamma(f, g), \beta(f), \beta(g))$ of the group $\langle f, g\rangle$ is independent of the choice of representative matrices in $\mathrm{Möb}(\overline{\mathbb{C}})$ for f and $g \in \mathrm{Isom}^+(\mathbb{H}^3)$. Further, Lemma 2.2 [17] gives the following theorem.

Theorem 2.3.2 *Let* $\Gamma = \langle f, g\rangle$ *be a group generated by* f *and* g *in* $\mathrm{Isom}^+(\mathbb{H}^3)$, *then* Γ *is determined uniquely up to conjugacy by its triple of parameters*

$$(\gamma(f, g), \beta(f), \beta(g))$$

with $\gamma(f, g) \neq 0$.

Remark 2.3.3 (1) *The restriction* $\gamma(f, g) \neq 0$ *is necessary. For instance if* $f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, *and* $h = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$, *then the parameters of two groups* $\langle f, g\rangle$ *and* $\langle f, h\rangle$ *are the same,*

$$(\gamma(f, g), \beta(f), \beta(g)) = (\gamma(f,h), \beta(f), \beta(h)) = (0,0,0).$$

But the first group is free abelian of rank 1 *and the other is free abelian of rank* 2 *and thus they are different groups and not conjugate with the same parameters (see introduction to free group at the end of Section* 3.1*).*

(2) *A two-generator group may have various presentations generated by different pairs of generators. For example, consider a two-generator group* $\Gamma = \langle f, g \rangle$ *with the triple of parameters* $(\gamma, \beta, -4)$*. Let* $u = f$ *and* $v = gf$ *then* $f = u, g = vu^{-1}$*, and hence* $\Gamma = \langle f, gf \rangle$ *with the triple of parameters* $(\gamma, \gamma - \beta - 4, -4)$ *(see the identity* (4.16) *in Section* 4.1*). Thus, the same group may have different triples of parameters accordingly. Theorem* 2.3.2 *above states that each group generated by each pair of two generators is determined uniquely up to conjugacy by its triple of parameters.*

Now one can use the complex parameter $\beta(f)$ to give an alternate definition of parabolic, elliptic, and loxodromic elements f in $\mathrm{Isom}^+(\mathbb{H}^3)$ that were defined by the traces in Definition 2.1.6 as elements of $\mathrm{Möb}^+(\overline{\mathbb{C}})$.

Definition 2.3.4 *Let* f *be a non-identity element in* $\mathrm{Isom}^+(\mathbb{H}^3)$*, then*

(*a*) f *is parabolic if* $\beta(f) = 0$.

(*b*) f *is elliptic if* $\beta(f) \in [-4, 0)$.

(*c*) f *is loxodromic if* $\beta(f) \in \mathbb{C} - [-4, 0]$.

Recall that an element a^p of a non-trivial cyclic group $G = \langle a \rangle$ of order n generates G if and only if $(p, n) = 1$, so any single element of the following set generates G :

$$\{a^p : 1 \leq p < n \text{ and } (p, n) = 1\}.$$

For example, if Γ is a cyclic group of order n generated by a rotation centered at the origin with the angle $\frac{2\pi}{n}$, then any single element of the following set generates

Γ :

$$\left\{\frac{2p\pi}{n} : 1 \leq p < n \text{ and } (p,n) = 1\right\}.$$

Theorem 2.3.5 *Let f be a non-trivial element in* $\mathrm{Isom}^+(\mathbb{H}^3)$*, then f is elliptic of order n if and only if*

$$\beta(f) \ = -4\sin^2(\frac{p\pi}{n}), \ \textit{for } 1 \leq p < n \textit{ and } (p,n) = 1.$$

Proof. (1) Suppose that $\beta(f) \ = -4\sin^2(\frac{p\pi}{n})$, for $1 \leq p < n$ and $(p,n) = 1$. Since $0 \leq \sin^2(\frac{p\pi}{n}) \leq 1$ and p and n are co-prime, $0 < \sin^2(\frac{p\pi}{n}) \leq 1$ and hence $\beta(f) \ \in [-4, 0)$. By Corollary 2.3.4, f is elliptic, then by Theorem 2.1.17, f is conjugate to the rotation $e = e^{i\theta}z$ for $\theta \in (0, 2\pi)$ and so $\cos\theta \neq 1$. Now the associated matrix of f in $\mathrm{PSL}(2,\mathbb{C})$ is

$$\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}.$$

It follows that

$$\begin{aligned} \beta(f) &= (e^{i\theta/2} + e^{-i\theta/2})^2 - 4 \\ &= e^{i\theta} + e^{-i\theta} - 2 \\ &= 2(\cos\theta - 1). \end{aligned}$$

On the other hand side, $\beta(f) \ = -4\sin^2(\frac{p\pi}{n}) = -2(1 - \cos\frac{2p\pi}{n})$. Thus,

$$\cos\theta = \cos\frac{2p\pi}{n},$$

which gives $\theta = \pm\frac{2p\pi}{n} + 2k\pi$, where $k \in \mathbb{Z}$. Therefore, the rotation angle $\theta = \frac{2p\pi}{n}$. Without losing generality, take $p = 1$ then the order of f is equal to n. In fact, if

$p \neq 1$, then $1 < p < n$ with $(p, n) = 1$, so $\frac{n}{p} \notin \mathbb{N}$. It follows that $\frac{2p\pi}{\frac{2p\pi}{n}} = n$.

(2) Suppose f is elliptic of order n, then by using Theorem 2.1.17 f is conjugate to $e^{i\theta}z$, and the rotation angle $\theta = \frac{2p\pi}{n}$ with $(p, n) = 1$. It is deduced that

$$\begin{aligned}\beta(f) &= (e^{i\theta/2} + e^{-i\theta/2})^2 - 4 \\ &= 4\left(\cos^2 \frac{\theta}{2} - 1\right) \\ &= -4\sin^2(\frac{p\pi}{n}).\end{aligned}$$

□

Theorem 2.3.5 gives some special cases in the following corollary that will be used frequently.

Corollary 2.3.6 *Let f be a non-identity element in* $\text{Isom}^+(\mathbb{H}^3)$.

(*a*) *f is elliptic of order* 2 *if and only if* $\beta(f) = -4$.

(*b*) *f is elliptic of order* 3 *if and only if* $\beta(f) = -3$.

(*c*) *f is elliptic of order* 4 *if and only if* $\beta(f) = -2$.

(*d*) *f is elliptic of order* 5 *if and only if* $\beta(f) = \frac{\sqrt{5}-5}{2}$, *or* $-\frac{\sqrt{5}+5}{2}$.

(*e*) *f is elliptic of order* 6 *if and only if* $\beta(f) = -1$.

Proof. By Theorem 2.3.5, f is elliptic of order n if and only if $\beta(f) = -4\sin^2(\frac{p\pi}{n})$, for $1 \leq p < n$ and $(p, n) = 1$. In particular,

(*a*) $n = 2 : (p, 2) = 1$ gives $p = 1$. Thus, $\beta(f) = -4\sin^2(\frac{\pi}{2}) = -4$.

(*b*) $n = 3 : (p, 3) = 1$ gives $p = 1$ and 2, and hence

$$\beta(f) = -4\sin^2(\frac{\pi}{3}) = -4\sin^2(\frac{2\pi}{3}) = -3.$$

(c) $n = 4 : (p, 4) = 1$ gives $p = 1$ and 3, so

$$\beta(f) = -4\sin^2(\frac{\pi}{4}) = -4\sin^2(\frac{3\pi}{4}) = -2.$$

(d) $n = 5 : (p, 5) = 1$ gives $p = 1, 2, 3,$ and 4. Therefore, there are the following two distinct values of $\beta(f)$:

$$\beta(f) = -4\sin^2(\frac{\pi}{5}) = -4\sin^2(\frac{4\pi}{5}) = \frac{\sqrt{5}-5}{2},$$
$$\beta(f) = -4\sin^2(\frac{2\pi}{5}) = -4\sin^2(\frac{3\pi}{5}) = -\frac{\sqrt{5}+5}{2}.$$

(e) $n = 6 : (p, 6) = 1$ gives $p = 1$ and 5, and hence

$$\beta(f) = -4\sin^2(\frac{\pi}{6}) = -4\sin^2(\frac{5\pi}{6}) = -1.$$

□

Note that Definition 2.3.1 gives $\mathrm{tr}^2(f) = 4 + \beta(f)$. Applying for Theorem 2.3.5, if f is elliptic of order n then $\mathrm{tr}^2(f) = 4 - 4\sin^2(\frac{p\pi}{n}) = 4\cos^2(\frac{p\pi}{n})$, for $1 \leq p < n$ and $(p, n) = 1$. This is the following corollary.

Corollary 2.3.7 *Let f be a non-trivial element in* $\mathrm{Isom}^+(\mathbb{H}^3)$, *if f is elliptic of order n then*

$$\mathrm{tr}(f) = \pm 2\cos(\frac{p\pi}{n}), \text{ for } 1 \leq p < n \text{ and } (p, n) = 1.$$

Using the elementary method prove the following trace identities in $\mathrm{SL}(2, \mathbb{C})$ including Fricke identity (see [10]). One can see that the second identity (2.3) is a kind of formula similar to the one for "integration by parts".

Proposition 2.3.8 *Let f and g be two Möbius transformations in* $\mathrm{SL}(2,\mathbb{C})$, *then the following three identities are held.*

$$\mathrm{tr}(f) = \mathrm{tr}(f^{-1}). \tag{2.2}$$

$$\mathrm{tr}(fg) = \mathrm{tr}(f)\mathrm{tr}(g) - \mathrm{tr}(fg^{-1}). \tag{2.3}$$

$$\mathrm{tr}[f,g] = \mathrm{tr}^2(f) + \mathrm{tr}^2(g) + \mathrm{tr}^2(fg) - \mathrm{tr}(f)\mathrm{tr}(g)\mathrm{tr}(fg) - 2. \tag{2.4}$$

Proof. (1) Let λ_1 and λ_2 be the eigenvalues of f, then the characteristic equation is $\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$. Notice that $\lambda_1 + \lambda_2 = \mathrm{tr}(f)$ and $\lambda_1\lambda_2 = \det(f)$, the characteristic equation becomes $\lambda^2 - \mathrm{tr}(f)\ \lambda + \det(f)\ = 0$.

Applying Cayley-Hamilton Theorem for f :

$$f^2 - \mathrm{tr}(f)f + \det(f)Id = 0. \tag{2.5}$$

Since $f \in \mathrm{SL}(2,\mathbb{C}), \det(f)\ = 1$. Multiplying (2.5) by f^{-1} gives

$$f + f^{-1} = \mathrm{tr}(f)Id. \tag{2.6}$$

It follows from taking trace for (2.6) that

$$\mathrm{tr}(\ f) + \mathrm{tr}(f^{-1}) = 2\mathrm{tr}(\ f).$$

Which gives the first well known trace identity (2.2) :

$$\mathrm{tr}(f) = \mathrm{tr}(f^{-1}).$$

(2) Multiplying the identity (2.6) by g from the right,

$$fg + f^{-1}g = \mathrm{tr}(f)g.$$

Taking the trace for this identity,

$$\mathrm{tr}(fg) + \mathrm{tr}(f^{-1}g) = \mathrm{tr}(f)\mathrm{tr}(g).$$

By the first trace identity(2.2),

$$\mathrm{tr}(f^{-1}g) = \mathrm{tr}(f^{-1}g)^{-1} = \mathrm{tr}(g^{-1}f) = \mathrm{tr}(fg^{-1}).$$

Thus, the previous identities implies

$$\mathrm{tr}(fg) = \mathrm{tr}(f)\mathrm{tr}(g) - \mathrm{tr}(fg^{-1}).$$

It is the second well known identity (2.3).

(3) Let $tr(f) = x$, $tr(g) = y$, and $tr(fg) = z$. Then using two trace identities (2.2) and (2.3) obtain the following expressions:

$$\mathrm{tr}(f^2) = \mathrm{tr}^2(f) - \mathrm{tr}(ff^{-1}) = x^2 - 2. \tag{2.7}$$

$$\begin{aligned} \mathrm{tr}(f^{-1}g) &= \mathrm{tr}(f^{-1})\mathrm{tr}(g) - \mathrm{tr}(f^{-1}g^{-1}) \\ &= \mathrm{tr}(f)\mathrm{tr}(g) - \mathrm{tr}((gf)^{-1}) \\ &= xy - \mathrm{tr}(gf) = xy - \mathrm{tr}(fg). \end{aligned} \tag{2.8}$$

and hence

$$\mathrm{tr}(f^{-1}g) = xy - z. \tag{2.9}$$

Now using the trace identity (2.3) and two expressions (2.7) and (2.9) ,

$$\begin{aligned}\mathrm{tr}(fgf^{-1}g) &= \mathrm{tr}(fg)\mathrm{tr}(f^{-1}g) - \mathrm{tr}\left(fgg^{-1}f\right)\\ &= \mathrm{tr}(fg)\mathrm{tr}(f^{-1}g) - \mathrm{tr}\left(f^2\right)\\ &= z(xy-z) - (x^2-2).\end{aligned}$$

Thus,

$$\mathrm{tr}(fgf^{-1}g) = xyz - z^2 - x^2 + 2.$$

It follows that

$$\begin{aligned}\mathrm{tr}[f,g] &= \mathrm{tr}\left(fgf^{-1}g^{-1}\right)\\ &= \mathrm{tr}\left(fgf^{-1}\right)\mathrm{tr}\left(g^{-1}\right) - \mathrm{tr}(fgf^{-1}g)\\ &= \mathrm{tr}^2\left(g\right) - \mathrm{tr}(fgf^{-1}g)\\ &= y^2 - xyz + z^2 + x^2 - 2.\end{aligned}$$

By substitution of $x, y,$ and z, it gives the Fricke identity (2.4) :

$$\mathrm{tr}[f,g] = \mathrm{tr}^2(f) + \mathrm{tr}^2(g) + \mathrm{tr}^2(fg) - \mathrm{tr}(f)\mathrm{tr}(g)\mathrm{tr}(fg) - 2.$$

□

Proposition 2.3.9 *Suppose that* $f, g \in \mathrm{Isom}^+(\mathbb{H}^3)$. *Then following identities are held.*

(a)

$$\beta\left(f\right) = \beta(f^{-1}).$$

(b)

$$\beta(fg) = \beta(gf).$$

(*c*)

$$\gamma\,(f,\,g) = \beta\,(f) + \beta(g) + \beta(fg) - \mathrm{tr}(f)\mathrm{tr}(g)\mathrm{tr}(fg) + 8. \tag{2.10}$$

(*d*)

$$\gamma\,(f,\,g) = \gamma(g, f) = \gamma(f, g^{-1}). \tag{2.11}$$

(*e*) *If* g *is elliptic of order* 2, *then*

$$\beta(fg) = \gamma\,(f,\,g) - \beta\,(f) - 4.$$

Proof. (*a*) Using Definition 2.3.1 and the identity (2.2),

$$\begin{aligned}\beta\,(f) &= \mathrm{tr}^2(f) - 4\\ &= \mathrm{tr}^2(f^{-1}) - 4 = \beta(f^{-1}).\end{aligned}$$

(*b*) Using the identities (2.3) and (2.2),

$$\begin{aligned}\mathrm{tr}(fg) &= \mathrm{tr}(f)\mathrm{tr}(g) - \mathrm{tr}(fg^{-1})\\ &= \mathrm{tr}(f)\mathrm{tr}(g) - \mathrm{tr}\left((fg^{-1})^{-1}\right)\\ &= \mathrm{tr}(g)\mathrm{tr}(f) - \mathrm{tr}(gf^{-1}) = \mathrm{tr}(gf).\end{aligned}$$

It follows from Definition 2.3.1 that

$$\begin{aligned}\beta(fg) &= \mathrm{tr}^2(fg) - 4\\ &= \mathrm{tr}^2(gf) - 4 = \beta(gf).\end{aligned}$$

(c) By Definition 2.3.1, Fricke identity (2.4) can be read as

$$\gamma(f, g) + 2 = \beta(f) + 4 + \beta(g) + 4 + \beta(fg) + 4 - \mathrm{tr}(f)\mathrm{tr}(g)\mathrm{tr}(fg) - 2,$$
$$\gamma(f, g) = \beta(f) + \beta(g) + \beta(fg) - \mathrm{tr}(f)\mathrm{tr}(g)\mathrm{tr}(fg) + 8.$$

(d) Applying for (c) and (b),

$$\begin{aligned}\gamma(f, g) &= \beta(f) + \beta(g) + \beta(fg) - \mathrm{tr}(f)\mathrm{tr}(g)\mathrm{tr}(fg) + 8 \\ &= \beta(g) + \beta(f) + \beta(gf) - \mathrm{tr}(g)\mathrm{tr}(f)\mathrm{tr}(gf) + 8 \\ &= \gamma(g, f).\end{aligned}$$

Applying for the trace identity (2.3),

$$\begin{aligned}\gamma(f, g) &= \mathrm{tr}(fgf^{-1}g^{-1}) - 2 \\ &= \mathrm{tr}(f)\mathrm{tr}\left(gf^{-1}g^{-1}\right) - \mathrm{tr}(f\left(gf^{-1}g^{-1}\right)^{-1}) \\ &= \mathrm{tr}(f)\mathrm{tr}\left(f^{-1}\right) - \mathrm{tr}(fgfg^{-1}). \\ \gamma(f, g^{-1}) &= \mathrm{tr}(fg^{-1}f^{-1}g) - 2 \\ &= \mathrm{tr}(f)\mathrm{tr}\left(g^{-1}f^{-1}g\right) - \mathrm{tr}(f\left(g^{-1}f^{-1}g\right)^{-1}) \\ &= \mathrm{tr}(f)\mathrm{tr}\left(f^{-1}\right) - \mathrm{tr}(fg^{-1}fg) \\ &= \mathrm{tr}(f)\mathrm{tr}\left(f^{-1}\right) - \mathrm{tr}(fgfg^{-1}).\end{aligned}$$

Thus, $\gamma(f, g) = \gamma(f, g^{-1})$ and hence $\gamma(f, g) = \gamma(g, f) = \gamma(f, g^{-1})$.

(e) Since g is elliptic of order two, $\mathrm{tr}(g) = 0$ and $\beta(g) = -4$. Now the identity (2.10) in (c) becomes:

$$\begin{aligned}\gamma(f, g) &= \beta(f) - 4 + \beta(fg) + 8 \\ &= \beta(f) + \beta(fg) + 4.\end{aligned}$$

That is, $\beta(fg) = \gamma(f, g) - \beta(f) - 4$. $\square$

Definition 2.3.10 *A non-trivial element f in* $\mathrm{Isom}^+(\mathbb{H}^3)$ *is called to be* primitive elliptic *of order n if* $\beta(f) = -4\sin^2(\frac{\pi}{n})$.

The part (a) of Proposition 2.3.9 implies immediately the following corollary.

Corollary 2.3.11 *Let f be a non-identity element in* $\mathrm{Isom}^+(\mathbb{H}^3)$. *Then f is primitive elliptic if and only if its inverse f^{-1} is primitive elliptic.*

A detailed proof for the following well-known fact is included now. The condition $\gamma(f, g) \neq 0$ is necessary for Kleinian groups in the next chapter, and hence $\mathrm{Fix}(f) \cap \mathrm{Fix}(g) = \varnothing$ is necessary for Kleinian groups as well.

Theorem 2.3.12 *Let Γ be generated by f and g in* $\mathrm{Isom}^+(\mathbb{H}^3)$ *with parameters* $(\gamma(f, g), \beta(f), \beta(g))$, *then* $\gamma(f, g) \neq 0$ *if and only if* $\mathrm{Fix}(f) \cap \mathrm{Fix}(g) = \varnothing$.

Proof. (1) The necessity is that $\gamma(f, g) \neq 0$ implies $\mathrm{Fix}(f) \cap \mathrm{Fix}(g) = \varnothing$. Equivalently, $\mathrm{Fix}(f) \cap \mathrm{Fix}(g) \neq \varnothing$ implies $\gamma(f, g) = 0$. Since f and g have a common fixed point in $\overline{\mathbb{C}}$, one may assume that f and g fix ∞. Apply for Theorem 2.1.13, $f = \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix}$. Thus,

$$\begin{aligned}[f, g] &= \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -b_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -b_2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

So $\gamma(f, g) = 0$.

(2) Suppose that $\mathrm{Fix}(f) \cap \mathrm{Fix}(g) = \varnothing$. There are two cases to consider.

Case 1: If f is parabolic, by using Theorem 2.1.17, one may assume the following

$$f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ where } ad - bc = 1 \text{ and } c \neq 0.$$

Computing the commutator,

$$\begin{aligned}[f, g] &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \begin{pmatrix} ac + ad - bc + c^2 & ad - ac - bc - a^2 \\ c^2 & ad - ac - bc \end{pmatrix}.\end{aligned}$$

Notice that $ad - bc = 1$,

$$\operatorname{tr}[f, g] = 2ad - 2bc + c^2 = 2 + c^2.$$

Since $c \neq 0$, $\operatorname{tr}[f, g] \neq 2$, and hence $\gamma(f, g) = \operatorname{tr}[f, g] - 2 \neq 0$.

Case 2: If f is not parabolic, one may assume the following by Theorem 2.1.17,

$$f = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}, \; g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ where } \lambda \neq 0, \pm 1, ad - bc = 1, \; b \text{ and } c \neq 0.$$

The commutator is

$$\begin{aligned}[f, g] &= \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \begin{pmatrix} ad - bc\lambda^2 & -ab + ab\lambda^2 \\ \frac{cd}{\lambda^2} - cd & \frac{-bc}{\lambda^2} + ad \end{pmatrix}.\end{aligned}$$

Using $ad = 1 + bc$,

$$\begin{aligned}\operatorname{tr}[f,g] &= ad - bc\lambda^2 + \frac{-bc}{\lambda^2} + ad \\ &= -bc(\lambda^2 + \frac{1}{\lambda^2}) + 2 + 2bc \\ &= -bc(\lambda - \frac{1}{\lambda})^2 + 2.\end{aligned}$$

Since b and $c \neq 0$, $\operatorname{tr}[f,g] \neq 2$, and hence $\gamma(f,\, g) = \operatorname{tr}[f,g] - 2 \neq 0$. □

Remark 2.3.13 *Suppose that two non-identity elements f and g in* $\mathrm{Isom}^+(\mathbb{H}^3)$ *have a common fixed point in* $\overline{\mathbb{C}}$, *i.e.,* $\mathrm{Fix}(f) \cap \mathrm{Fix}(g) \neq \varnothing$; *then of course* $\gamma(f,\, g) = 0$ *by the previous theorem. Furthermore, Theorem* 4.3.5 *in [2] gives the following two specific cases:*

(*a*) *If* $\mathrm{Fix}(f) = \mathrm{Fix}(g)$, *then* $[f,g] = Id$ *and* $fg = gf$.

(*b*) *If* $\mathrm{Fix}(f) \neq \mathrm{Fix}(g)$, *then* $[f,g]$ *is parabolic and* $fg \neq gf$.

The triple of parameters for a two-generator group conveniently encode various other important geometric quantities such as the following translation length (see [30]), holonomy, and complex hyperbolic distance.

Definition 2.3.14 *Let f and g be elliptic or loxodromic elements of* $\mathrm{M\ddot{o}b}^+(\overline{\mathbb{C}})$, *and suppose that p is a hyperbolic line perpendicular to $axis(f)$.*

(1) *The hyperbolic distance between two hyperbolic lines p and $f(p)$ is called the* translation length *of f, denoted by τ_f.*

(2) *The dihedral angle between the plane containing $axis(f)$ and p and the plane containing $axis(f)$ and $f(p)$ is called the* holonomy *of f, denoted by θ_f.*

(3) *The complex number $\delta + i\theta$ is called the* complex hyperbolic distance *between the axes of f and g if δ is the hyperbolic distance between $axis(f)$ and*

axis(g) and θ is the holonomy of the element of Möb$^+(\overline{\mathbb{C}})$ *whose natural extension moves axis(f) to axis(g), and whose axis contains the common perpendicular between axis(f) and axis(g).*

Remark 2.3.15 *(a) The translation length τ_f and holonomy θ_f of f are independent of choice of the perpendicular hyperbolic line p.*

(b) The easiest way to see the holonomy θ_f is to use a conjugacy to arrange things so that axis(f) lies on x_3-axis, then it is simply the angle between the vertical projections to $\overline{\mathbb{C}}$ of p and $f(p)$ at the origin.

(c) Martin [19] established a way to find the parameters of a two-generator group $\langle f, g\rangle$ in terms of geometric quantities τ_f, θ_f, and $\delta + i\theta$ as following:

$$\beta(f) = 4\sinh^2\left(\frac{\tau_f + i\theta_f}{2}\right), \tag{2.12}$$

$$\beta(g) = 4\sinh^2\left(\frac{\tau_g + i\theta_g}{2}\right), \tag{2.13}$$

$$\gamma(f, g) = \frac{\beta(f)\,\beta(g)}{4}\sinh^2(\delta + i\theta). \tag{2.14}$$

Corollary 2.3.16 *Let f and g be two non-parabolic Möbius transformations in* Möb$^+(\overline{\mathbb{C}})$. *If $axis(f) \cap axis(g) \neq \varnothing$, then*

$$\gamma(f, g) = \frac{-\beta(f)\,\beta(g)}{4}\sin^2(\theta).$$

Proof. Let δ be the hyperbolic distance between $axis(f)$ and $axis(g)$. Since $axis(f) \cap axis(g) \neq \varnothing$, then $\delta = 0$, and hence the formula (2.14) becomes

$$\gamma(f, g) = \frac{\beta(f)\,\beta(g)}{4}\sinh^2(i\theta).$$

Notice that

$$\sinh(i\theta) = i\sin(\theta).$$

Finally,

$$\gamma\,(f,\,g) = \frac{-\beta\,(f)\,\beta(g)}{4}\sin^2(\theta).$$

□

Recall the following useful formulas from Lemma 4.4 in [15] that are derived from above identities (2.12)-(2.14):

$$\cosh(\tau_f) = \frac{|\beta\,(f)+4|+|\beta\,(f)\,|}{4} \tag{2.15}$$

$$\cos(\theta_f) = \frac{|\beta\,(f)+4|-|\beta\,(f)\,|}{4} \tag{2.16}$$

$$\cosh(2\delta) = \left|\frac{4\gamma(f,g)}{\beta\,(f)\,\beta(g)}+1\right|+\left|\frac{4\gamma(f,g)}{\beta\,(f)\,\beta(g)}\right| \tag{2.17}$$

$$\cos(2\theta) = \left|\frac{4\gamma(f,g)}{\beta\,(f)\,\beta(g)}+1\right|-\left|\frac{4\gamma(f,g)}{\beta\,(f)\,\beta(g)}\right|. \tag{2.18}$$

It is often concerned with the case where one of the isometries, say g is of order 2, in which case $\beta(g) = -4$, and (2.17) and (2.18) take the simpler form

$$\cosh(2\delta) = |1-\frac{\gamma\,(f,\,g)}{\beta\,(f)}|+|\frac{\gamma\,(f,\,g)}{\beta\,(f)}|. \tag{2.19}$$

$$\cos(2\theta) = |1-\frac{\gamma\,(f,\,g)}{\beta\,(f)}|-|\frac{\gamma\,(f,\,g)}{\beta\,(f)}|. \tag{2.20}$$

Notice that for fixed $\beta\,(f) \in \mathbb{C}$ and fixed $\cosh(2\delta)$ at (2.19), the set of possible values for $\gamma\,(f,\,g)$ form an ellipse, while for fixed $\cos(2\theta)$ and $\beta\,(f) \in \mathbb{C}$ at (2.20) one can get hyperbola. Thus δ and θ give very appealing geometric orthogonal coordinates on $\mathbb{C} \setminus [\beta, 0]$.

2.4 Parameters of elementary groups

One will see that Kleinian groups are non-elementary discrete groups in the first section of the next chapter. Thus, the dissertation pays the attention first to elementary groups in the last section of the current chapter. A number of important results about elementary groups are obtained, such as the clarification about different definitions of elementary groups in the literature (Theorem 2.4.6), the classification of elementary groups (Theorem 2.4.9), the features of elementary groups (Theorem 2.4.6), and the list of the possible parameters for two-generator discrete elementary groups with non-zero parameter $\gamma(f, g)$ in three tables (Tables 1, 2, and 3). These characterizations and tables play key roles in enhancing the theory of Kleinian groups in Chapter 3 and in establishing the universal constraints for Kleinian groups in Chapter 4.

Definition 2.4.1 *A topological group is a* discrete group *if its topology is discrete. In particular, a subgroup of a topological group is a discrete subgroup if its induced topology is discrete. Otherwise, the group is called a* non-discrete group *.*

It is equivalent that if a topological group G is a non-discrete group then there is an infinite sequence of distinct elements in G converging to the identity. Every subgroup of a discrete group is of course discrete. In addition, conjugations preserve the discreteness that is stated in the following proposition.

Proposition 2.4.2 *Let G and H be topological groups acting on the topological space X. Suppose that Γ and Γ' are subgroups of G. Suppose that Γ' is conjugate to Γ in H. If Γ is a discrete subgroup of G then Γ' is a discrete subgroup of G.*

Proof. Assume that Γ' is a non-discrete subgroup, then there is an infinite sequence $\{g'_n\}$ of distinct elements of Γ' such that

$$\lim_{n\to\infty} g'_n = Id.$$

Since Γ' is conjugate to Γ in H, there exists $h \in H$ such that

$$\Gamma' = h \circ \Gamma \circ h^{-1} = \left\{ g' = hgh^{-1} : g \in \Gamma \right\}.$$

It follows that there is an infinite sequence $\{g_n = h^{-1} g'_n h\}$ of distinct elements of Γ. Notice that h^{-1} is continuous,

$$\lim_{n \to \infty} g_n = \lim_{n \to \infty} h^{-1} g'_n h = Id.$$

Thus, Γ is a non-discrete group, a contradiction. □

Definition 2.4.3 *Suppose that a topological group G of homeomorphisms acts on a topological space X and let S be a subset of X. The* set stabilizer *of S in G, denoted by $Stab_G(S)$, is defined by*

$$\mathrm{Stab}_G(S) = \{g \in G : g(S) = S\} \subseteq G.$$

If $S = \{x\}$: $Stab_G(\{x\})$ is called the stabilizer *of the point x in X, denoted by G_x.*

If $Stab_G(S) = G$: S is called to be invariant *under G, or simply say S is* G-invariant.

For each point x in X, the following set is called an orbit *of x under G and denoted by $G(x)$:*

$$G(x) = \{g(x) \in X : g \in G\} \subseteq X.$$

It is clear that the following lemma is given immediately by the definition of orbit.

Lemma 2.4.4 *Let G be a topological group acting on the topological space X, then every orbit of G is invariant under G. In particular, if the orbit $G(x) = \{x\}$ then x is a fixed point of the group G.*

Definition 2.4.5 *Let G be a discrete group of isometries acting on $\overline{\mathbb{H}^3}$. The set of all accumulation points of the following orbit is called the* limit set *of G, denoted by $L(G)$:*

$$\{g((0,0,1)) : g \in G\},$$

where $(0,0,1)$ is a point in $\mathbb{H}^3$. The discrete group G is said to be elementary *if the limit set $L(G)$ contains at most two points. Otherwise, G is said to be* non-elementary *.*

Notice that discreteness will imply that the orbit can only accumulate on the boundary $\overline{\mathbb{C}}$ and hence

$$L(G) \subseteq \overline{\mathbb{C}}.$$

There are three characterizations of an elementary group in the following theorem.

Theorem 2.4.6 *Let G be a discrete group of hyperbolic isometries acting on $\overline{\mathbb{H}^3}$, then the followings are equivalent:*

(a) G is elementary.

(b) G has a finite orbit in $\overline{\mathbb{H}^3}$.

(c) G has a finite orbit fixed point-wise by each element of G up to an integer power.

(d) G is one of the following three types:

$i)$ If the finite orbit meets $\mathbb{H}^3$ or it is in $\overline{\mathbb{C}}$ with at least three distinct points (Type I), then every non-trivial element is elliptic of finite order.

ii) If the singleton orbit $\subseteq \overline{\mathbb{C}}$ *(Type II), then* G *is conjugate to a subgroup of* $\text{Möb}(\overline{\mathbb{C}})$ *fixing* ∞ *whose every element is parabolic of the form* $az+b$ $(a \neq 0 \in \mathbb{C})$.

iii) If the doubleton orbit $\subseteq \overline{\mathbb{C}}$ *(Type III), then* G *is conjugate to a subgroup of* $\text{Möb}(\overline{\mathbb{C}})$ *leaving the set* $\{0, \infty\}$ *invariant whose every element is of the form* az^s $(a \neq 0 \in \mathbb{C}, s = \pm 1)$.

Proof. $(a) \Longrightarrow (b)$: Since G is elementary, the limit set $L(G)$ contains at most two accumulation points. Notice that Proposition 3.1.10 states later that the limit set $L(G)$ is invariant under G. Thus, $L(G)$ includes a orbit containing one or two points in $\overline{\mathbb{H}^3}$.

$(b) \Longrightarrow (c)$: Suppose there exists a finite orbit of G, say $O = \{x_1, x_2, \ldots, x_n\}$. By Lemma 2.4.4, the orbit O is invariant under G. In particular, for each element g of G and each $x_i \in O$ satisfies

$$\left\{g^k(x_i) : k = 0, 1, 2, \cdots\right\} \subseteq \{x_1, x_2, \ldots, x_n\}.$$

Thus, there exist two distinct non-negative integers $k_i > l_i$ such that $g^{k_i}(x_i) = g^{l_i}(x_i)$, i.e., $g^{k_i - l_i}(x_i) = x_i$. Let $m_i = k_i - l_i$ then $g^{m_i}(x_i) = x_i$. Set $m = m_1 m_2 \cdots m_n$, then

$$g^m(x_i) = \underbrace{g^{m_i} \circ \cdots \circ g^{m_i}(x_i)}_{m_1 \cdots m_{i-1} m_{i+1} \cdots m_n \text{ times}} = x_i,$$

and hence g^m fixes all the points $x_1, x_2, \ldots,$ and x_n.

$(c) \Longrightarrow (d)$: Obviously, G falls into the following three types.

Type I: If the finite orbit has one point in $\mathbb{H}^3$, say x_0. then, by (c), there exists a non-negative m for each $g \in G$ such that g^m fixes x_0. By Theorem 2.2.7, g^m is elliptic or identity, and hence g^m has a finite order, say $(g^m)^k = Id$ for some $k \in \mathbb{N}$. So $g^{mk} = Id$ gives that g is elliptic (order mk) or identity.

If the finite orbit does not meet $\mathbb{H}^3$, then it has three distinct points in $\overline{\mathbb{C}}$. Thus, there exists a non-negative m for each $g \in G$ such that g^m fixes those three distinct points, so $g^m = Id$ gives that g is of order up to m and hence g is elliptic or identity.

Type II: Since the singleton orbit in $\overline{\mathbb{C}}$, by Lemma 2.4.4, G fixes the singleton, say z_0. Let $h = \frac{1}{z-z_0}$, then G is conjugate to subgroup hGh^{-1} that fixes $h(z_0) = \infty$. By Corollary 2.1.16, every element of hGh^{-1} is parabolic of the form $az + b \ (a \neq 0, b \in \mathbb{C})$.

Type III: Since the doubleton finite orbit in $\overline{\mathbb{C}}$, by Lemma 2.4.4, the doubleton, say $\{z_1, z_2\}$, is invariant under G. Let $h = \frac{z-z_1}{z-z_2}$ then G is conjugate to subgroup hGh^{-1} that leaves the set $\{h(z_1), h(z_2)\} = \{0, \infty\}$ invariant. Now applying for Corollary 2.1.16. If $\{0, \infty\}$ is a fixed point set of hGh^{-1} then every element of hGh^{-1} is of the form $az \ (a \neq 0 \in \mathbb{C})$. Otherwise, hGh^{-1} interchanges 0 and ∞, so every element is of the form $az^{-1} \ (a \neq 0 \in \mathbb{C})$.

$(d) \Longrightarrow (a)$: If G has a finite orbit of Type I, then $L(G) = \varnothing$ by Theorem 5.7 in [25] and hence G is elementary. If G has a finite orbit of Type II, then $L(G)$ is a singleton by applying for Theorem 5.10 in [25] and hence G is elementary. If G has a finite orbit of Type III, then $L(G)$ contains two accumulation points by Theorem 5.11 in [25] and hence G is elementary. □

Now a discrete group is elementary if and only if it has a finite orbit, it is the characterization of an elementary group provided in Theorem 2.4.6. Now using this characterization show the features of elementary groups in the following theorem.

Theorem 2.4.7 *The following subgroups of* $\text{Möb}(\overline{\mathbb{C}})$ *acting on* $\overline{\mathbb{H}^3}$ *are elementary:*

(a) *Finite subgroup.*

(b) *Stabilizer of each point in* $\overline{\mathbb{H}^3}$.

(c) *Subgroup with finitely many common fixed points in* $\overline{\mathbb{H}^3}$.

(d) Subgroup $\langle f,\, g\rangle$ *with* $g\,(\mathrm{Fix}\,(f)) = \mathrm{Fix}\,(f)$.

(e) Subgroup of an elementary group.

Proof. (a) Let G be a finite subgroup, say $G = \{g_1, g_2, \cdots, g_n\}$. Let x be a point in $\overline{\mathbb{H}^3}$, then there is a finite orbit

$$G\,(x) = \{g_1\,(x)\,, g_2\,(x)\,, \cdots, g_n\,(x)\}\,.$$

By Theorem 2.4.6, G is elementary.

(b) Let G be the stabilizer of a point x in $\overline{\mathbb{H}^3}$, then there is a singleton orbit $G\,(x) = \{x\}$. So Theorem 2.4.6 gives that G is elementary.

(c) Let G be a subgroup with finitely many common fixed point in $\overline{\mathbb{H}^3}$, say $z_1, z_2, \cdots$, and z_n. So there are finite orbits $\{z_1\}, \{z_2\}, \cdots$, and $\{z_n\}$, and hence G is elementary by Theorem 2.4.6.

(d) Notice that f has either one or two fixed points in $\overline{\mathbb{C}}$. If f has one unique fixed point, say $\mathrm{Fix}\,(f) = \{z_1\}$, then $g\,(\mathrm{Fix}\,(f)) = \mathrm{Fix}\,(f)$ gives $g\,(z_1) = z_1$ and hence $\langle f,\, g\rangle$ has a common fixed point in $\overline{\mathbb{H}^3}$. So $\langle f,\, g\rangle$ is elementary by the previous part (c). Otherwise, f has two fixed points, say $\mathrm{Fix}\,(f) = \{z_1, z_2\}$, then g interchanges the fixed points of f, i.e., $g\,(z_1) = z_2$ and $g\,(z_2) = z_1$, then there exists a finite orbit

$$G\,(z_1) = G\,(z_2) = \{z_1, z_2\}\,.$$

Thus, G is elementary by using Theorem 2.4.6.

(e) Let S be a subgroup of an elementary group G, and let x be a point in $\overline{\mathbb{H}^3}$, then the orbits satisfy

$$S(x) \subseteq G(x).$$

Thus, there exists a finite orbit under G implies there exists a finite orbit under S. Now Theorem 2.4.6 guarantees S is elementary. □

A group G is said to be *virtually abelian* if there is a finite index abelian subgroup of G. In particular, an abelian group is of course a virtually abelian.

The following proposition tells that every virtually abelian is elementary.

Proposition 2.4.8 *If G is virtually abelian subgroups of* Möb$(\overline{\mathbb{C}})$, *then G is elementary.*

Proof. Let G be an abelian subgroup, then either G contains only elliptic elements and the identity Id or G contains some parabolic or loxodromic elements. If G is the former case, by Theorem 4.3.7 [2], G has a common fixed point in $\mathbb{H}^3$, so G is elementary by Corollary 2.4.7 part (c). If G is the latter case, by Theorem 4.3.6 (iii) [2, Section 5.1], either G has two common fixed point in $\overline{\mathbb{C}}$ and hence G is elementary by Corollary 2.4.7 part (c), or there are at least a pair f and g such that $\mathrm{Fix}\,(f) \cap \mathrm{Fix}\,(g) = \varnothing$ and they have a common fixed point in $\mathbb{H}^3$. $\square$

Note that Euclidean triangle groups can be regarded as two-generator groups

$$\Delta(p,q,r) = \langle f, g : f^p = g^q = (fg)^r = Id \rangle,$$

where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ and hence

$$(p,q,r) = (2,3,6), (2,4,4), (3,3,3),$$

Assume that $a, b,$and c are the sides of the triangle counterclockwise on the complex plan $\mathbb{C}$ opposite the angles $\frac{\pi}{p}, \frac{\pi}{q}$, and $\frac{\pi}{r}$, respectively. Recall that $a, b,$and c are the corresponding reflections about the sides and hence $a^2 = b^2 = c^2 = Id$. It follows that $f = bc$, $g = ca$, and $fg = ba$ are the rotations of order $p, q,$ and r, centered at the corresponding vertices of the triangle, counterclockwise from the corresponding axis to the other axis, respectively. Thus, the counterclockwise rotation angles of $f, g,$ and fg are double two interior angles $\frac{2\pi}{p}, \frac{2\pi}{q}$, and double exterior angle $2\pi\left(1 - \frac{1}{r}\right)$, respectively. Similarly, the comments here are valid for finite spherical triangle groups on the unit sphere S^2 just replacing $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ by $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$. Furthermore, Hagelberg, MacLachlan, and Rosenberger studied discrete generalized triangle groups, see [36] for the more details.

It is discussed the classification of discrete elementary groups in Section 5.1 of Beardon's book [2] that is one of main reference books. The classification of discrete elementary groups will be reformulated in the following theorem.

Theorem 2.4.9 (Classification of discrete elementary groups) *Let G be an elementary discrete group of* $\mathrm{Isom}^+(\mathbb{H}^3)$. *Then G is isomorphic to one of the following group. In case where $p = \infty$, no relation of the form $a^p = Id$ or $b^p = Id$ should be imposed.*

(a) A cyclic group $\mathbb{Z}_p = \langle a : a^p = Id \rangle$, for some $p = 1, 2, \cdots, \infty$.

(b) A dihedral group $D_p \cong \mathbb{Z}_p \ltimes \mathbb{Z}_2 = \langle a, b : a^p = b^2 = Id, bab^{-1} = a^{-1} \rangle$, for some $p = 1, 2, \cdots, \infty$.

(c) The group $(\mathbb{Z}_p \times \mathbb{Z}) \ltimes \mathbb{Z}_2$ or $\mathbb{Z}_p \times \mathbb{Z}$ for some $p = 1, 2, \cdots, \infty$:

$$\langle a, b, c : aba^{-1}b^{-1} = b^p = c^2 = Id, cac^{-1} = a^{-1}, cbc^{-1} = b^{-1} \rangle.$$

(d) A Euclidean translation group $\mathbb{Z} \times \mathbb{Z}$.

(e) A Euclidean triangle group $\Delta(2, 3, 6)$, or $\Delta(2, 4, 4)$, or $\Delta(3, 3, 3)$.

(f) A finite spherical triangle group $A_4 = \Delta(2, 3, 3)$, or $S_4 = \Delta(2, 3, 4)$, or $A_5 = \Delta(2, 3, 5)$.

The following lemma shows that if the parameter $\gamma(f, g) = 0$ then the two-generator group $\langle f, g \rangle$ is elementary, but the converse need not be true. One can identify the two-generator elementary groups between $\gamma(f, g) = 0$ and $\gamma(f, g) \neq 0$.

Lemma 2.4.10 *Let $\Gamma = \langle f, g \rangle$ be a two-generator group with the parameters (γ, β, β'). If $\gamma = 0$ then Γ is elementary. Equivalently, if Γ is non-elementary then $\gamma \neq 0$.*

Proof. It is a direct consequence of the result of Theorem 2.3.12 which implies that if $\gamma = 0$, then f and g have a common fixed point z_0 in $\bar{\mathbb{C}}$. It follows that Γ fixes z_0 and has a finite orbit $\{z_0\}$. Therefore Γ is elementary. $\square$

Lemma 2.4.11 *Suppose that* $\Gamma = \langle f, g \rangle$ *is a group generated by* f *and* g *in* $\text{Isom}^+(\mathbb{H}^3)$. *If* Γ *is a cyclic group then the parameter* $\gamma(f, g) = 0$.

Proof. Since $\Gamma = \langle f, g \rangle$ is a cyclic group, one may assume $\Gamma = \left\{a^k : k \in \mathbb{Z}\right\}$ for some $a \in \text{Isom}^+(\mathbb{H}^3)$. Thus, $f = a^k$ and $g = a^l$ for some $k, l \in \mathbb{Z}$. Therefore,

$$\begin{aligned}
\gamma(f, g) &= tr\left(fgf^{-1}g^{-1}\right) - 2 \\
&= tr\left(a^k a^l a^{-k} a^{-l}\right) - 2 \\
&= tr(Id) - 2 = 0.
\end{aligned}$$

$\square$

Lemma 2.4.12 *Let* $\Gamma = \langle f, g \rangle$ *be a discrete group with a triple of parameters* $(\gamma, \beta, -4)$. *If* $\gamma = \beta$ *then* Γ *is elementary. Moreover,* Γ *is isomorphic to a dihedral group* D_p, *for some* $p = 1, 2, 3, \cdots, \infty$.

Proof. There are two cases to consider: Suppose f is parabolic. Then $\beta = 0 = \gamma$ and therefore by lemma 2.4.10 Γ is elementary.

Suppose f is not parabolic. One may assume the following by Theorem 2.1.17,

$$f = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}, \; g = \begin{pmatrix} \alpha & \mu \\ \delta & \nu \end{pmatrix} \text{ where } \lambda \neq 0, \lambda \neq \pm 1, \alpha\nu - \mu\delta = 1$$

Computing the commutator,

$$fgf^{-1}g^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \begin{pmatrix} \alpha & \mu \\ \delta & \nu \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \nu & -\mu \\ -\delta & \alpha \end{pmatrix}$$
$$= \begin{pmatrix} \alpha\nu - \lambda^2\mu\delta & \alpha\lambda^2\mu - \alpha\mu \\ \frac{1}{\lambda^2}\delta\nu - \delta\nu & \alpha\nu - \frac{1}{\lambda^2}\mu\delta \end{pmatrix},$$

and hence the parameters are

$$\begin{aligned} \gamma &= \mathrm{tr}\,[f,g] - 2 \\ &= 2\alpha\nu - \lambda^2\mu\delta - \frac{1}{\lambda^2}\mu\delta - 2 \\ &= 2\mu\delta - \mu\lambda^2\delta - \frac{1}{\lambda^2}\mu\delta \\ &= -\mu\delta(\lambda - \frac{1}{\lambda})^2, \\ \beta &= \mathrm{tr}^2(f) - 4 = (\lambda - \frac{1}{\lambda})^2. \end{aligned}$$

Assuming $\gamma = \beta$

$$-\mu\delta(\lambda - 1/\lambda)^2 = (\lambda - \frac{1}{\lambda})^2$$
$$-\mu\delta = 1$$

Hence $\alpha\nu - \mu\delta = 1$, so $\alpha\nu = 0$. As assumed, $\beta(g) = -4$, and hence $\mathrm{tr}(g) = 0$. It follows that $\alpha + \nu = 0$, hence $\alpha = 0$. Then

$$g = \begin{pmatrix} 0 & \mu \\ \delta & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \mu \\ -1/\mu & 0 \end{pmatrix}$$

Therefore, $g(z) = -\frac{\mu^2}{z}$. As g interchanges fixed points of f, hence Γ has a finite orbit and is elementary.

Since $\beta(g) = -4$, g is order 2. Recall, f and g

$$f = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \text{ and } g = \begin{pmatrix} 0 & \mu \\ \frac{-1}{\mu} & 0 \end{pmatrix} \text{where } \lambda \neq 0, \mu \neq 0.$$

Computing

$$\begin{aligned} gfg^{-1} &= \begin{pmatrix} 0 & \mu \\ \frac{-1}{\mu} & 0 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \begin{pmatrix} 0 & -\mu \\ \frac{1}{\mu} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{pmatrix} = f^{-1} \end{aligned}$$

Thus, $gfg^{-1} = f^{-1}$. By Theorem 2.4.9 Γ is dihedral group D_p, for some $p = 1, 2, 3, \cdots, \infty$. $\square$

Let f and g be elliptic of order $m \in \{2, 3, 4, 5\}$ in $\text{Möb}^+(\overline{\mathbb{C}})$, and let θ be the angle subtended at the origin between axis(f) and axis(g) and hence $\delta = 0$. Then one can compute $\sin^2(\theta)$ by using Lemmas 6.19, 6.20, and 6.21 in [41], find the parameters $\beta(g)$ and $\beta(f)$ from Corollary 2.3.6, and calculate the parameter $\gamma(f, g)$ by using Corollary 2.3.16. Thus, with these elementary observations and spherical trigonometry, the list of triples of parameters with non-zero parameter $\gamma(f, g)$ can be obtained in the following Tables 1, 2, and 3 (including the cases in the Zhang's dissertation [56]). The list of parameters occurring in these three tables is called the *exceptional set of parameters*.

Table 1: Commutator parameter: 2, m.

m	$\sin^2(\theta)$	γ	Group	Parameters
3	$\frac{2}{3}$	-2	A_4	$(-2,-3,-4)$
3	$\frac{1}{3}$	-1	S_4	$(-1,-3,-4)$
3	$\frac{3-\sqrt{5}}{6}$	$-\frac{3-\sqrt{5}}{2}$	A_5	$(-\frac{3-\sqrt{5}}{2},-3,-4)$
3	$\frac{3+\sqrt{5}}{6}$	$-\frac{3+\sqrt{5}}{2}$	A_5	$(-\frac{3+\sqrt{5}}{2},-3,-4)$
3	1	-3	D_3	$(-3,-3,-4)$
4	$\frac{1}{2}$	-1	S_4	$(-1,-2,-4)$
4	1	-2	D_4	$(-2,-2,-4)$
5	$\frac{5-\sqrt{5}}{10}$	$-\frac{3-\sqrt{5}}{2}$	A_5	$(-\frac{3-\sqrt{5}}{2},-\frac{5-\sqrt{5}}{2},-4)$
5	$\frac{5-\sqrt{5}}{10}$	-1	A_5	$(-1,-\frac{5+\sqrt{5}}{2},-4)$
5	$\frac{5+\sqrt{5}}{10}$	-1	A_5	$(-1,-\frac{5-\sqrt{5}}{2},-4)$
5	$\frac{5+\sqrt{5}}{10}$	$-\frac{3+\sqrt{5}}{2}$	A_5	$(-\frac{3+\sqrt{5}}{2},-\frac{5+\sqrt{5}}{2},-4)$
5	1	$-\frac{5-\sqrt{5}}{2}$	D_5	$(-\frac{5-\sqrt{5}}{2},-\frac{5-\sqrt{5}}{2},-4)$
5	1	$-\frac{5+\sqrt{5}}{2}$	D_5	$(-\frac{5+\sqrt{5}}{2},-\frac{5+\sqrt{5}}{2},-4)$

Table 2: Commutator parameter: 3, m.

m	$\sin^2(\theta)$	γ	Group	Parameters
3	$\frac{4}{9}$	-1	A_5	$(-1,-3,-3)$
3	$\frac{8}{9}$	-2	A_4	$(-2,-3,-3)$
4	$\frac{2}{3}$	-2	S_4	$(-2,-2,-3)$
5	$\frac{10-2\sqrt{5}}{15}$	$-\frac{3-\sqrt{5}}{2}$	A_5	$(-\frac{3-\sqrt{5}}{2},-\frac{5-\sqrt{5}}{2},-3)$
5	$\frac{10-2\sqrt{5}}{15}$	-1	A_5	$(-1,-\frac{5+\sqrt{5}}{2},-3)$
5	$\frac{10+2\sqrt{5}}{15}$	-1	A_5	$(-1,-\frac{5-\sqrt{5}}{2},-3)$
5	$\frac{10+2\sqrt{5}}{15}$	$-\frac{3-\sqrt{5}}{2}$	A_5	$(-\frac{3-\sqrt{5}}{2},-\frac{5+\sqrt{5}}{2},-3)$

Notice that the angle between intersecting axes of elliptics of order 4 in a discrete group is always either 0 when they meet on the Riemann sphere $\overline{\mathbb{C}}$ or $\pi/2$. This yields the additional parameter $(-1,-2,-2)$ for the elementary group

S_4 when generated by two elements of order 4.

Furthermore, the angle between intersecting axes of elliptics of order 5 in a discrete group is either arcsin $\frac{2}{\sqrt{5}}$ or its complement arcsin $\frac{-2}{\sqrt{5}}$. After possibly taking powers of the generator of order 5, the three additional parameters can be obtained in the following table.

Table 3: Commutator parameter: $4, 4$ and $5, 5$.

m, m	$\sin(\theta)$	γ	Group	Parameters
$4, 4$	1	-1	S_4	$(-1, -2, -2)$
$5, 5$	$\frac{2\sqrt{5}}{5}$	$-\frac{3-\sqrt{5}}{2}$	A_5	$(-\frac{3-\sqrt{5}}{2}, -\frac{5-\sqrt{5}}{2}, -\frac{5-\sqrt{5}}{2})$
$5, 5$	$\frac{2\sqrt{5}}{5}$	-1	A_5	$(-1, -\frac{5+\sqrt{5}}{2}, -\frac{5-\sqrt{5}}{2})$
$5, 5$	$\frac{2\sqrt{5}}{5}$	$-\frac{3+\sqrt{5}}{2}$	A_5	$(-\frac{3+\sqrt{5}}{2}, -\frac{5+\sqrt{5}}{2}, -\frac{5+\sqrt{5}}{2})$

Remark 2.4.13 (1) *The axes of elliptics both of order* 2 *can intersect at an angle* $\frac{k\pi}{n}$ *for any* k *and* $n \geq 2$ *giving the Dihedral group* D_n *with parameters* $\left(-4\sin^2 \frac{k\pi}{n}, -4, -4\right)$.

(2) *The axes of elliptics of order* p *and* q, $p \leq q$, *in a discrete group meet on the Riemann sphere* $\overline{\mathbb{C}}$, *i.e., meeting with angle* 0, *if and only if*

$$(p, q) \in \{(2,2), (2,3), (2,4), (2,6), (3,3), (3,6), (4,4), (6,6)\}$$

For all of these groups $\gamma\,(f,\, g) = 0$.

In particular, Euclidean triangle groups $\Delta(2,3,6)$, $\Delta(3,3,3)$, *and* $\Delta(2,4,4)$ *have* $\gamma\,(f,\, g) = 0$. *Furthermore, Lemma* 2.4.11 *below tells that cyclic groups have* $\gamma\,(f,\, g) = 0$. *Euclidean translation groups have* $\gamma = 0$ *as well. Thus, the two-generator elementary groups with* $\gamma\,(f,\, g) \neq 0$ *are the dihedral groups (if* $\gamma\,(f,\, g) = \beta\,(f)$ *and* $\beta(g) = -4$) *or the finite spherical triangle groups.*

(3) *The axes of elliptics elements of orders* 2 *and* p, $p \geq 3$, *can meet at right-angles. In this case, the dihedral group* D_p *with parameters are*

$$\left(-4\sin^2\frac{\pi}{p}, -4\sin^2\frac{\pi}{p}, -4\right).$$

At the end of this section, it is observed that the group generated by two distinct elements of order 2 is elementary.

Theorem 2.4.14 *Let* $\Gamma = \langle f, g\rangle$ *be a discrete subgroup of* $\text{Isom}^+(\mathbb{H}^3)$, *where* f *and* g *are distinct elements of order* 2. *Then* Γ *is elementary.*

Proof. (1) Assume $axis(f) \cap axis(g) = \varnothing$. Let α be the common perpendicular between two axes $axis(f)$ and $axis(g)$. Since f and g are both rotation of order 2, each of them interchanges the ending points of α and fixes α setwise. Thus, the product fg fixes the ending points of α and fixes α setwise. It follows that $axis(fg) = \alpha$ and hence $axis(fg)$ is fixed setwise by fg. Therefore, fg is loxodromic. Again since f and g are both rotation of order 2, by using Proposition 2.3.9,

$$\begin{aligned}\gamma(f, fg) &= \gamma(fg, f)\\ &= \beta(f^2g) + \beta(fg) + 4\\ &= \beta(g) + \beta(fg) + 4\\ &= \beta(fg).\end{aligned}$$

That is, $\gamma(f, fg) = \beta(fg)$. Notice that $\langle f, g\rangle = \langle f, fg\rangle \cong D_\infty$. This shows that $\langle f, g\rangle$ is an elementary group.

(2) Suppose $axis(f) \cap axis(g) \neq \varnothing$ in $\mathbb{H}^3$, then f and g have a common fixed point in $\mathbb{H}^3$. Since f and g are rotations of order 2 about their fixed axes $axis(f)$ and $axis(g)$ in $\mathbb{H}^3$, the product fg is a rotation of order 2 about $axis(fg)$

that is perpendicular to $axis(f)$ and $axis(g)$ and passing through the common fixed point. It follows that $gfg^{-1} = f^{-1}$ and hence $\langle f, g\rangle = \langle f, fg\rangle \cong D_n$. That is $\langle f, g\rangle$ is an elementary group.

(3) Suppose $axis(f) \cap axis(g) \neq \varnothing$ in $\overline{\mathbb{C}}$, then by (2.1), f and g have one common fixed point in $\overline{\mathbb{C}}$ and hence by Remark 2.3.13, $fg = gf$. There are two cases to consider for the common fixed point, say $z_0 \in \overline{\mathbb{C}}$.

Case 1: $z_0 \in \mathbb{C}$: Since both f and g are rotations of order 2 with the common center in $\mathbb{C}$, the product fg is a rotation of order 2. Thus, $gfg^{-1} = f^{-1}$ and hence $\langle f, g\rangle \cong D_\infty$. This shows that $\langle f, g\rangle$ is an elementary group.

Case 2: $z_0 = \infty$: Now f and g fix ∞, so $axis(f)$ and $axis(g)$ are vertical hyperbolic lines. Since both f and g are rotations of order 2 with distinct centers. On the other hand, both rotation angles are π and the sum is 2π. It follows that the product fg is a translation and hence fg is parabolic fixing ∞. It follows that the group $\langle f, g\rangle$ fixes ∞. Thus, by Theorem 2.4.6 $\langle f, g\rangle$ is elementary of type II.

□

Chapter 3: Moduli Space of Kleinian Groups

One can describe the space of two-generator groups by three complex dimensional space $\mathbb{C}^3$ via the mapping

$$\langle f,\, g\rangle \longmapsto \left(\gamma\left(f,\, g\right), \beta\left(f\right), \beta(g)\right).$$

Indeed, every two-generator Kleinian group $\langle f,\, g\rangle$ can be determined uniquely up to conjugacy by a triple of complex parameters $\left(\gamma\left(f,\, g\right), \beta\left(f\right), \beta(g)\right)$. Thus, the space of two-generator Kleinian groups can be identified with a subspace $\mathcal{D}$ of $\mathbb{C}^3$. Note that conjugations preserve the triples of complex parameters for two-generator Kleinian groups, one can normalize a two-generator Kleinian group and even more a sequence of two-generator Kleinian groups or passing to a subsequence at any stage if necessary.

A fundamental result about the spaces of two-generator Kleinian groups is that they are closed in the topology of algebraic convergence (Jørgensen Theorem 3.2.13). This dissertation extends that the set $\mathcal{D}$ of triples of parameters for Kleinian groups is closed subspace in $\mathbb{C}^3$ in the usual topology (in Theorem 3.2.15), and that the set $\mathcal{D}_2$ of the pairs of the first two parameters for Kleinian groups is a closed subspace in two complex dimensional space $\mathbb{C}^2$ in the usual topology (Theorem 3.3.4) by considering two projections, one is from $\mathcal{D}$ to $\mathcal{D}_2$ and the other is from $\mathcal{D}$ to the subspace on the slice $z_3 = -4$ in $\mathbb{C}^3$. So that there is an alternate proof of Jørgensen's inequality (Theorem 3.4.1) based on the closed subspace $\mathcal{D}_2$ in $\mathbb{C}^2$ before looking at the more general cases in the next chapter.

3.1 Kleinian groups

Nowadays the term "Kleinian group" is being often used for a discrete subgroup of hyperbolic isometries, a Kleinian group is adopted as a non-elementary discrete group of hyperbolic isometries in this dissertation. Kleinian groups were introduced by Poincaré in the 1880's as subgroups of the Möbius group $\text{Möb}(\overline{\mathbb{C}})$ acting discontinuously on some domain of $\overline{\mathbb{C}}$. In this section, the discontinuous groups are characterized in Theorem 3.1.11 and the key concepts with different definitions in the literature are clarified, such as discontinuity in Theorem 3.1.14, and the limit set $L(G)$ in Lemma 3.1.6.

As mentioned at the beginning of Chapter 2, there are three different ways of thinking about subgroups with the same concept of discreteness: as subgroups of $\text{Isom}^+(\mathbb{H}^3)$, as subgroups of $\text{Möb}^+\left(\overline{\mathbb{C}}\right)$, and as subgroups of $\text{PSL}(2,\mathbb{C})$. Thus, one can define Kleinian groups as subgroups of $\text{Isom}^+(\mathbb{H}^3)$ as follows.

Definition 3.1.1 *The subgroup G of* $\text{Isom}^+(\mathbb{H}^3)$ *is called a* Kleinian group *if it is discrete and non-elementary.*

Thus, Kleinian groups are not elementary discrete groups studied in Section 2.4. Applying for Theorem 2.4.9 there, the hyperbolic triangle groups

$$\Delta(p,q,r) = \left\langle f,g : f^p = g^q = (fg)^r = Id,\ \frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1 \right\rangle$$

are two-generator Kleinian groups.

Example 3.1.2 *The hyperbolic triangle group*

$$\Delta(2,4,5) = \left\langle f,g : f^2 = g^4 = (fg)^5 = Id \right\rangle$$

is a Kleinian group. By Corollary 2.3.6, $\beta(f) = -4, \beta(g) = -2$, and $\beta(fg) =$

$\frac{\sqrt{5}-5}{2}$ *or* $\frac{-\sqrt{5}-5}{2}$. *It follows that* $\operatorname{tr}(f) = 0$ *and then Proposition 2.3.9 gives*

$$\begin{aligned}
\gamma(f,\, g) &= \beta(f) + \beta(g) + \beta(fg) - \operatorname{tr}(f)\operatorname{tr}(g)\operatorname{tr}(fg) + 8 \\
&= \beta(g) + \beta(fg) + 4 \\
&= 2 + \frac{\sqrt{5}-5}{2} \ \textit{or}\ 2 - \frac{\sqrt{5}+5}{2} \\
&= \frac{\sqrt{5}-1}{2} \ \textit{or}\ -\frac{\sqrt{5}+1}{2}.
\end{aligned}$$

Thus, the triple of parameters for the Kleinian group $\Delta(2,4,5)$ *is* $\left(\frac{\sqrt{5}-1}{2}, -2, -4\right)$ *or* $\left(-\frac{\sqrt{5}+1}{2}, -2, -4\right)$.

Example 3.1.3 *There are infinitely many Kleinian groups given by the hyperbolic triangle groups*

$$\Delta(2,3,p) = \left\langle f, g : f^p = g^2 = (fg)^3 = Id,\ p \geq 7 \right\rangle.$$

Then the corresponding triples of parameters are $(1+\beta, \beta, -4)$, *where* $\beta = \beta(f)$. *In fact, applying for Corollary 2.3.6,* $\beta(g) = -4$ (*so* $\operatorname{tr}(g) = 0$) *and* $\beta(fg) = -3$. *So Proposition 2.3.9 gives*

$$\begin{aligned}
\gamma(f,\, g) &= \beta(f) + \beta(g) + \beta(fg) - \operatorname{tr}(f)\operatorname{tr}(g)\operatorname{tr}(fg) + 8 \\
&= 1 + \beta.
\end{aligned}$$

A necessary condition of two-generator Kleinian groups $\langle f,\, g\rangle$ is that the parameter $\gamma(f,\, g) \neq 0$ in the following corollary, which is directly from Lemma 2.4.10. Moreover, by Lemma 2.3.12, the condition $\gamma(f,\, g) \neq 0$ is equivalent to the disjoint fixed point sets: $\operatorname{Fix}(f) \cap \operatorname{Fix}(g) = \varnothing$.

Corollary 3.1.4 *Let* $\Gamma = \langle f, g \rangle$ *be a two-generator Kleinian group with the parameters* (γ, β, β'), *then* $\gamma \neq 0$.

Next introduce discontinuous groups and some properties.

Definition 3.1.5 *Let* X *be a topological space and* G *be the group of homeomorphisms acting on* X. *The group* G *is called to be* discontinuous *at* $x \in X$ *if there is a neighborhood* U *of* x *such that*

$$g(U) \cap U = \varnothing, \text{ for all but finitely many } g \in G.$$

The set of all $x \in X$ *at which* G *is discontinuous is called the* ordinary set *of* G *and denoted by* $O(G)$. *A group* G *is called a* discontinuous group *acting on* X *if* $O(G) \neq \varnothing$, *i.e., there is a point* $x \in X$ *at which* G *is discontinuous.*

Maskit defined freely discontinuous in his book [46], it is clear that a group G is freely discontinuous at a point x then it is discontinuous at x. Thus, the discontinuity here generalizes the free discontinuity.

Clearly, if G is discontinuous at all points of an open subset D of X then $D \subseteq O(G)$. Furthermore, the ordinary set $O(G)$ can not contain any points fixed by the group G. Thus,

$$\mathrm{Fix}\,(G) \cap O(G) = \varnothing.$$

Lemma 3.1.6 *Let* G *be a discrete group of isometries acting on* $\overline{\mathbb{H}^3}$. *Then the limit set is the complement set of the ordinary set* $O(G)$:

$$L(G) = \overline{\mathbb{H}^3} - O(G).$$

Proof. Notice that $L(G)$ is set of all accumulation points of the orbit $\{g((0,0,1)) : g \in G\}$.

(1) Let $x \in L(G)$, and let U be any neighborhood of x. Then there is a sequence $\{g_j\}$ in G such that

$$g_j\left((0,0,1)\right) \in U, \text{ for all } j.$$

In particular, $g_1\left((0,0,1)\right) \in U$, and hence $(0,0,1) \in g_1^{-1}(U)$. Let $h_j = g_j \circ g_1^{-1}$, then $g_j\left((0,0,1)\right) \in g_j \circ g_1^{-1}(U) = h_j(U)$. It follows that

$$h_j(U) \cap U \neq \varnothing,$$

for all j. Thus, the group G is not discontinuous at x, so $x \in \overline{\mathbb{H}^3} - O(G)$ and hence $L(G) \subseteq \overline{\mathbb{H}^3} - O(G)$.

(2) Let $x \in \overline{\mathbb{H}^3} - O(G)$, then the group G is not discontinuous at x. Apply for Lemma 2.1.19 and Lemma 4.1 in [25], there exists a point x' in $\overline{\mathbb{H}^3} - O(G)$ and a sequence $\{g_j\}$ in G such that

$$\lim_{j\to\infty} g_j = x \tag{3.1}$$

uniformly on all compact subsets of $\overline{\mathbb{H}^3} \backslash \{x'\}$. It's clear that the point $(0,0,1)$ is not fixed under G. There are two cases to consider.

If $x' \neq (0,0,1)$, then $\lim_{j\to\infty} g_j\left((0,0,1)\right) = x$, so $x \in L(G)$ and hence $\overline{\mathbb{H}^3} - O(G) \subseteq L(G)$.

If $x' = (0,0,1)$, then there exists $g \in G$ such that $g\left((0,0,1)\right) \neq x'$, and hence $\lim_{j\to\infty} g_j \circ g\left((0,0,1)\right) = \lim_{j\to\infty} g_j\left(g\left((0,0,1)\right)\right) = x$, so $x \in L(G)$ and hence $\overline{\mathbb{H}^3} - O(G) \subseteq L(G)$.

Therefore, it is proved that $L(G) = \overline{\mathbb{H}^3} - O(G)$. $\square$

Thus, the set of points fixed by the group G is included in the limit set $L(G)$. Actually, the limit set $L(G)$ is defined in the Beardon's book [2] as the closure of the set of points fixed by some loxodromic element of G in $\overline{\mathbb{C}}$, and hence

$$\text{Fix}\,(G) \subseteq L(G) \subseteq \overline{\mathbb{C}}.$$

It follows that if G is a discrete subgroup of $\text{Isom}^+(\mathbb{H}^3)$ then $\mathbb{H}^3 \subseteq O(G)$ and hence $\mathbb{H}^3 \cap \text{Fix}\,(G) = \varnothing$. Moreover, Lemma 3.1.6 gives that the limit set $L(G)$ is the set of all accumulation points of the orbit of a point of $\mathbb{H}^3$ under the group G.

Next, show that the discreteness is necessary for a discontinuous group in the following proposition.

Proposition 3.1.7 *Let X be a topological space and G be a discontinuous group acting on X. Then G is a discrete group.*

Proof. Suppose that G is a non-discrete group, then there is a sequence $\{g_j\}$ of elements of G converging to the identity:

$$\lim_{j\to\infty} g_j = Id.$$

Thus for every $x \in X$, $\lim_{j\to\infty} g_j(x) = x$, so for every neighborhood U of x there are infinitely many $g_k \in \{g_j\}$ such that

$$g_k(U) \cap U \neq \varnothing.$$

So G is not discontinuous at any point, and hence G is not a discontinuous group, it is a contradiction. □

The following proposition shows that the discontinuity is preserved under a conjugation.

Proposition 3.1.8 *Let G be topological group acting on the topological space X. Suppose that Γ and Γ' are conjugate subgroups of G. If Γ is a discontinuous* *subgroup of G then Γ' is a discontinuous subgroup of G.*

Proof. Since Γ' is conjugate to Γ, there exists $h \in G$ such that

$$\Gamma' = h \circ \Gamma \circ h^{-1} = \left\{ g' = hgh^{-1} : g \in \Gamma \right\}.$$

On the other hand, Γ is a discontinuous group, so there is $x \in X$ such that Γ is discontinuous at x, and hence there is a neighborhood U of x such that $g(U) \cap U = \varnothing$, for all but finitely many $g \in \Gamma$. It follows that $h\,(g(U) \cap U) = \varnothing$. Thus, there is a neighborhood $h\,(U)$ of $h\,(x)$ such that

$$\begin{aligned} g'(h\,(U)) \cap h\,(U) &= hgh^{-1}(h\,(U)) \cap h\,(U) \\ &= h\,(g\,(U)) \cap h\,(U) \\ &= h\,(g(U) \cap U) = \varnothing, \end{aligned}$$

for all but finitely many $g' \in \Gamma'$. Therefore Γ' is discontinuous at $h\,(x) \in X$, so Γ' is a discontinuous group. □

Lemma 3.1.9 *Let X be a topological space and G be the group of homeomorphisms acting on X. If G is discontinuous at x, then G is discontinuous at $g(x)$ for all $g \in G$.*

Proof. Since G is discontinuous at x, there exists a neighborhood U of x such that $h(U) \cap U = \varnothing$, for all but finitely many $h \in G$. Therefore, there exists a neighborhood $g(U)$ of $g(x)$ such that $h(g(U)) \cap g(U) = \varnothing$, for all but finitely many $h \in G$ and hence G is discontinuous at $g(x)$. In fact, if $h(g(U)) \cap g(U) \neq \varnothing$, for infinitely many $h \in G$ then there exists $t_0 \in h(g(U)) \cap g(U)$ such that $t_0 \in h(g(U))$ and $t_0 \in g(U)$. Thus, there exists $x_1 \in U$ such that $t_0 = g(x_1)$ and there exists $x_2 \in U$ such that $t_0 = h(g(x_2))$, so $g(x_1) = h(g(x_2))$ and then $x_1 = g^{-1}h(g(x_2))$ where $g^{-1}hg \in G$, thus $g^{-1}h\,(g(U)) \cap U \neq \varnothing$ for infinitely many $h \in G$. Let $g' =$

$g^{-1}hg$, then $g^{-1}h_1g = g^{-1}h_2g$ if and only if $h_1 = h_2$ and hence $g'(U) \cap U \neq \varnothing$ for infinitely many $g' \in G$, it is a contradiction to that G is discontinuous at x. $\square$

Proposition 3.1.10 *Let X be a topological space and G be the group of homeomorphisms acting on X. Then*

(1) *the ordinary set $O(G)$ is an open set in X and is G-invariant.*

(2) *the limit set $L(G)$ is a closed set in X and is G-invariant.*

Proof. (1) Assume $O(G) \neq \varnothing$, otherwise $O(G)$ is already open. Let $x_0 \in O(G)$, then there exists a neighborhood U of x_0 such that $g(U) \cap U = \varnothing$, for all but finitely many $g \in G$. Thus, for all $x \in U : g(U) \cap U = \varnothing$, for all but finitely many $g \in G$, so $x \in O(G)$ and hence $U \subseteq O(G)$. Thus, $O(G)$ is open in X. Now applying for Lemma 3.1.9, if $x \in O(G)$ then $g(x) \in O(G)$, i.e., $g(O(G)) \subseteq O(G)$, for all $g \in G$. On the other hand, since $g^{-1} \in G$, $g^{-1}(O(G)) \subseteq O(G)$ and hence $O(G) = g\left(g^{-1}(O(G))\right) \subseteq g\left(O(G)\right)$. Thus, $g(O(G)) = O(G)$, for all $g \in G$. So $O(G)$ is G-invariant.

(2) It is clear that $L(G)$ is closed in X as the complement of an open set is closed. Since the entire space X and $O(G)$ are G-invariant, then $L(G) = X - O(G)$ is G-invariant. $\square$

Remark that the limit set $L(G)$ of a non-elementary group is a *perfect set* (i.e., it is closed and has no isolated points). Thus, if a limit set $L(G)$ contains three distinct points then it has uncountably many points. It is that motivates Definition 2.4.5 of an elementary group. The following theorem gives two characterizations of discontinuous groups.

Theorem 3.1.11 *Let X be a topological space and G be a group of homeomorphisms on X, then the following statements are equivalent.*

(1) *G is a discontinuous group on X.*

(2) *There exists a point x in X and a neighborhood U of x such that*

$$f(U) \cap g(U) = \varnothing,$$

for all but finitely many pairs of distinct f and $g \in G$.

(3) *There exists a point x in X and a neighborhood U of x such that*

$$g(U) \cap U = \varnothing, \text{ for } g \in G - G_x$$

where the stabilizer G_x is finite.

Proof. (1) $\implies$ (2) : Since G is a discontinuous group on X, there is $x \in X$ at which G is discontinuous, i.e., there exists a neighborhood U of x such that $g(U) \cap U = \varnothing$, for all but finitely many $g \in G$. Suppose for infinitely many pairs of distinct f and $g \in G : f(U) \cap g(U) \neq \varnothing$. Taking $g^{-1} \in G$,

$$g^{-1}f(U) \cap U = g^{-1}f(U) \cap g^{-1}g(U) = g^{-1}\left(f(U) \cap g(U)\right) \neq g^{-1}\left(\varnothing\right) = \varnothing.$$

Let $h = g^{-1}f \in G$, then $h \neq Id$ as f and g are distinct. It follows that $h(U) \cap U \neq \varnothing$ for infinitely many $h \in G$, it is a contradiction to the discontinuity of G at x.

(2) $\implies$ (1) : Suppose there exists a point x in X and a neighborhood U of x such that $f(U) \cap g(U) = \varnothing$, for all but finitely many pairs of distinct f and $g \in G$. In particular, taking $g = Id$, then $f(U) \cap U = \varnothing$ for all but finitely many $f \in G$. Thus G is discontinuous at $x \in X$ and hence G is a discontinuous group on X.

(1) $\implies$ (3) : Since G is a discontinuous group on X, there exists a point x in X, where G is discontinuous. Thus, there exists a neighborhood U of x such that $g(U) \cap U = \varnothing$, for all but finitely many $g \in G$.

Let $g \in G_x$ then $g(x) = x$ and hence $x \in g(U) \cap U$, so $g(U) \cap U \neq \varnothing$.

Therefore,

$$G_x \subseteq \{g \in G : g(U) \cap U \neq \varnothing\} = S.$$

Notice that S is finite and hence the stabilizer G_x is finite.

Moreover, $S - G_x$ is finite, so there is $n \in \mathbb{N}$ such that $S - G_x = \{g_1, g_2, \cdots, g_n\}$. Set the following interior set

$$U' = (U - \cup_{k=1}^{n} (g_k(U) \cap U))^{\circ}$$

then U' is a neighborhood of x such that $g(U') \cap U' = \varnothing$, for $g \in G - G_x$.

(3) $\implies$ (1) : Suppose there exists a point x in X and a neighborhood U of x such that $g(U) \cap U = \varnothing$, for $g \in G - G_x$ where G_x is finite. Thus, G is discontinuous at $x \in X$ and hence G is a discontinuous group on X. $\square$

Next turn the attention to the groups which are properly discontinuous in the Riemann sphere $\overline{\mathbb{C}}$.

Definition 3.1.12 *Let G be the group of self-homeomorphisms acting on $\overline{\mathbb{C}}$, and let D be an open subset of $\overline{\mathbb{C}}$. The group G is called to be* properly discontinuous *in D if for every compact subset K of D satisfies*

$$g(K) \cap K = \varnothing,$$

for all but finitely many $g \in G$.

Proposition 3.1.13 *Let G be the group of self-homeomorphisms acting on $\overline{\mathbb{C}}$, and let D be an open subset of $\overline{\mathbb{C}}$. If G is properly discontinuous in D then it is discontinuous at each point in D.*

Proof. Suppose that x is an arbitrary point in the open set D, then there exists an open disk V centered at x such that $x \in V \subseteq \overline{V} \subseteq D$ and hence the closed disk $\overline{V}$ centered at x is a compact subset of D. Since G is properly discontinuous in D, then

$$\varnothing \subseteq g(V) \cap V \subseteq g(\overline{V}) \cap \overline{V} = \varnothing,$$

for all but finitely many $g \in G$, i.e., $g(V) \cap V = \varnothing$, for all but finitely many $g \in G$. Thus, G is discontinuous at $x \in D$. □

Notice that the converse in Proposition 3.1.13 is not true in general, however, the converse holds for subgroups of Möb($\overline{\mathbb{C}}$) in the next theorem.

Theorem 3.1.14 *A subgroup G of* Möb($\overline{\mathbb{C}}$) *is properly discontinuous in $\overline{\mathbb{C}} \backslash L(G)$ if and only if G is a discontinuous group in $\overline{\mathbb{C}}$.*

Proof. (1) Suppose that G is properly discontinuous in $\overline{\mathbb{C}} \backslash L(G)$. By Proposition 3.1.13, G is a discontinuous at each point in $\overline{\mathbb{C}} \backslash L(G) \subseteq \overline{\mathbb{C}}$ and hence G is a discontinuous group in $\overline{\mathbb{C}}$.

(2) Suppose that G is a discontinuous group in $\overline{\mathbb{C}}$. By Proposition 3.1.7, G is discrete. Notice that G is a subgroup of Möbius group Möb($\overline{\mathbb{C}}$), by Lemma 2.1.19, G is a convergence group. Now apply for Theorem 4.8 in [25], since G is discrete, convergence, discontinuous group, G is properly discontinuous in $\overline{\mathbb{C}} \backslash L(G)$. □

Proposition 3.1.15 *Every Kleinian group is countable.*

Proof. Let G be a Kleinian group acting on $\mathbb{H}^3$, then there is $x \in \mathbb{H}^3$ such that G is discontinuous at x.

Let $G(x)$ and G_x be the orbit of x and the stabilizer of x, respectively. Since G is discontinuous at x, by Theorem 3.1.11, G_x is finite and hence all cosets are finite. Thus, G is countable if and only if G/G_x is countable.

It is well known in Abstract Algebra that G/G_x and $G(x)$ have the same cardinality. Thus, G is countable if and only if $G(x)$ is countable. It needs to show $G(x)$ is countable. If $G(x)$ is uncountable subset in $\mathbb{H}^3 \subseteq \mathbb{R}^3$, then it is well known in Topology that $G(x)$ contains a convergent distinct sequence $\{g_j(x)\}$ to a limit point x_0, where $g_j \in G$ and $x_0 \in G(x)$. It follows that the set $K = \cup_{j=1}^{\infty} \{g_j(x)\} \cup \{x_0\}$ is compact and $g_j(K) \cap K \neq \varnothing$ for infinitely many j. G can not be properly discontinuous in $\mathbb{H}^3$. $\square$

Now turn the attention to a finitely generated *free group* that is a group generated with no non-trivial relations on at least one generator, i.e., a group is free if there exists at least one *free generator.* The number of free generators is called *rank*. For example, if a group has n free generators then it is a free group of rank n. Thus, a non-trivial finite group cannot be a free group. Martin [41] has discussed a very interesting family of polynomial trace identities which can be used to obtain geometric information about Kleinian groups. Also Marshall and Martin [45] give a thorough account and complete proofs of the polynomial trace identities by relating them to a determinant condition in a quaternion algebra. Recall the following definition and theorem from [41] at the end of this section.

Definition 3.1.16 *Let $\langle a, b\rangle$ be the free group on the two letters a and b. Any written product of a, b, a^{-1}, and b^{-1} is called a* word *in $\langle a, b\rangle$, and denote $w(a, b)$ by a word starting and ending in a. A word $w(a, b)$ is called a* good word *if it can be written as*

$$w(a, b) = a^{s_1} b^{r_1} a^{s_2} b^{r_2} \cdots a^{s_{m-1}} b^{r_{m-1}} a^{s_m} \tag{3.2}$$

where the exponents of a alternate in sign, i.e., $s_1 \in \{\pm 1\}$ and $s_j = (-1)^{j+1} s_1$, and $r_j \neq 0$ but are otherwise unconstrained.

In particular, if one assumes that $a^2 = Id$, then the alternating sign condition is redundant since $a^k = a = a^{-1}$ or Id for $k \in \mathbb{Z}$ and hence every word $w(a, b)$ is good. For example, in the free group $\langle f,\, g\rangle$ with the triple $(\gamma, \beta, -4)$, every word $w(g,\, f)$ is good, where $f, g \in \mathrm{PSL}(2, \mathbb{C})$. Recall the following well known theorem that is a key tool used in the study of the moduli spaces of discrete groups.

Theorem 3.1.17 *Suppose that $\langle f, g\rangle$ is free group with $\beta = \beta(f)$ and $\gamma = \gamma(f, g)$. Let $w = w(f,g)$ be a good word in $\langle f, g\rangle$. Then there is a monic polynomial p_w of two complex variables having integer coefficients with the following property:*

$$p_w(\gamma, \beta) = \gamma(f, w(g, f)), \tag{3.3}$$

where $w(g, f)$ is the good word in $\langle f, g\rangle$ by interchanging f and g in $w(f, g)$.

The monic polynomial $\gamma(f, w(g, f))$ is referred to *trace polynomial* in this dissertation.

Lemma 3.1.18 *Let $\langle f, g\rangle$ be a free group with $(\gamma, \beta) = (\gamma(f, g), \beta(f))$, and let $w = w(f, g)$ be a good word. Then*

$$p_w(\gamma, \beta) = \gamma(f, w') = \gamma(f, f^m w' f^n), \text{ for } m, n \in \mathbb{Z}$$

where $w' = w(g, f)$ is the good word in $\langle f, g\rangle$ by interchanging f and g in $w(f, g)$.

Proof. Applying for Theorem 3.1.17, $p_w(\gamma, \beta) = \gamma(f, w')$. It needs to show $\gamma(f, w') = \gamma(f, f^m w' f^n)$ only.

By Definition 2.3.1,

$$\begin{aligned}
\gamma(f, f^m w' f^n) &= \operatorname{tr}[f, f^m w' f^n] - 2 \\
&= \operatorname{tr}\left(f f^m w' f^n f^{-1} f^{-n} w'^{-1} f^{-m}\right) - 2 \\
&= \operatorname{tr}\left(f^m f w' f^{-1} w'^{-1} f^{-m}\right) - 2.
\end{aligned}$$

Since $f^m f w' f^{-1} w'^{-1} f^{-m}$ is conjugate to $f w' f^{-1} w'^{-1}$ by f^m, then by Remark 2.1.5, $\operatorname{tr}(f^m f w' f^{-1} w'^{-1} f^{-m}) = \operatorname{tr}(f w' f^{-1} w'^{-1})$. Thus, $\gamma(f, f^m w' f^n) = \gamma(f, w')$, for $m, n \in \mathbb{Z}$. □

3.2 Space of two-generator Kleinian groups

The dissertation pays the attention to two-generator Kleinian groups from this section. A fundamental result concerning spaces of finitely generated Kleinian groups is that they are closed in the topology of algebraic convergence (see Jørgensen Theorem 3.2.13). The approach here is to use the fundamental result extending the closedness of the space of two-generator Kleinian groups to that the set $\mathcal{D}$ of triples of parameters for Kleinian groups is a closed subspace in three complex dimensional space $\mathbb{C}^3$ in the usual topology (see Theorem 3.2.15).

One of the most important subgroups of a Kleinian group $\langle f,\, g\rangle$ is $\langle f,\, gfg^{-1}\rangle$ that generated by two elements of the same trace and Theorem 3.2.6 guarantees that $\langle f,\, gfg^{-1}\rangle$ is a Kleinian group if f is not elliptic of order $p \leq 6$. There are a further two Kleinian groups $\Gamma^{\phi} = \langle f, \phi\rangle$ and $\Gamma^{\psi} = \langle f, \psi\rangle$ in Corollary 3.2.9 once $\langle f,\, gfg^{-1}\rangle$ is Kleinian. These three Kleinian groups play a significant role later on.

A group is said to have a *property virtually* if it has a subgroup with a finite index that has the property. Thus, G is a *virtually Kleinian group* if G has a Kleinian subgroup of finite index. Lemma 3.2.3 and Theorem 3.2.4 state that discrete groups and Kleinian groups are equivalent to the virtually discrete groups and the virtually Kleinian groups, respectively. Furthermore, Theorem 3.2.5 shows that every non-trivial subgroup of finite index in a Kleinian group remains Kleinian.

Starting with the next lemma in part from [16] include an alternate proof as some of the ideas that suggests an approach to the related problems in this dissertation.

Lemma 3.2.1 *Let $\langle f,\, g\rangle$ be a two-generator group, and let $\gamma = \gamma\,(f,\, g)$ and $\beta = \beta\,(f)$. Then the subgroup $\langle f,\, gfg^{-1}\rangle$ of $\langle f,\, g\rangle$ has*

(1) *the triple of parameters*

$$\left(\gamma(f, gfg^{-1}), \beta\,(f)\,, \beta\left(gfg^{-1}\right)\right) = \left(\gamma\,(\gamma - \beta)\,, \beta, \beta\right). \tag{3.4}$$

(2) *index two in $\langle f,\, g\rangle$ whenever g is elliptic of order 2, and the $\mathbb{Z}_2$-extension of $\langle f,\, gfg^{-1}\rangle$ can be expressed as the union of the right cosets of $\langle f,\, gfg^{-1}\rangle$:*

$$\langle f,\, g\rangle = \left\langle f,\, gfg^{-1}\right\rangle \cup g\left\langle f,\, gfg^{-1}\right\rangle.$$

Proof. (1) By the definition of the parameters β and γ,

$$\mathrm{tr}^2(f) = \beta + 4 \text{ and } \mathrm{tr}\,[f, g] = \gamma + 2.$$

Let $h = gfg^{-1}$ then $\mathrm{tr}(f) = \mathrm{tr}(h)$ and hence $\beta\,(gfg^{-1}) = \beta\,(h) = \beta\,(f) = \beta.$

Applying for the identities (2.3) and (2.4),

$$\begin{aligned}
\mathrm{tr}(fh) &= \mathrm{tr}(f)\mathrm{tr}(h) - \mathrm{tr}(fh^{-1}) \\
&= \mathrm{tr}^2(f) - \mathrm{tr}\,[f, g] = \beta - \gamma + 2, \\
\gamma(f, gfg^{-1}) &= \mathrm{tr}\,[f, h] - 2 = 2\mathrm{tr}^2(f) + \mathrm{tr}^2(fh) - \mathrm{tr}^2\,(f)\,\mathrm{tr}\,(fh) - 4 \\
&= 2\,(\beta + 4) + (\beta - \gamma + 2)^2 - (\beta + 4)\,(\beta - \gamma + 2) - 4 \\
&= \gamma(\gamma - \beta).
\end{aligned}$$

(2) Since g is elliptic of order 2, $(gfg^{-1})^p = gf^pg^{-1} = gf^pg$ for each $p \in \mathbb{Z}$. So every element w in $\langle f,\, gfg^{-1}\rangle$ can be expressed as

$$w = f^{p_1}gf^{p_2}gf^{p_3}g \ldots gf^{p_n},$$

where $p_1, p_2, p_3, \ldots, p_{n-1}, p_n \in \mathbb{Z}$. Since the generator $gfg^{-1} = gfg$ consists of two time of g, each element w in $\langle f, gfg^{-1}\rangle$ if and only if w consists of even number of g. If w consists of odd number of g then $gw \in \langle f, gfg^{-1}\rangle$ and hence $w \in g^{-1}\langle f, gfg^{-1}\rangle = g\langle f, gfg^{-1}\rangle$. Thus $\langle f, g\rangle = \langle f, gfg^{-1}\rangle \cup g\langle f, gfg^{-1}\rangle$ and $\langle f, gfg^{-1}\rangle$ has index two in $\langle f, g\rangle$. □

Lemma 3.2.2 *Let $\langle f, g\rangle$ be a two-generator group. If g is not elliptic of order* 2, *then*

$$\mathrm{Fix}(f) \cap \mathrm{Fix}(gfg^{-1}) = \varnothing.$$

Proof. Observe that by Corollary 2.1.11 $\mathrm{Fix}(gfg^{-1}) = g(\mathrm{Fix}(f))$. There are two cases to consider.

(a) If $\mathrm{Fix}(f)$ has one element, say z, i.e., $\mathrm{Fix}(f) = \{z\}$, then $\mathrm{Fix}(gfg^{-1}) = \{g(z)\}$. Since $\langle f, g\rangle$ is Kleinian, $g(z) \neq z$. Thus, $\mathrm{Fix}(f) \cap \mathrm{Fix}(gfg^{-1}) = \varnothing$.

(b) The second case is that $\mathrm{Fix}(f)$ has two elements, say z_1 and z_2, then $\mathrm{Fix}(gfg^{-1}) = \{g(z_1), g(z_2)\}$. Since $\langle f, g\rangle$ is Kleinian, f and g cannot share any fixed points, thus

$$g(z_1) \neq z_1, g(z_2) \neq z_2. \tag{3.5}$$

By Beardon's Theorem 5.1.2 in [2], f and gfg^{-1}cannot share exactly one fixed point. Thus, if $\mathrm{Fix}(f) \cap \mathrm{Fix}(gfg^{-1}) \neq \emptyset$ then $\mathrm{Fix}(f) = \mathrm{Fix}(gfg^{-1})$, the only possibility is that g interchange the fixed points of f, so $g(z_1) = z_2$ and $g(z_2) = z_1$ and hence

$$g^2(z_1) = g(g(z_1) = g(z_2) = z_1$$
$$g^2(z_2) = g(g(z_2) = g(z_1) = z_2.$$

This implies that g^2 fixes $z_1$and z_2 but that are not fixed by g. By using Proposition 2.1.8, g^2 and g fix the same points, so g has at least one fixed point z_0 different from z_1 and z_2. It is concluded that g^2 has at least three fixed points being z_1, z_2, and z_0. This implies that g^2 is the identity and hence g has order 2. It is contradicting to the assumption that g does not have order 2. Therefore $\mathrm{Fix}(f) \cap \mathrm{Fix}(gfg^{-1}) = \varnothing$ and the proof is completed. □

It is natural and interesting to ask if a group has a property when it has a subgroup with the property. The following lemma and theorem give the confirmative answers to discrete groups and Kleinian groups in the case of a subgroup with a finite index.

Lemma 3.2.3 *A group G is discrete if and only if it is a virtually discrete group.*

Proof. (1) If G is a discrete group then it is a virtually discrete group because that the discrete group G is the subgroup of G of order 1.

(2) Since G is a virtually discrete group, there is a discrete subgroup H of G of finite index. Suppose that G is not discrete, then there is a sequence $\{g_n\}$ of elements of G that converges to the identity Id :

$$\lim_{n\to\infty} g_n = Id.$$

As H has a finite index in G, one may assume

$$G = \phi_1 H \cup \phi_2 H \cup \cdots \cup \phi_n H,$$

where $\phi_1 = Id, \phi_{2,} \cdots, \phi_n \in G$ for some n. Thus, there exists at least one coset, say $\phi_i H$ for some $1 \leq i \leq n$, has a subsequence $\{g_{n_k}\}$ converges to Id. On the other hand side, $\phi_1, \phi_{2,} \cdots, \phi_n$ are homeomorphisms, $H, \phi_2 H, \cdots, \phi_n H$ are discrete and hence there does not exist any convergent sequence. It follows a contradiction. So G is a discrete group. □

Theorem 3.2.4 *Let G be a subgroup of* $\mathrm{Isom}^+(\mathbb{H}^3)$, *then G is a Kleinian group if and only if it is a virtually Kleinian group.*

Proof. The sufficiency is trivial because that the Kleinian group G is the subgroup of G of order 1. First show the necessity. Suppose that G is a virtually Kleinian group, so there is a subgroup H that is Kleinian with a finite index and hence H is discrete. One may assume

$$G = H \cup H_1 \cup \cdots \cup H_n,$$

where $H, H_1, \cdots, H_n$ are the all left cosets of G under H.

It follows from Lemma 3.2.3 that G is discrete. Suppose that G is elementary, then there exists a finite orbit of a point $x_0 \in \overline{\mathbb{H}^3}$, say $G(x_0)$, thus

$$G(x_0) = H(x_0) \cup H_1(x_0) \cup \cdots \cup H_n(x_0).$$

Therefore, H has a finite orbit $H(x_0)$. This contradicts that H is Kleinian. Thus, G is non-elementary and hence G is a Kleinian group. □

Notice that the sufficiency of the proof for the previous theorem is obtained by the trivial subgroup G. Do non-trivial subgroups of finite index in Kleinian groups remain Kleinian? It is confirmed in the following lemma.

Lemma 3.2.5 *Suppose that G is a Kleinian group. If H is a two-generator non-trivial subgroups of finite index in G, then H is a Kleinian group.*

Proof. Since G is a Kleinian group, according to Theorem 3.2.14, the two-generator subgroup H is discrete.

Claim that subgroup H of non-elementary group G is non-elementary as well and hence H is a Kleinian group.

Suppose that H is an elementary subgroup of G, then there exists a finite orbit $H(x_0)$ of a point $x_0 \in \overline{\mathbb{H}^3}$. Since H has a finite index in G, one may assume

$$G = H \cup g_1 H \cup g_2 H \cup \cdots \cup g_n H,$$

where $g_1, g_2, \cdots, g_n \in G$ for some $n \in \mathbb{N}$. Thus,

$$G(x_0) = H(x_0) \cup g_1 H(x_0) \cup g_2 H(x_0) \cup \cdots \cup g_n H(x_0).$$

Let $H(x_0) = \{x_1, x_2, \cdots, x_k\}$ some $k \in \mathbb{N}$, then

$$g_j H(x_0) = \{g_j(x_1), g_j(x_2), \cdots, g_j(x_k)\}$$

is finite for $j = 1, 2, \cdots, n$. It follows that $G(x_0)$ has a finite orbit of a point $x_0 \in \overline{\mathbb{H}^3}$ and hence G is an elementary group, it contradicts to G is a Kleinian group. $\square$

Theorem 3.2.6 *Suppose that $\langle f,\, g\rangle$ is a Kleinian group. If f is loxodromic or parabolic or elliptic of order $p \geq 6$, then $\langle f,\, gfg^{-1}\rangle$ is a Kleinian group.*

Proof. (a) Suppose that g is elliptic of order 2. By Lemma 3.2.1 $\langle f,\, gfg^{-1}\rangle$ has index 2 in $\langle f,\, g\rangle$. Thus applying for Lemma 3.2.5, Kleinian group $\langle f,\, g\rangle$ implies that $\langle f,\, gfg^{-1}\rangle$ is a Kleinian group.

(b) Suppose that the order of g is not 2. The Kleinian group $\langle f,\, g\rangle$ gives the following two facts. First, it is non-elementary and hence it is not abelian by Proposition 2.4.8. Second, applying for Theorem 3.2.14 $\langle f,\, gfg^{-1}\rangle$ is discrete. It needs to show that $\langle f,\, gfg^{-1}\rangle$ is non-elementary. Otherwise, if $\langle f,\, gfg^{-1}\rangle$ is elementary, then there are three types by Theorem 2.4.6:

(i) Suppose $\langle f,\, gfg^{-1}\rangle$ is an elementary group of type I, each non-trivial element of G is elliptic. Since the order of f is not $2, 3, 4$ or 5, the order of gfg^{-1} is not $2, 3, 4$ or 5, and hence, by using the Tables 1, 2 and 3, $\langle f,\, gfg^{-1}\rangle$ is not a finite spherical triangle group A_4, S_4, and A_5 or the dihedral groups D_3, D_4, and D_5. Suppose $\langle f,\, gfg^{-1}\rangle$ is a finite cyclic group. By Lemma 2.4.11, $\gamma(f, gfg^{-1}) = 0$, and then Theorem 2.3.12 gives $\mathrm{Fix}(f) \cap \mathrm{Fix}(gfg^{-1}) \neq \emptyset$, but this contradicts to Lemma 3.2.2.

(ii) Suppose $\langle f,\, gfg^{-1}\rangle$ is an elementary group of type II. By Corollary 2.4.7, $\langle f,\, gfg^{-1}\rangle$ is conjugate to a subgroup of Möb($\overline{\mathbb{C}}$) fixing ∞ whose every element is parabolic of the form $az+b$ $(a \neq 0 \in \mathbb{C})$. Thus, the group $\langle f,\, gfg^{-1}\rangle$ is abelian and hence $g\,(\mathrm{Fix}(f)) = \mathrm{Fix}(gfg^{-1}) = \mathrm{Fix}(f)$, it is a contradiction to Kleinian group $\langle f,\, g\rangle$.

(iii) Suppose $\langle f,\, gfg^{-1}\rangle$ is an elementary group of type III. Then both f and gfg^{-1} are elliptic or both are loxodromic. In either case $\langle f,\, gfg^{-1}\rangle$ is abelian and as above this is a contradiction. It is now shown that $\langle f,\, gfg^{-1}\rangle$ cannot be elementary if g does not have order 2. Hence in all cases $\langle f,\, gfg^{-1}\rangle$ is a Kleinian group. □

In a Kleinian group $\langle f,\, g\rangle$, if f is non-parabolic, then so is $h = gfg^{-1}$ and hence the axes $axis(f)$ and $axis(h)$ exist. Thus, there are two elliptic elements ϕ and ψ of order 2 such that $\varphi f\phi^{-1} = h$ and $\psi f\psi^{-1} = h^{-1}$. Actually, ϕ and ψ have their axes $axis(\phi)$ and $axis(\psi)$ as the fixed point sets at right-angles to one and other and also their axes bisect the common perpendicular between the axes $axis(f)$ and $axis(gfg^{-1})$ (see [15]). This is a useful tool that plays an important role in this chapter. For convenience, the tool is summarized in the following lemma.

Lemma 3.2.7 *Let $\langle f,\, g\rangle$ be a Kleinian group and let f be a non-parabolic element. Then there are two elliptic conjugators ϕ and ψ of order 2 such that ϕ acts on f via conjugating f by g and ψ acts on f via inverting f and then conjugating by g :*

$$\phi f\phi^{-1} = gfg^{-1} \text{ and } \psi f\psi^{-1} = gf^{-1}g^{-1},$$

where ϕ and ψ have their axes $axis(\phi)$ and $axis(\psi)$ as the fixed point sets at right-angles to one and other and also their axes bisect the common perpendicular between the axes $axis(f)$ and $axis(gfg^{-1})$.

Lemma 3.2.8 *Let $\langle f,\ g\rangle$ be a Kleinian group with the triple of parameters (γ, β, β'), where f is non-parabolic. Then there are two elliptic ϕ and ψ of order 2 such that $\Gamma^{\phi} = \langle f, \phi\rangle$ and $\Gamma^{\psi} = \langle f, \psi\rangle$ are discrete groups containing $\langle f,\ gfg^{-1}\rangle$ with index 2 and their triples of parameters are $(\gamma, \beta, -4)$ and $(\beta - \gamma, \beta, -4)$, respectively.*

Proof. Let $\langle f,\ g\rangle$ be a Kleinian group with parameters (γ, β, β').

Since f is either elliptic or loxodromic, so there exists $axis(f)$ in $\mathbb{H}^3$. Let $h = gfg^{-1}$ then $h^{-1} = gf^{-1}g^{-1}$ and h has the same trace as f and hence $axis(h)$ exists in $\mathbb{H}^3$. By Lemma 3.2.7, there are two elliptic ϕ and ψ of order 2 such that $\phi f \phi^{-1} = h$ and $\psi f \psi^{-1} = h^{-1}$.

First, claiming that two groups $\Gamma^{\phi} = \langle f, \phi\rangle$ and $\Gamma^{\psi} = \langle f, \psi\rangle$ are discrete. Let $\Gamma = \langle f, h\rangle$, then

$$\Gamma = \langle f, h\rangle = \left\langle f, \phi f \phi^{-1}\right\rangle,$$
$$\Gamma = \langle f, h\rangle = \left\langle f, h^{-1}\right\rangle = \left\langle f, \psi f \psi^{-1}\right\rangle.$$

It follows that Γ is a subgroup of each Γ^{φ} and Γ^{ψ}. By Lemma 3.2.1, Γ is of index two in Γ^{ϕ} and

$$\Gamma^{\phi} = \Gamma \cup \phi\Gamma. \tag{3.6}$$

Since $\langle f,\ g\rangle$ is a Kleinian group, by Theorem 3.2.14, the subgroup Γ of $\langle f,\ g\rangle$ is discrete and hence $\phi\Gamma$ is discrete as ϕ is is an homeomorphism. It follows from (3.6) that Γ^{ϕ} is discrete. Similarly, Γ^{ψ} is discrete.

Second, claiming that the parameters for Γ^{ϕ} and Γ^{ψ} are one $(\gamma, \beta, -4)$ and the other one $(\beta - \gamma, \beta, -4)$.

The parameters for Γ^{ϕ} are $(\gamma_1, \beta, -4)$ and those of Γ^{ψ} are $(\gamma_2, \beta, -4)$. Now using the identity (3.4) and Proposition 2.3.9 imply that

$$\gamma(\gamma-\beta)=\gamma(f,h)=\gamma(f,\phi f\phi^{-1})=\gamma_1(\gamma_1-\beta),$$
$$\gamma(\gamma-\beta)=\gamma(f,h)=\gamma(f,h^{-1})=\gamma(f,\psi f\psi^{-1})=\gamma_2(\gamma_2-\beta).$$

It gives the following two quadratic equations

$$\gamma_1^2-\beta\gamma_1-\gamma(\gamma-\beta)=0$$
$$\gamma_2^2-\beta\gamma_2-\gamma(\gamma-\beta)=0.$$

Solving the first quadratic equation: $\gamma_1=\frac{\beta\pm\sqrt{\beta^2+4\gamma(\gamma-\beta)}}{2}=\frac{\beta\pm(\beta-2\gamma)}{2}=\gamma$, or $\beta-\gamma$.

Similarly, by the second quadratic equation: $\gamma_2=\gamma$, or $\beta-\gamma$. That is, $\{\gamma_1,\gamma_2\}=\{\gamma,\beta-\gamma\}$ as (2.19) shows both possibilities occur.

Thus, after relabeling, the parameters for Γ^ϕ are $(\gamma,\beta,-4)$ and those of Γ^ψ are $(\beta-\gamma,\beta,-4)$.

By Lemma 3.2.7, $\phi f\phi^{-1}=gfg^{-1}$ and $\psi f\psi^{-1}=gf^{-1}g^{-1}$, so

$$\langle f,\, gfg^{-1}\rangle=\langle f,\, \phi f\phi^{-1}\rangle=\langle f,\, \psi f\psi^{-1}\rangle.$$

Since ϕ and ψ are elliptic of order 2, by Lemma 3.2.1, $\langle f,\, gfg^{-1}\rangle$ is subgroup of each Γ^ϕ and Γ^ψ with index 2. This completes the proof. □

Remark that $\beta-\gamma\neq 0$ in Lemma 3.2.8. Otherwise, it is a dihedral group by Lemma 2.4.12. It contradicts to Kleinian group $\langle f,\, g\rangle$.

Now the natural question is to determine whether the two discrete groups Γ^ϕ and Γ^ψ produced by Lemma 3.2.8 are actually Kleinian. Since they contain $\langle f,\, gfg^{-1}\rangle$ that is of index 2, it only needs to decide if $\langle f,\, gfg^{-1}\rangle$ is Kleinian. By Theorem 3.2.4, if $\langle f,\, gfg^{-1}\rangle$ is a Kleinian group then the following corollary is obtained immediately that Γ^ϕ and Γ^ψ are Kleinian groups.

Corollary 3.2.9 *Suppose that* $\langle f,\, g\rangle$ *is a Kleinian group. If that* f *is loxodromic or parabolic or elliptic of order* $p \geq 6$*, then the groups* $\Gamma^{\phi} = \langle f, \phi\rangle$ *and* $\Gamma^{\psi} = \langle f, \psi\rangle$ *produced in Lemma 3.2.8 are Kleinian groups.*

Lemma 3.2.10 *Let* $\langle f,\, g\rangle$ *be a Kleinian group with* f *not of order* 2*. Suppose that the triple of parameters* $(\gamma(f,\, g), \beta(f), -4)$ *is not one of those exceptional groups listed in Table* 1*. Then the subgroup* $\langle f,\, gfg^{-1}\rangle$ *is Kleinian.*

Proof. Applying for Theorem 3.2.14 the subgroup $\langle f,\, gfg^{-1}\rangle$ of Kleinian group $\langle f,\, g\rangle$ is discrete. It needs to show that $\langle f,\, gfg^{-1}\rangle$ can not be elementary. Let $\gamma = \gamma(f,\, g)$ and $\beta = \beta(f)$. By Lemma 3.2.8 there is an elliptic ϕ of order 2 such that $\Gamma^{\phi} = \langle f, \phi\rangle$ is discrete group containing $\langle f,\, gfg^{-1}\rangle$ with index 2 and the triple of parameters for Γ^{ϕ} is $(\gamma, \beta, -4)$. By the hypothesis, $(\gamma, \beta, -4)$ is not one of those exceptional groups listed in Table 1, then Γ^{ϕ} is not a finite spherical triangle group A_4, S_4, and A_5. Thus, $\langle f,\, gfg^{-1}\rangle$ can only be elementary if $\langle f,\, gfg^{-1}\rangle = \gamma(\gamma - \beta) = 0$, i.e., $\gamma = 0$ or $\gamma = \beta$. Since $\langle f,\, g\rangle$ is Kleinian, $\gamma \neq 0$. So it can only be $\gamma = \beta$ and hence it is the dihedral group. In this case, since f is not of order two,

$$gfg^{-1} = f^{\pm 1}.$$

In the case $gfg^{-1} = f$, it gives $g(\mathrm{Fix}(f)) = \mathrm{Fix}(gfg^{-1}) = \mathrm{Fix}(f)$ and hence g is elliptic of order 2 (Lemma 3.2.2) fixing or interchanging the fixed points of f in $\overline{\mathbb{C}}$. In the case $gfg^{-1} = f^{-1}$, g might be a power of f. In either case $\langle f,\, g\rangle$ is not Kleinian, contradiction. $\square$

It has been showed in Example 3.1.3 that each Kleinian group given by the hyperbolic triangle group $\Delta(2,3,p) = \left\langle f, g : f^p = g^2 = (fg)^3 = Id,\ p \geq 7\right\rangle$ has the triple of parameters $(1+\beta, \beta, -4)$, where $\beta = \beta(f)$. Thus, Lemma 3.2.1 gives the triple of parameters $(1+\beta, \beta, \beta)$ for the subgroup $\langle f,\, gfg^{-1}\rangle$.

Theorem 3.2.11 *Let* $\Gamma = \langle f,\, g\rangle$ *be a Kleinian group with the parameters* $(1+\beta,\beta,\beta)$*. Then there are two elliptics elements of orders* 3 *and* 2 *generating* Γ.

Proof. Let $u = fg^{-1}$and $v = fg^{-2}$, then $f = uv^{-1}u$ and $g = v^{-1}u$. Thus,

$$\Gamma = \langle f,\, g\rangle = \langle u, v\rangle\,.$$

Now show that u and v are elliptics elements of orders 3 and 2, respectively.

By hypothesis the parameters $(\gamma\,(f,\,g)\,,\beta\,(f)\,,\beta\,(g)) = (1+\beta,\beta,\beta)$, the following equations are obtained,

$$\gamma\,(f,\,g) = 1+\beta\,(f)\ \text{ and }\beta\,(f) = \beta\,(g)\,,$$

i.e., $\mathrm{tr}[f,g]-2 = 1+\mathrm{tr}^2(f)\ -4$ and $\mathrm{tr}^2(f)\ -4 = \mathrm{tr}^2(g)-4$. Thus,

$$\mathrm{tr}[f,g] = \mathrm{tr}^2(f)-1, \tag{3.7}$$

and $\mathrm{tr}^2(f) = \mathrm{tr}^2(g)$. One may assume that $\mathrm{tr}(f) = \mathrm{tr}(g)$ by replacing f by $-f$ if necessary. Thus, the identity (2.3) becomes

$$\mathrm{tr}(fg)+\mathrm{tr}(fg^{-1}) = \mathrm{tr}^2(f). \tag{3.8}$$

Applying for the Friche's identity (3.7),

$$\begin{aligned}\mathrm{tr}^2(f)-1 &= \mathrm{tr}[f,g]\\ &= \mathrm{tr}^2(f)+\mathrm{tr}^2(g)+\mathrm{tr}^2(fg)-\mathrm{tr}(f)\mathrm{tr}(g)\mathrm{tr}(fg)-2\\ &= 2\mathrm{tr}^2(f)+\mathrm{tr}^2(fg)-\mathrm{tr}^2(f)\mathrm{tr}(fg)-2.\end{aligned}$$

Rearranging and using the identity (3.8),

$$\begin{aligned} 1 &= \mathrm{tr}^2(f) + \mathrm{tr}^2(fg) - \mathrm{tr}^2(f)\mathrm{tr}(fg) \\ &= \mathrm{tr}^2(f) + \mathrm{tr}^2(fg) - \left(\mathrm{tr}(fg) + \mathrm{tr}(fg^{-1})\right)\mathrm{tr}(fg) \\ &= \mathrm{tr}^2(f) - \mathrm{tr}(fg^{-1})\mathrm{tr}(fg), \end{aligned}$$

which gives

$$\mathrm{tr}(fg)\mathrm{tr}(fg^{-1}) = \mathrm{tr}^2(f) - 1. \tag{3.9}$$

Using the above identities (3.9) and (3.8),

$$\begin{aligned} \left(\mathrm{tr}(fg) - \mathrm{tr}(fg^{-1})\right)^2 &= \left(\mathrm{tr}(fg) + \mathrm{tr}(fg^{-1})\right)^2 - 4\mathrm{tr}(fg^{-1})\mathrm{tr}(fg) \\ &= \mathrm{tr}^4(f) - 4\mathrm{tr}^2(f) + 4 \\ &= \left(\mathrm{tr}^2(f) - 2\right)^2. \end{aligned}$$

By replacing g by $-g$ if necessary, one may assume that

$$\mathrm{tr}(fg) - \mathrm{tr}(fg^{-1}) = \mathrm{tr}^2(f) - 2. \tag{3.10}$$

Using the equations (3.8) and (3.10),

$$\mathrm{tr}(fg^{-1}) = 1 \tag{3.11}$$

and hence $\beta\left(fg^{-1}\right) = -3$, so $u = fg^{-1}$ is elliptic of order 3.

Finally, using the identities (2.3), (2.2), and (3.11):

$$\begin{aligned}
\mathrm{tr}(fg^{-2}) &= \mathrm{tr}(fg^{-1}g^{-1}) \\
&= \mathrm{tr}(fg^{-1})\mathrm{tr}(g^{-1}) - \mathrm{tr}(fg^{-1}g) \\
&= \mathrm{tr}(fg^{-1})\mathrm{tr}(g) - \mathrm{tr}(f) \\
&= \mathrm{tr}(fg^{-1})\mathrm{tr}(g) - \mathrm{tr}(g) \\
&= \mathrm{tr}(g)\left(\mathrm{tr}(fg^{-1}) - 1\right) = 0,
\end{aligned}$$

which gives $tr(fg^{-2}) = 0$ and hence $\beta\left(fg^{-2}\right) = -4$, therefore, $v = fg^{-2}$ is elliptic of order 2.

By Proposition 2.3.9 (d), $\gamma\left(u, v\right) = \gamma\left(u, v^{-1}\right)$

$$\begin{aligned}
\gamma\left(u, v^{-1}\right) &= \mathrm{tr}\left(uv^{-1}u^{-1}v\right) - 2 \\
&= \mathrm{tr}\left(fg^{-1}g^{2}fgf^{-1}fg^{-2}\right) - 2 \\
&= \mathrm{tr}\left(fgf^{-1}g^{-1}\right) - 2 \\
&= \gamma\left(f,\, g\right).
\end{aligned}$$

Hence, $\gamma\left(u, v^{-1}\right) = 1 + \beta$. □

Now turn the attention to “algebraic convergence” for n-generator groups. However two-generator groups are mainly concerned in this dissertation.

Definition 3.2.12 *One say a sequence* Γ_j *of subgroups of* Möb($\overline{\mathbb{C}}$) converges algebraically *to a subgroup* Γ *of* Möb($\overline{\mathbb{C}}$) *provided that each* Γ_j *may be expressed as* $\langle f_{j,1}, f_{j,2}, \cdots, f_{j,n}\rangle$*, that* Γ *may be expressed as* $\langle f_1, f_2, \cdots, f_n\rangle$*, and that for each* $k = 1, 2, \cdots, n$ *the sequence* $f_{j,k}$ *converges uniformly to* f_k *in the spherical metric of* $\overline{\mathbb{C}}$.

A fundamental result concerning spaces of finitely generated Kleinian groups is that they are closed in the topology of algebraic convergence due to

Jørgensen. Recall the result as the following Jørgensen Theorem. Here note the fact that if instead of a sequence $\{\Gamma_j\}_{j=1}^{\infty}$ of Kleinian groups it is a continuous family $\{\Gamma_t\}_{t\in[0,1]}$, then all the groups are in fact isomorphic and this isomorphism is induced by the map back.

Theorem 3.2.13 (Jørgensen) *The space of n-generator Kleinian groups is closed in the topology of algebraic convergence. Equivalently, if a sequence of n-generator Kleinian subgroups $\Gamma_j = \langle f_{j,1}, f_{j,2}, \cdots, f_{j,n}\rangle$ converges algebraically to a n-generator subgroup $\Gamma = \langle f_1, f_2, \cdots, f_n\rangle$ in* $\mathrm{M\ddot{o}b}(\overline{\mathbb{C}})$ *then Γ is a Kleinian group.*

Moreover, the map back is an eventual homomorphism. That is for all sufficiently large j the map $\Gamma \to \Gamma_j$ defined by $f_k \longmapsto f_{j,k}$ extends to a homomorphism of the groups.

Typically the proof of the above Jørgensen Theorem is one of the first applications of Jørgensen's inequality (see Section 3.4). Another important application of Jørgensen inequality is the characterization of a Kleinian group by two-generator subgroups that is recalled from Martin [41] in the following theorem.

Theorem 3.2.14 *A subgroup G of* $\mathrm{Isom}^+(\mathbb{H}^3)$ *is Kleinian if and only if every two-generator subgroup of G is discrete.*

It is going to extend the closedness to the set of triples of complex parameters of Kleinian two-generator in the following important theorem. One can describe the space of Kleinian groups generated by two generators f and $g \in G$ (up to conjugacy) as a subset of the three complex dimensional space $\mathbb{C}^3$ via the map

$$\langle f, g\rangle \longmapsto (\gamma(f, g), \beta(f), \beta(g)).$$

Theorem 3.2.15 *Let subset $\mathcal{D}$ be defined by*

$$\mathcal{D} = \left\{(\gamma, \beta, \beta') \in \mathbb{C}^3 : (\gamma, \beta, \beta') \text{ are the parameters of a Kleinian group } \langle f,\, g\rangle\right\},$$

then $\mathcal{D}$ is closed in the three complex dimensional space $\mathbb{C}^3$ in the usual topology. Equivalently, let $(\gamma_j, \beta_j, \beta'_j)$ be a sequence of parameters for two-generator Kleinian groups $\langle f_j, g_j\rangle$. If $(\gamma_j, \beta_j, \beta'_j) \to (\gamma, \beta, \beta')$, then (γ, β, β') is a triple of parameters for a two-generator Kleinian group.

Proof. The proof is broken down into two cases. For each case, it needs to construct a sequence of two-generator Kleinian groups $\langle f_j, g_j\rangle$ with the parameters $(\gamma_j, \beta_j, \beta'_j)$ and then, by Theorem 3.2.13, the algebraic convergence limit group $\langle f,\, g\rangle$ is a Kleinian group. It will be shown that the limit triple (γ, β, β') is the parameters for the limit group $\langle f,\, g\rangle$

(a) Suppose $(\beta, \beta') \neq (0, 0)$. One may assume $\beta \neq 0$, then $\beta_j \neq 0$, for all but finitely many j. For the sequence of parameters $(\gamma_j, \beta_j, \beta'_j)$, by Theorem 2.3.2, there is a sequence of Kleinian groups $\langle f_j, g_j\rangle$ up to conjugacy for all but finitely many f_j are non-parabolic elements. Passing to a subsequence if necessary, recall $\langle f_j, g_j\rangle$, for all f_j are non-parabolic elements, and $\beta_j \neq 0$ for all j. Thus, applying for Theorem 2.1.17, one may assume the following with $\lambda_j \neq 0, \pm 1$:

$$f_j = \begin{pmatrix} \lambda_j & 0 \\ 0 & \frac{1}{\lambda_j} \end{pmatrix} \text{ and } g_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C}),$$

such that $(\gamma_j, \beta_j, \beta'_j)$ is a triple of parameters for $\langle f_j, g_j\rangle$. Further, $b_j \neq 0$ and $c_j \neq 0$ for all j, otherwise f_j and g_j share a common fixed point and $\langle f_j, g_j\rangle$ is elementary, if it is discrete.

First, by Definition 2.3.1,

$$\beta_j = (\lambda_j + \frac{1}{\lambda_j})^2 - 4 = (\lambda_j - \frac{1}{\lambda_j})^2,$$

which gives $\lambda_j^2 - \sqrt{\beta_j}\lambda_j - 1 = 0$. Solving the quadratic equation gives

$$\lambda_j = \frac{\sqrt{\beta_j} \pm \sqrt{\beta_j + 4}}{2}.$$

Since β_j converges to β, certainly λ_j converges to $\lambda = \frac{\sqrt{\beta} \pm \sqrt{\beta+4}}{2}$ and hence $\frac{1}{\lambda_j}$ converges to $\frac{1}{\lambda}$. Thus,

$$f_j \to f = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \text{ and } \beta_j \to \beta = (\lambda - \frac{1}{\lambda})^2. \tag{3.12}$$

Second, consider the conjugacy of group $\langle f_j, g_j \rangle$ by a diagonal matrix conjugator

$$h_j = \begin{pmatrix} \mu_j & 0 \\ 0 & \frac{1}{\mu_j} \end{pmatrix}.$$

Since h_j commutes with f_j and with f, f_j and f are conjugate to themselves. The conjugacy of g_j is also given,

$$h_j g_j h_j^{-1} = \begin{pmatrix} \mu_j & 0 \\ 0 & \frac{1}{\mu_j} \end{pmatrix} \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \begin{pmatrix} \frac{1}{\mu_j} & 0 \\ 0 & \mu_j \end{pmatrix} = \begin{pmatrix} a_j & b_j\mu_j^2 \\ c_j\mu_j^{-2} & d_j \end{pmatrix}.$$

Now choose $\mu_j = \frac{1}{\sqrt{b_j}}$, then $b_j\mu_j^2 = 1$ and $c_j\mu^{-2} = c_j b_j$. One assume recall that

$$g_j = \begin{pmatrix} a_j & 1 \\ c_j & d_j \end{pmatrix}.$$

Then, by Definition 2.3.1,

$$\beta_j' + 4 = (a_j + d_j)^2. \tag{3.13}$$

Computing the commutator,

$$[f_j, g_j] = f_j g_j f_j^{-1} g_j^{-1} = \begin{pmatrix} a_j d_j - c_j \lambda_j^2 & -a_j + a_j \lambda_j^2 \\ \frac{c_j d_j}{\lambda_j^2} - c_j d_j & -\frac{c_j}{\lambda_j^2} + a_j d_j \end{pmatrix},$$

and hence the parameter is

$$\begin{aligned} \gamma_j &= \operatorname{tr}[f_j, g_j] - 2 \\ &= a_j d_j - c_j \lambda_j^2 - \frac{c_j}{\lambda_j^2} + a_j d_j - 2 \\ &= -c_j \left(\lambda_j^2 + \frac{1}{\lambda_j^2} \right) + 2(1 + c_j) - 2 \\ &= -(\lambda_j - \frac{1}{\lambda_j})^2 c_j \\ &= -\beta_j c_j. \end{aligned}$$

Finally, since $c_j = \frac{\gamma_j}{-\beta_j}$, $\gamma_j = -\beta_j c_j \to \gamma$, and $\beta_j \to \beta \neq 0$,

$$c_j \to c, \quad \text{for some } c \in \mathbb{C}.$$

Notice that $\det(g_j) = a_j d_j - c_j = 1$, gives the product $a_j d_j$ is convergent and hence bounded

$$a_j d_j = 1 + c_j \to 1 + c.$$

Since the sequence β_j' is convergent, β_j' is bounded. The identity (3.13) gives the sum $a_j + d_j = \sqrt{\beta_j' + 4}$ is bounded. Let the product $a_j d_j = s_j$ and the sum $a_j + d_j = t_j$, then $a_j = \frac{s_j \pm \sqrt{s_j^2 - 4t_j}}{2}$ and $d_j = s_j - \frac{s_j \pm \sqrt{s_j^2 - 4t_j}}{2}$, so a_j and d_j are in terms of sum $a_j + d_j$ and product $a_j d_j$ by the continuous operations a_j and d_j are bounded. So a_j and d_j admit convergent subsequences, say the limits are a

and d, respectively. It follows that g_j is convergent to $g = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$, passing to a corresponding subsequence if necessary.

The calculation of parameter γ :

$$\begin{aligned}\gamma &= \operatorname{tr}[f, g] - 2 \\ &= ad - c\lambda^2 - \frac{c}{\lambda^2} + ad - 2 \\ &= -c\left(\lambda^2 + \frac{1}{\lambda^2}\right) + 2(1+c) - 2 \\ &= -(\lambda - \frac{1}{\lambda})^2 c \\ &= -\beta c.\end{aligned}$$

In summary of the part (a), by the identities (3.12), (3.13).

$$\begin{aligned}&\beta_j \to \beta = (\lambda - \frac{1}{\lambda})^2 \\ &\beta_j' = (a_j + d_j)^2 - 4 \to (a+d)^2 - 4 = \beta'. \\ &\gamma_j = -\beta_j c_j \to -\beta c = \gamma.\end{aligned}$$

Therefore, the triple of parameters of $\langle f, g\rangle$ is (γ, β, β'), and hence $\langle f, g\rangle$ is Kleinian by Theorem 3.2.13.

(b) Suppose that $(\beta, \beta') = (0, 0)$. For each f_j and g_j has at least one fixed point, to avoid common fixed points, say f_j fixes z_1 and g_j fixes z_2 with $z_1 \neq z_2$. Let $z_3 \in \overline{\mathbb{C}} - \{z_1, z_2\}$. Since Möb$\left(\overline{\mathbb{C}}\right)$ acts transitively on triples of points in $\overline{\mathbb{C}}$, there exists $h \in$ Möb$\left(\overline{\mathbb{C}}\right)$ taking (z_1, z_2, z_3) to $(\infty, 0, 1)$. Thus, hf_jh^{-1} fixes $h(z_1) = \infty$ and hg_jh^{-1} fixes $h(z_2) = 0$. Recall hf_jh^{-1} and hg_jh^{-1} as f_j and g_j, respectively. So one may assume by conjugacy that for each j a fixed point of each f_j is ∞ and that a fixed point of each g_j is 0.

Then, by Corollary 2.1.16, choose f_j and g_j as following,

$$f_j = \begin{pmatrix} \lambda_j & a_j \\ 0 & \frac{1}{\lambda_j} \end{pmatrix} \text{ and } g_j = \begin{pmatrix} \mu_j & 0 \\ b_j & \frac{1}{\mu_j} \end{pmatrix},$$

where $\lambda_j \to 1, \mu_j \to 1$, and $b_j \to b$, with $a_j \neq 0$ and $b_j \neq 0$, otherwise f_j and g_j would share a fixed point and would thus generate an elementary group.

Notice that $f_j(z) = \lambda_j^2 z + \lambda_j a_j$, one may normalize that $f_j(0) = \lambda_j a_j = 1$, so $a_j = \frac{1}{\lambda_j} \to 1$. It follows that

$$f_j \to f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } g_j \to g = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$$

and hence

$$\begin{aligned} &\beta(f) = \beta(g) = 0, \\ &\beta(f_j) = \left(\lambda_j + \frac{1}{\lambda_j}\right)^2 - 4 = \left(\lambda_j - \frac{1}{\lambda_j}\right)^2 \to 0 = \beta(f), \text{ as } \lambda_j \to 1 \\ &\beta(g_j) = \left(\mu_j + \frac{1}{\mu_j}\right)^2 - 4 = \left(\mu_j - \frac{1}{\mu_j}\right)^2 \to 0 = \beta(g), \text{ as } \mu_j \to 1. \end{aligned}$$

Then the parameters of commutators can be computed by using Proposition 2.3.9,

$$\begin{aligned}
\gamma = \gamma(f,\, g) &= \beta(f) + \beta(g) + \beta(fg) - \mathrm{tr}(f)\mathrm{tr}(g)\mathrm{tr}(fg) + 8 \\
&= b^2 + 4b - 4b - 8 + 8 = b^2. \\
\gamma_i &= \beta(f_i) + \beta(g_i) + \beta(f_i g_i) - \mathrm{tr}(f_i)\mathrm{tr}(g_i)\mathrm{tr}(f_i g_i) + 8 \\
&= \frac{\left(\lambda_i^2+1\right)^2}{\lambda_i^2} + \frac{(\mu_i^2+1)^2}{\mu_i^2} + \frac{(\lambda_i\mu_i\,(\lambda_i\mu_i + a_i b_i) + 1)^2}{\lambda_i^2\mu_i^2} \\
&- \frac{\left(\lambda_i^2+1\right)(\mu_i^2+1)\,(\lambda_i\mu_i\,(\lambda_i\mu_i + a_i b_i) + 1)}{\lambda_i^2\mu_i^2} - 4 \\
&= \frac{a_i b_i}{\lambda_i\mu_i}\left(\lambda_i^2\mu_i^2 - \lambda_i^2 + a_i b_i \lambda_i \mu_i - \mu_i^2 + 1\right) \\
&= a_i b_i \left(\frac{(\mu_i^2-1)(\lambda_i^2-1)}{\lambda_i\mu_i} + a_i b_i\right) \to b^2 = \gamma, \text{ as } \lambda_i \to 1, \mu_i \to 1, \text{and } b_i \to b.
\end{aligned}$$

So it can be deduced that $\gamma_i \to \gamma$ and the result follows exactly as before. By Theorem 3.2.13 the triple of parameters of $\langle f,\, g\rangle$ is (γ, β, β'), and hence $\langle f,\, g\rangle$ is Kleinian. □

3.3 Projections of Kleinian groups

Every two-generator Kleinian group can be represented by a triple of complex parameters $(\gamma(f,\, g), \beta(f), \beta(g))$. Those who familiar with Jørgensen's inequality for Kleinian groups $\langle f,\, g\rangle$ know that it involves only two parameters $\gamma(f,\, g)$ and $\beta(f)$ of the triple of complex parameters. It can not immediately follow from Theorem 3.2.15 that the subspace $\mathcal{D}_2$ of pairs $(\gamma(f,\, g), \beta(f))$ of the first two complex parameters of a Kleinian group is closed in two complex dimensional space $\mathbb{C}^2$.

The current approach is to consider two projections: one is from the subspace $\mathcal{D}$ in three complex dimensional space $\mathbb{C}^3$ to the subspace $\mathcal{D}_2$ in two complex dimensional $\mathbb{C}^2$ and the other is from $\mathcal{D}$ to the subspace on the slice $z_3 = -4$ in $\mathbb{C}^3$ (see Proposition 3.3.1 and Theorem 3.3.2). Then show that the set $\mathcal{D}_2$ is a closed subspace of $\mathbb{C}^2$ in the usual topology in Theorem 3.3.4 by two different proofs. That $\mathcal{D}_2$ is a closed subspace in $\mathbb{C}^2$ is an essential result for the approach to establish the inequalities in the scheme.

Consider the projection from the subspace $\mathcal{D}$ in three complex dimensional space $\mathbb{C}^3 = \{(z_1, z_2, z_3) : z_1, z_2, z_3 \in \mathbb{C}\}$ to the subspace $\mathcal{D}^*$ on the slice $z_3 = -4$ of the space $\mathbb{C}^3$:

$$\mathcal{D}^* = \{(\gamma, \beta, -4) : \text{all these triples for two-generator Kleinian groups}\}.$$

The following proposition shows directly that $\mathcal{D}^*$ is closed in $\mathbb{C}^3$ in the usual topology.

Proposition 3.3.1 *Let $\{\Gamma_j\}$ be a sequence of two-generator Kleinian groups such that the corresponding sequence of parameters $\{(\gamma_j, \beta_j, -4)\}$ converges to $(\gamma, \beta, -4)$. Then, up to conjugation and subsequence, $\{\Gamma_j\}$ converge algebraically to a group Γ with parameters $(\gamma, \beta, -4)$.*

Proof. The approach is to show that one can find a sequence of pairs of two-generators $\{(f_j, g_j)\}$ and a pair of two-generators (f, g) such that $f_j \to f$, $g_j \to g$, $\Gamma = \langle f,\, g\rangle$, and $\Gamma_j = \left\langle f_j, g_j\right\rangle$ with the triple of parameters $(\gamma_j, \beta_j, -4)$ for each j. The fact that the limit $(\gamma, \beta, -4)$ is the triple of parameters of $\langle f,\, g\rangle$ follows immediately by construction.

First, it needs to show that the generators for the groups $\Gamma_j = \left\langle f_j, g_j\right\rangle$ converge to $\Gamma = \langle f,\, g\rangle$. It is proceed by considering two cases: up to subsequence, f_j is parabolic for all j or not.

(a) Suppose f_j is parabolic for all j. By Theorem 2.1.17, conjugate each Γ_j so that the first generator is now represented by the matrix

$$f_j = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then f_j is constant and thus converges to f, say.

$$f_j \to f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \beta_j \to \beta = 0 = \beta(f).$$

Recall the resulting group $\Gamma_j = \langle f_j, g_j \rangle$ also, it remains to show that the sequence $\{g_j\}$ also converges. Suppose the matrix for second generator is

$$g_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}.$$

Since $\beta'_j = \beta(g_j) = \mathrm{tr}^2(g_j) - 4 = -4$, $\mathrm{tr}(g_j) = 0$ and hence $a_j + d_j = 0$, i.e.,

$$d_j = -a_j.$$

Also, since the group Γ_j is Kleinian, $c_j \neq 0$, otherwise f_j and g_j would share a fixed point and would thus generate an elementary group.

Finally, as $a_j d_j - b_j c_j = 1$, $a_j d_j - 1 = b_j c_j$. Then

$$b_j = \frac{a_j d_j - 1}{c_j} = -\frac{a_j^2 + 1}{c_j}.$$

In total,

$$g_j = \begin{pmatrix} a_j & -\frac{a_j^2+1}{c_j} \\ c_j & -a_j \end{pmatrix}.$$

Consider the conjugacy of the group $\langle f_j, g_j \rangle$ by the following form of upper triangular matrix conjugator

$$h_j = \begin{pmatrix} 1 & s_j \\ 0 & 1 \end{pmatrix}.$$

Since f_j commutes with h_j, f_j is left unchanged under conjugation. It is expected that the conjugacy of g_j becomes

$$h_j g_j h_j^{-1} = \begin{pmatrix} * & 1 \\ c_j & -* \end{pmatrix}.$$

Set $s_j = \frac{-a_j + i\sqrt{1+c_j}}{c_j}$, then

$$\begin{aligned} h_j &= \begin{pmatrix} 1 & \frac{-a_j+i\sqrt{1+c_j}}{c_j} \\ 0 & 1 \end{pmatrix}, \\ h_j g_j h_j^{-1} &= \begin{pmatrix} 1 & \frac{-a_j+i\sqrt{1+c_j}}{c_j} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_j & -\frac{a_j^2+1}{c_j} \\ c_j & -a_j \end{pmatrix} \begin{pmatrix} 1 & \frac{a_j - i\sqrt{c_j+1}}{c_j} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} i\sqrt{1+c_j} & 1 \\ c_i & -i\sqrt{1+c_j} \end{pmatrix}. \end{aligned}$$

Without loss of generality, recall the resulting group as $\Gamma_j = \langle f_j, g_j \rangle$. Thus, the commutator is

$$[f_j, g_j] = f_j g_j f_j^{-1} g_j^{-1} = \begin{pmatrix} c_j i\sqrt{1+c_j} + c_j^2 + 1 & 2 - ic_j\sqrt{1+c_j} + c_j \\ c_j^2 & -c_j i\sqrt{1+c_j} + 1 \end{pmatrix}.$$

Hence the parameter γ_j is

$$\begin{aligned} \gamma_j &= \operatorname{tr}([f_j, g_j]) - 2 \\ &= c_j i\sqrt{1+c_j} + c_j^2 + 2 - c_j i\sqrt{1+c_j} - 2 \\ &= c_j^2. \end{aligned}$$

Since γ_j converges to γ, c_j^2 is convergent. So $\left\{c_j^2\right\}$ is bounded and hence $\{c_j\}$ is bounded. It follows that c_j converges to a complex number, up to a subsequence, say c, and hence $\gamma_j \to c^2$. Thus passing to a corresponding subsequence if necessary, g_j converges to an element g in $\mathrm{PSL}(2,\mathbb{C})$:

$$g = \begin{pmatrix} i\sqrt{1+c} & 1 \\ c & -i\sqrt{1+c} \end{pmatrix},$$

and hence $\beta' = \beta(g) = \mathrm{tr}^2(g) - 4 = -4$. Calculating

$$\begin{aligned} \gamma(f, g) &= \mathrm{tr}([f,g]) - 2 \\ &= ci\sqrt{1+c} + c^2 + 2 - ci\sqrt{1+c} - 2 \\ &= c^2 = \gamma. \end{aligned}$$

In summary of the part (a),

$$\begin{aligned} &\beta_j \to \beta = 0 \\ &\beta'_j \to \beta' = -4 \\ &\gamma_j \to \gamma = c^2. \end{aligned}$$

Therefore, the triple of parameters of $\langle f, g\rangle$ is $(\gamma, \beta, -4)$.

(b) Suppose that now f_j is not parabolic for all j. By Theorem 2.1.17 one may assume that f_j is represented by the matrix up to conjugation each Γ_j,

$$f_j = \begin{pmatrix} \lambda_j & 0 \\ 0 & \frac{1}{\lambda_j} \end{pmatrix},$$

where $\lambda_j \neq 0, \pm 1$. Thus the parameter $\beta_j = (\lambda_j - \frac{1}{\lambda_j})^2$, which gives $\lambda_j^2 - \sqrt{\beta_j}\lambda_j - 1 = 0$ and hence $\lambda_j = \frac{\sqrt{\beta_j} \pm \sqrt{\beta_j + 4}}{2}$. Since $\beta_j \to \beta$, λ_i converges to $\lambda = \frac{\sqrt{\beta} \pm \sqrt{\beta+4}}{2}$. Noting that $\lambda \neq 0, \pm 1$ for either choice of λ, it is concluded that f_j converges to an element f represented by a matrix

$$f_j \to f = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \text{ and } \beta_j \to \beta = (\lambda - \frac{1}{\lambda})^2.$$

It remains to show that the order 2 elements g_j also converge. As in the previous case the matrix for g_j can be written as

$$g_j = \begin{pmatrix} a_j & -\frac{1+a_j{}^2}{c_j} \\ c_j & -a_j \end{pmatrix}.$$

Consider the conjugacy of the group $\langle f_j, g_j \rangle$ by the following form of a diagonal matrix conjugator.

$$\phi_j = \begin{pmatrix} \mu_j & 0 \\ 0 & \frac{1}{\mu_j} \end{pmatrix}.$$

Since f_j commutes with ϕ_j, f_j is left unchanged under the conjugation. Set $\mu_j = i\sqrt{\frac{c_j}{1+a_j^2}}$, then ϕ_j becomes

$$\phi_j = \begin{pmatrix} i\sqrt{\frac{c_j}{1+a_j^2}} & 0 \\ 0 & -i\sqrt{\frac{1+a_j^2}{c_j}} \end{pmatrix},$$

and recall g_j the conjugacy $\phi_j g_j \phi_j^{-1}$:

$$g_j = \begin{pmatrix} a_j & 1 \\ -(1+a_j^2) & -a_j \end{pmatrix}.$$

Recall the resulting group $\Gamma_j = \langle f_j, g_j \rangle$. Thus, the commutator is

$$[f_j, g_j] = f_j g_j f_j^{-1} g_j^{-1} = \begin{pmatrix} -a_j^2 + \lambda_j^2(1+a_j^2) & -a_j + \lambda_j^2 a_j \\ \frac{a_j\left(1+a_j^2\right)}{\lambda_j^2} - a_j(1+a_j^2) & \frac{1+a_j{}^2}{\lambda_j^2} - a_j^2 \end{pmatrix}.$$

Hence the parameter γ_j for the group $\langle f_j, g_j \rangle$ is

$$\begin{aligned}
\gamma_j &= \operatorname{tr}([f_j, g_j]) - 2 \\
&= a_j^2 \left(\lambda_j^2 + \frac{1}{\lambda_j^2} - 2 \right) + \lambda_j^2 + \frac{1}{\lambda_j^2} - 2 \\
&= a_j^2 \left(\lambda_j - \frac{1}{\lambda_j} \right)^2 + \left(\lambda_j - \frac{1}{\lambda_j} \right)^2 \\
&= \frac{a_j^2 \left(\lambda_j^2 - 1 \right)^2 + \left(\lambda_j^2 - 1 \right)^2}{\lambda_j^2}.
\end{aligned}$$

Thus, $\gamma_j = \frac{a_j^2 \left(\lambda_j^2 - 1 \right)^2 + \left(\lambda_j^2 - 1 \right)^2}{\lambda_j^2}$. Solving for a_j yields

$$a_j^2 = \frac{\gamma_j \lambda_j^2 - \left(\lambda_j^2 - 1 \right)^2}{\left(\lambda_j^2 - 1 \right)^2},$$

As $\lambda_j \to \lambda$ and $\gamma_j \to \gamma$,

$$a_j^2 \to \frac{\gamma \lambda^2 - \left(\lambda^2 - 1 \right)^2}{\left(\lambda^2 - 1 \right)^2}.$$

Let $a^2 = \frac{\gamma \lambda^2 - \left(\lambda^2 - 1 \right)^2}{\left(\lambda^2 - 1 \right)^2}$, then $a_j^2 \to a^2$. So $\left\{ a_j^2 \right\}$ is bounded and hence a_j is bounded. It follows up to a subsequence, that a_j converges to a and thus g_j converges to the element g of $\mathrm{PSL}(2, \mathbb{C})$ represented by the matrix

$$\begin{pmatrix} a & 1 \\ -(1 + a^2) & -a \end{pmatrix}.$$

Since $\gamma_j \to \gamma$ and $\gamma_j = \frac{a_j^2 \left(\lambda_j^2 - 1 \right)^2 + \left(\lambda_j^2 - 1 \right)^2}{\lambda_j^2} \to \frac{a^2 \left(\lambda^2 - 1 \right)^2 + \left(\lambda^2 - 1 \right)^2}{\lambda^2}$, $\gamma = \frac{a^2 \left(\lambda^2 - 1 \right)^2 + \left(\lambda^2 - 1 \right)^2}{\lambda^2}$.

Calculating

$$\begin{aligned}\gamma(f,\,g) &= \operatorname{tr}([f,g]) - 2\\ &= a^2\left(\lambda^2 + \frac{1}{\lambda^2} - 2\right) + \lambda^2 + \frac{1}{\lambda^2} - 2\\ &= a^2\left(\lambda - \frac{1}{\lambda}\right)^2 + \left(\lambda - \frac{1}{\lambda}\right)^2\\ &= \frac{a^2\left(\lambda^2-1\right)^2 + \left(\lambda^2-1\right)^2}{\lambda^2} = \gamma.\end{aligned}$$

In summary of the part (b),

$$\begin{aligned}&\beta_j \to \beta = (\lambda - \frac{1}{\lambda})^2 = \beta(f)\\ &\beta'_j \to \beta' = -4 = \beta(g)\,.\\ &\gamma_j \to \gamma = \frac{a^2\left(\lambda^2-1\right)^2 + \left(\lambda^2-1\right)^2}{\lambda^2} = \gamma(f,\,g)\,.\end{aligned}$$

Therefore, the triple of parameters of $\langle f,\, g\rangle$ is $(\gamma, \beta, -4)$ in the part (b).

Now it is shown in both cases (a) and (b) that $f_j \to f$, $g_j \to g$, and the triple of parameters for the group $\langle f,\, g\rangle$ is $(\gamma, \beta, -4)$. □

In the proof of above Proposition 3.3.1 is not based on Jørgensen's Theorem 3.2.13. However, if one applies for Jørgensen's Theorem 3.2.13, it can be obtained directly from Theorem 3.2.15 and the limit group Γ is a Kleinian group. Furthermore, it is clear that if there is no control on one of the generators, then one cannot apply directly for Jørgensen's Theorem 3.2.13. It is the case in the following theorem.

Theorem 3.3.2 *Suppose that* $\Gamma_j = \langle f_j, g_j\rangle$ *is a sequence of two-generator Kleinian groups with the parameters* $\left(\gamma_j, \beta_j\right)$ *and* $\left(\gamma_j, \beta_j\right) \to (\gamma, \beta)$, *where* $\beta \neq -4$. *Then either* $\gamma_j = \gamma$ *and* $\beta_j = \beta$ *for some* j, *or there is a sequence of Kleinian groups* $\Gamma_j^* = \langle f_j, h_j\rangle$ *with the parameters* $\left(\gamma_j, \beta_j, -4\right)$ *and a Kleinian group* $\Gamma = \langle f, h\rangle$

with the parameters $(\gamma, \beta, -4)$ *such that* Γ_j^* *converges algebraically to* Γ.

Proof. Notice that $\gamma_j \neq 0$ for all j as Γ_j is Kleinian. Since $\beta_j \to \beta \neq -4$, $\beta_j \neq -4$ for all but finitely many j. One may assume, passing to a subsequence if necessary, $\beta_j \neq -4$ for all j, that is f_j is not of order two for all j. For each j, by Lemma 3.2.7, there is an elliptic conjugator h_j of order 2 such that

$$h_j f_j h_j = g_j f_j g_j^{-1},$$

where h_j is essentially a rotation of order two (i.e., $\beta(h_j) = -4$) through the bisector of the common perpendicular between the axes of f_j and $g_j f_j g_j^{-1}$. It follows that

$$h_j f_j^{-1} h_j^{-1} = g_j f_j^{-1} g_j^{-1}.$$

In fact, $h_j f_j h_j^{-1} = g_j f_j g_j^{-1}$ and then $h_j f_j^{-1} h_j^{-1} = \left(h_j f_j h_j^{-1}\right)^{-1} = \left(g_j f_j g_j^{-1}\right)^{-1} = g_j f_j^{-1} g_j^{-1}$. Thus,

$$\gamma(f_j, h_j) = \gamma_j$$

as $[f_j, h_j] = f_j \left(h_j f_j^{-1} h_j^{-1}\right) = f_j \left(g_j f_j^{-1} g_j^{-1}\right) = [f_j, g_j]$. Now there is a sequence of groups $\Gamma_j^* = \langle f_j, h_j \rangle$ with the triple of parameters $\left(\gamma_j, \beta_j, -4\right)$. The rest of the proof is broken down into two parts.

(a) If $\widetilde{\Gamma}_j$ is not Kleinian. One may assume for infinitely many j, otherwise, if $\tilde{\Gamma}_j$ is not Kleinian for finitely many j, it can go to the part (b) (passing to a subsequence): $\widetilde{\Gamma}_j$ is Kleinian for all j, pass to a subsequence if necessary. By Theorem 3.2.6, f_j is elliptic and has order $p \leq 6$. It is already excluded the possibility $p = 2$ at the beginning. In this case, $\beta_j \in \left\{-3, -2, -1, \frac{\sqrt{5}-5}{2}, -\frac{\sqrt{5}+5}{2}\right\}$ for infinitely many j. One may pass to a subsequence and, after a conjugacy, assume the sequences $\beta_j = \beta$ and $f_j = f$ of order $p \in \{3, 4, 5, 6\}$.

On the other hand side, if $\tilde{\Gamma}_j$ is not Kleinian, then it is one of the finitely many finite spherical triangle groups, because that $\gamma_j \neq 0$ implies the Euclidean triangle groups are eliminated by Remark 2.4.13. This means that

$\gamma(f_j, g_j f_j g_j^{-1}) = \gamma_j(\gamma_j - \beta)$ can only take finitely many values, it follows that γ_j can take finitely many values, so one may assume the sequence $\gamma_j = \gamma$ (passing to a subsequence if necessary). It is shown that $\beta_j = \beta$ and $\gamma_j = \gamma$ for infinitely many j. This case gives the required result.

(b) Now a case has to deal with is that $\tilde{\Gamma}_j$ is Kleinian for each j.

By Theorems 3.2.14, 2.3.12, and 2.4.14, $\tilde{\Gamma}_j$ is discrete, $\gamma(f_j, g_j f_j g_j^{-1}) \neq 0$, and $\beta_j(f_j) = \beta_j\left(g_j f_j g_j^{-1}\right) \neq -4$, respectively. Thus, apply for Theorem 4.1 [6, Theorem 4.1], the following inequality holds

$$\gamma(f_j, g_j f_j g_j^{-1}) = \gamma_j(\gamma_j - \beta_j) \geq 2 - 2\cos(\frac{\pi}{7}) = 0.198\,06\cdots .$$

Therefore, $\lim_{j\to\infty} \gamma_j(\gamma_j - \beta_j) = \gamma(\gamma - \beta) \geq 0.198\,06\cdots$ and hence $\gamma \neq 0$ and $\gamma \neq \beta$. It follows that $\gamma_j \nrightarrow 0$ and $\gamma_j \nrightarrow \beta$, so $\gamma_j \neq \beta_j$ for all but finitely many j.

Now $\Gamma_j^* = \langle f_j, h_j \rangle$ contains $\tilde{\Gamma}_j = \langle f_j, h_j f_j h_j^{-1} \rangle$ with index two (at most) and by Theorem 3.2.4 Γ_j^* is Kleinian. Now conjugate each $\langle f_j, h_j \rangle$ so that $h_j = h$ and $h(z) = -z$, a fixed element of order two. Then

$$h_j = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ f_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C}).$$

Further conjugacy by a diagonal matrix for each j :

$$\phi_j = \begin{pmatrix} \mu_j & 0 \\ 0 & \frac{1}{\mu_j} \end{pmatrix}$$

leaves h_j fixed and the conjugacy of f_j is

$$\phi_j f_j \phi_j^{-1} = \begin{pmatrix} a_j & b_j \mu_j^2 \\ c_j \mu_j^{-2} & d_j \end{pmatrix}.$$

Now choose $\mu_j = \frac{1}{\sqrt{b_j}}$, then $b_j \mu_j^2 = 1$ and $c_j \mu^{-2} = c_j b_j$. Recall that

$$f_j = \begin{pmatrix} a_j & 1 \\ c_j & d_j \end{pmatrix},$$

where $a_j d_j - c_j = 1$, i.e., $a_j d_j = 1 + c_j$. By computing, the commutator is

$$[f_j, h_j] = f_j h_j f_j^{-1} h_j^{-1} = \begin{pmatrix} a_j d_j + c_j & 2a_j \\ 2c_j d_j & a_j d_j + c_j \end{pmatrix},$$

and the parameter is

$$\begin{aligned} \gamma_j &= 2a_j d_j + 2c_j - 2 \\ &= 2(1 + c_j) + 2c_j - 2 = 4c_j \neq 0. \end{aligned}$$

Hence, $c_j = \frac{\gamma_j}{4} \to c = \frac{\gamma}{4}$. Then by Definition 2.3.1, $(a_j + d_j)^2 = \beta_j + 4$ and hence $a_j + d_j = \sqrt{\beta_j + 4} \to \sqrt{\beta + 4}$. Also $a_j d_j - c_j = 1$ gives $a_j d_j \to 1 + \frac{\gamma}{4}$. Since a_j and d_j can be written in terms of $a_j + d_j$ and $a_j d_j$ by the continuous operations, a_j and d_j admit convergent subsequences, say the limits are a and d, respectively. After doing so (and after all the normalizations by conjugacy) find the following f and h such that $f_j \to f$, $h_j \to h$, and $\Gamma = \langle f, h \rangle$ with the triple of parameters $(\gamma, \beta, -4)$:

$$f = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \text{ and } h = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Now the result reduces to Jørgensen's Theorem 3.2.13. □

Lemma 3.3.3 *Let $\{(\gamma_j, \beta_j, \beta'_j)\}$ be a sequence of parameters for two-generator Kleinian groups $\langle f_j, g_j \rangle$, and let (γ, β, β') be parameters for two-generator group*

$\langle f, g \rangle$. *Suppose that* (γ_j, β_j) *converges to* (γ, β) *and* f *is not elliptic of order* $p \leq 6$. *Then* $\gamma \neq 0$ *and* $\gamma \neq \beta$.

Proof. Since f is not elliptic of order $p \leq 6$, f_j is not elliptic of order $p \leq 6$ for all j but finitely many. Otherwise, if f_j is elliptic of order $p \leq 6$ for infinitely many j, then by Corollary 2.3.6 there is a constant sequence $\{\beta(f_j)\}$ and hence the limit $\beta(f)$ has the same constant. So f is elliptic of order $p \leq 6$, it is a contradiction.

Now applying for Theorem 3.2.6, there is a sequence $\langle f_j, g_j f_j g_j^{-1} \rangle$ of Kleinian groups with corresponding parameters $(\gamma_j(\gamma_j - \beta_j), \beta_j, \beta_j)$ which converge to $(\gamma(\gamma - \beta), \beta, \beta)$. On the other hand $\gamma_j \neq 0$ by using Corollary 3.1.4 and $\gamma_j \neq \beta_j$ by using Lemma 2.4.12, so one can apply for a result by C.Cao (see Theorem 5.1 in [6]) which gives a lower bound:

$$\gamma_j(\gamma_j - \beta_j) \geq 0.198.$$

Thus, $\lim_{j\to\infty} \gamma_j(\gamma_j - \beta_j) = \gamma(\gamma - \beta) \geq 0.198$. It follows that $\gamma \neq 0$ and $\gamma \neq \beta$. $\square$

Next consider the projection from subspace $\mathcal{D}$ in three complex dimensional space $\mathbb{C}^3$ to subspace $\mathcal{D}_2$ in two complex dimensional $\mathbb{C}^2$. Note that the image of the projection of a closed set in $\mathbb{C}^3$ to $\mathbb{C}^2$ need not be closed. For example, consider the projection of the set F in $\mathbb{C}^3$ onto the set E in $\mathbb{C}^2$, where

$$F = \{\left(\frac{1}{n}, 0, n\right) : n \in \mathbb{N}\} \text{ and } E = \{\left(\frac{1}{n}, 0\right) : n \in \mathbb{N}\}.$$

The set F is closed in $\mathbb{C}^3$ because there are not any limit points in F and hence the closure $\overline{F} = F$. But the set E is not closed in $\mathbb{C}^2$ because the limit point $(0, 0)$ is not in E. However, it is true by the following theorem for the projection of the closed subspace $\mathcal{D}$ in $\mathbb{C}^3$ onto the subspace $\mathcal{D}_2$ in $\mathbb{C}^2$. Also notice that if a triple $(\gamma_0, \beta_0, \beta_0')$ is not the parameters of a Kleinian group, but the triple $(\gamma_0, \beta_0, \beta_1')$

could be the parameters of a Kleinian group for the same γ_0 and β_0. Two different proofs of the following theorem are provided below and the information revealed in the proofs plays an important role in studying Kleinian groups.

Theorem 3.3.4 *Let subspace $\mathcal{D}_2$ be defined by*

$$\mathcal{D}_2 = \{(\gamma,\, \beta) : \textit{for some } \beta' \textit{ such that } (\gamma, \beta, \beta') \textit{ are the parameters of a Kleinian group}\}.$$

then $\mathcal{D}_2$ is closed in the two complex dimensional space $\mathbb{C}^2$ in the usual topology.

Proof 1. Suppose $(\gamma_j, \beta_j) \to (\gamma,\, \beta)$ with $(\gamma_j, \beta_j) \in \mathcal{D}_2$. Consider the sequence of triples $\left(\gamma_j, \beta_j, -4\right)$.

If there are infinitely many j such that $\left(\gamma_j, \beta_j, -4\right)$ are the parameters of Kleinian groups, after passing to a subsequence, recall $\left(\gamma_j, \beta_j, -4\right)$, by Theorem 3.2.15, the limit $(\gamma, \beta, -4)$ is the triple of parameters for a Kleinian group and hence $(\gamma, \beta) \in \mathcal{D}_2$.

Otherwise, there are infinitely many j such that $\left(\gamma_j, \beta_j, -4\right)$ are not the parameters of Kleinian groups. Since $(\gamma_j, \beta_j) \in \mathcal{D}_2$, Lemma 3.2.8 tells that $\left(\gamma_j, \beta_j, -4\right)$ are the parameters of discrete groups and hence there are infinitely many $\left(\gamma_j, \beta_j\right)$ in a finite list (at Table 1, in fact) where $\left(\gamma_j, \beta_j, -4\right)$ are the parameters of discrete elementary groups. Consequently, there is a triple of parameters, say $(\gamma, \beta, -4)$, in that finite list such that (γ, β) taking by infinitely many $\left(\gamma_j, \beta_j\right)$. Thus, after passing to a subsequence if necessary one may assume $\left(\gamma_j, \beta_j\right) = (\gamma, \beta)$ for all j. Since each (γ_j, β_j) in the subsequence is already assumed in $\mathcal{D}_2$, so $(\gamma,\, \beta) \in \mathcal{D}_2$. $\square$

Proof 2. Suppose $\{(\gamma_j, \beta_j)\}$ is a sequence in $\mathcal{D}_2$ with limit $(\gamma,\, \beta)$ in $\mathbb{C}^2$. According to the definition of $\mathcal{D}_2$, there is a sequence $\left\{(\gamma_j, \beta_j, \beta'_j)\right\}$ of triples of parameters of two-generator Kleinian groups, and hence there is a sequence $\{\Gamma_j = \langle f_j, g_j\rangle\}$ of Kleinian groups with the triples of parameters $\left\{(\gamma_j, \beta_j, \beta'_j)\right\}$ by Theorem 2.3.2.

It needs to exhibit a two-generator Kleinian group with parameters (γ, β, β') for some $\beta' \in \mathbb{C}$. There are three cases breakdown to prove that the limit gives the parameters γ and β for some two-generator Kleinian group $\Gamma = \langle f,\, g\rangle$. In the Cases II & III, first show that Γ is discrete, and then to show it is impossible that Γ is elementary. By Theorem 2.4.6 about the classification of elementary groups, Γ falls into Type I, II, and III.

Case I: Suppose that, up to subsequence, the parameter entries β_j and γ_j are constant for all j, say $(\gamma_j, \beta_j) = (\gamma,\, \beta)$ for all j. Let $\Gamma = \Gamma_j$ for some j. Then Γ is a Kleinian group with parameters (γ, β, β'), as required.

Case II: Suppose that, up to subsequence, the parameter entries β_j are constant for all j and the parameter entries γ_j are distinct for all j. Since $(\gamma_j, \beta_j, \beta'_j)$ are the parameters of a two-generator Kleinian group Γ_j, by Lemma 3.2.8 there exists a sequence of discrete groups $\{\Gamma_j^{\phi} = \langle f_j, \phi_j\rangle\}$ with corresponding parameters $\{(\gamma_j, \beta_j, -4)\}$. Thus using the finite exceptional set of parameters of discrete elementary groups with $\beta' = -4$, one may assume that all of the groups Γ_j^{ϕ} are Kleinian groups, passing to a subsequence if necessary. Since (γ_j, β_j) converges to $(\gamma,\, \beta)$, $\{(\gamma_j, \beta_j, -4)\}$ converges to $(\gamma, \beta, -4)$. Applying for Proposition 3.3.1, the sequence $\{\Gamma_j^{\phi}\}$ converges algebraically to a group Γ. Jørgensen showed in Proposition 2 [35] that if Γ is non-elementary then Γ must be discrete, hence Γ is Kleinian. Now assume to the contrary that Γ is elementary, then there are the contradictions in the following three subcases.

(i) Assume that $\Gamma = \langle f,\, g\rangle$ is of Type I, then each non-trivial element of Γ is elliptic. Then f can't be an irrational rotation as assumed that $\beta = \beta_j$, which implies that f_j is an irrational rotation, hence contradicting the assumption that $\Gamma_j = \langle f_j, g_j\rangle$ is discrete and then countable. So, f can only be an elliptic of finite order. If one assumes that f is of order 2, then $\beta_j = \beta = -4$ for each j and $\Gamma_j^{\phi} = \langle f_j, \phi_j\rangle$ is a two-generator group with both generators of order 2. By Theorem 2.4.14, Γ_j^{ϕ} is elementary, it contradicts that Γ_j^{ϕ} is Kleinian. What left is to consider the case when Γ is generated by f an elliptic of order $n \geq 3$ and g an elliptic of order 2. In this case, by Beardon [2, Section 5.1] Theorem 4.3.7, there is a point in $\mathbb{H}^3$ fixed by every element of Γ, thus it is a point of intersection of $\operatorname{ax}is(f)$ and $\operatorname{axis}(g)$ in $\mathbb{H}^3$ and hence $\delta = 0$. Also one can assume that $\beta_j = \beta$ for all j. Hence Γ_j^ϕ is generated by an elliptic f_j of order $n \geq 3$ and an elliptic ϕ_j of order 2 for all j also. Since Γ_j^ϕ is non-elementary, $\delta(f_j, \phi_j) \neq 0$ for all j. But by F. Gehring-G. Martin [15] (Theorem 4.20) there is a lower bound: $\delta(f_j, \phi_j) \geq \frac{b(n)}{2}$. Thus, $\lim_{j\to\infty} \delta(f_j, \phi_j) = \delta(f, g) \geq \frac{b(n)}{2}$ which leads to a contradiction to $\delta = 0$.

(ii) Assume that $\Gamma = \langle f,\, g\rangle$ is of Type II, then Γ is conjugate to a subgroup of Möb $\left(\overline{\mathbb{C}}\right)$ that every element is parabolic and Γ has a common fixed point. Thus, f is parabolic and hence f can't be of order $2, 3, 4, 5$ or 6; and both f and g have one fixed point in common then $\gamma = 0$ by using Theorem 2.3.12. It contradicts to Lemma 3.3.3.

(iii) Assume that $\Gamma = \langle f,\, g\rangle$ is of Type III, then Γ is conjugate to a subgroup of Möb $\left(\overline{\mathbb{C}}\right)$ that every element of which leaves the set $\{0, \infty\}$ invariant under Γ. If f is a loxodromic element which shares its axis with g, then f are not of order $2, 3, 4, 5$ or 6 and $\operatorname{axis}(f) = \operatorname{axis}(g)$ and hence $\operatorname{Fix}(f) = \operatorname{Fix}(g)$. By Theorem 2.3.12, $\gamma = 0$. It is a contradiction from Lemma 3.3.3.

Otherwise f is a loxodromic element and g is an elliptic of order 2 which interchanges two fixed points of f. One may assume by Theorem 2.1.17 that:

$$f = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \text{ and } g = \begin{pmatrix} 0 & \mu \\ \frac{1}{\mu} & 0 \end{pmatrix},$$

where $\lambda \neq 0, \mu \neq 0$. Moreover,

$$\gamma = tr([f, g]) - 2 = (\lambda - \frac{1}{\lambda})^2,$$
$$\beta = tr^2(f) \;\; - 4 = (\lambda - \frac{1}{\lambda})^2.$$

Thus, $\gamma = \beta$. It is also a contradiction from Lemma 3.3.3.

Case III: Since there are infinitely many β_j that are the same in Cases I & II, it is left with the possibility that, there are infinitely many distinct β_j. One may assume that the β_j are distinct for all j, passing to a subsequence if necessary. In particular, one can assume that the generators f_j do not have order $2, 3, 4, 5$ or 6 for all j. By Theorem 3.2.6 $\Gamma_j' = \langle f_j, g_j f_j g_j^{-1} \rangle$ is a Kleinian group for each j, and by Lemmas 3.2.7, there exists an element ϕ_j of order 2 such that $\phi_j f_j \phi_j^{-1} = g_j f_j g_j^{-1}$ for each j. Thus, by Corollary 3.2.9 $\Gamma_j^{\phi} = \langle f_j, \phi_j \rangle$ is a Kleinian group for each j and its triple of parameters is $(\gamma_j, \beta_j, -4)$. By Proposition 3.3.1, it is known that the sequence $\{\Gamma_j^{\phi}\}$ converges algebraically to a group Γ. By Theorem 3.2.14 showed that if Γ is non-elementary then Γ must be discrete, hence Γ is Kleinian. Assume to the contrary that Γ is elementary.

(i) Assume Γ is elementary of Type 1, then f is elliptic. Since Γ_j^{ϕ} is Kleinian and hence non-elementary and discrete, by Lemma 2 in [35], the sequence of β_j is constant for all large indices which contradicts the assumptions that β_j are distinct for all j.

(ii) Assume Γ is elementary of Type II, then it reaches a contradiction as in (ii) of Case II.

(iii) Assume Γ is elementary of Type III, then it is a contradiction as in (iii) of Case II. □

Remark 3.3.5 (1) *The consequence of Theorem* 3.3.4 *is that the complement set* $\mathbb{C}^2 \backslash \mathcal{D}_2$ *is open. Thus, there is* $\delta > 0$ *such that the open ball*

$$B((0,0),\delta) = \left\{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < \delta^2\right\} \subseteq \mathbb{C}^2 \backslash \mathcal{D}_2,$$

i.e., $\mathcal{D}_2 \subseteq \mathbb{C}^2 \backslash B((0,0),\delta)$, *where* $(\gamma, \beta) = (0,0)$ *are the parameters of the identity group. It is showing that there is a neighborhood of the parameters* $(0,0)$ *for the identity group which cannot contain the parameters* (γ, β) *for any two-generator Kleinian group. It is Jørgensen's inequality which gives a precise bound such that* *there is not any two-generator Kleinian groups fitting the inequality* $|z| + |w| < 1$. *It follows that there are no Kleinian groups in the unit open "ball" in* $\mathbb{C}^2$ *regarding to the distance* $d = d((z, w), (0, 0)) = |z| + |w| : \{(z, w) \in \mathbb{C}^2 : d < 1\}$.

(2) $\mathcal{D}_2$ *can be embedded into* $\mathbb{C}^3$:

$$\mathbb{C}^2 \supseteq \mathcal{D}_2 \cong \{(\gamma, \beta, 0) : (\gamma, \beta) \in \mathcal{D}_2\} \subseteq \mathbb{C}^3.$$

3.4 Jørgensen's inequality

Jørgensen's inequality as he established in [35] is the first important universal constraint in studying the geometry of Kleinian groups [49]. Two important applications have already been introduced: one is the fundamental result in Jørgensen Theorem 3.2.13 and the other is Kleinian group's characterization in Theorem 3.2.14.

In this section an alternate proof of Jørgensen's inequality is given based on Theorem 3.3.4 before looking at the more general cases in Chapter 4. There are a few reasons for this. First, one will see the most elementary trace polynomial and then show how it can be used to generate an inequality. The steps in the proof will identify results that will have to be generalized and potential exceptions lying in lower dimensional subspaces, which will have to be dealt with using other ideas. In fact using the trace polynomial in (3.4), a number of inequalities for discrete groups have been generated in [16]. Earlier, Brooks and Matelski considered Jørgensen's matrix iteration procedure for different initial configurations producing inequalities [5].

Starting with the simple but important trace polynomial $p_w = \gamma(f, gfg^{-1}) = \gamma(\gamma - \beta)$ for the word $w = gfg^{-1}$ from Lemma 3.2.1 and apply for the essential result Theorem 3.3.4 to give an alternate proof of Jørgensen's inequality.

Theorem 3.4.1 (Jørgensen inequality) *Let $\langle f,\, g\rangle$ be a Kleinian group. Then*

$$|\gamma\,(f,\, g)| + |\beta\,(f)| \geq 1.$$

This inequality is sharp for infinitely many distinct Kleinian groups.

Proof. The proof is broken up into the following four general steps.

(1) Compactness:

Let $\gamma = \gamma\,(f,\, g)$ and $\beta = \beta\,(f)$, and let $B = \{(z, w) \in \mathbb{C}^2 : |z| + |w| \leq 1\}$, then B is closed and bounded and hence compact in $\mathbb{C}^2$ and

$$\mathcal{D}_2 = (\mathcal{D}_2 \cap B) \cup (\mathcal{D}_2 \cap B^c)\,,$$

where the complement set $B^c = \{(z, w) \in \mathbb{C}^2 : |z| + |w| > 1\}$. Thus,

$$|\gamma| + |\beta| > 1, \text{ for } (\gamma,\, \beta) \in \mathcal{D}_2 \cap B^c. \tag{3.14}$$

Now set

$$\delta = \min\{|\gamma| + |\beta| : (\gamma,\, \beta) \in \mathcal{D}_2 \cap B\}\,.$$

Since $\mathcal{D}_2$ is closed in $\mathbb{C}^2$ by Theorem 3.3.4, the bounded closed set $\mathcal{D}_2 \cap B$ is compact in $\mathbb{C}^2$. It follows that the minimum is achieved in $\mathcal{D}_2 \cap B$, say at $(\gamma_0, \beta_0) \in \mathcal{D}_2 \cap B$:

$$\delta = |\gamma_0| + |\beta_0| \leq 1. \tag{3.15}$$

(2) Trace polynomials:

Let $\langle f_0,\, g_0\rangle$ be the Kleinian group whose parameters (γ_0, β_0) achieved the minimum above, and let the word $w = g_0 f_0 g_0^{-1}$ and set $p_w(z) = z(z - \beta)$.

One can assume that $\langle f_0, w\rangle$ is Kleinian, then $\beta\,(f_0) = \beta_0$ and Theorem

3.1.17 and the identity (3.4) give

$$\gamma(f_0, w) = p_w(\gamma_0) = \gamma_0 \left(\gamma_0 - \beta_0\right). \tag{3.16}$$

and using the identity (3.15),

$$\begin{aligned} |\gamma(f_0, w)| + |\beta_0| &= |\gamma_0 \left(\gamma_0 - \beta_0\right)| + |\beta_0| \\ &\leq |\gamma_0| \left(|\gamma_0| + |\beta_0|\right) + |\beta_0| \\ &\leq |\gamma_0| + |\beta_0| \leq 1. \end{aligned}$$

then $(\gamma(f_0, w), \beta_0) \in \mathcal{D}_2 \cap B$ and hence

$$|\gamma_0| + |\beta_0| \leq |p_w(\gamma_0)| + |\beta_0| = |\gamma_0||\gamma_0 - \beta_0| + |\beta_0|.$$

Notice that $\gamma_0 \neq 0$ as it is the parameter for a Kleinian group, which gives

$$|\gamma_0 - \beta_0| \geq 1. \tag{3.17}$$

Applying for the triangle inequality for (3.17),

$$|\gamma_0| + |\beta_0| \geq |\gamma_0 - \beta_0| \geq 1. \tag{3.18}$$

Thus, $\delta \geq 1$, so $\delta = 1$ by the identity (3.15). Therefore $|\gamma| + |\beta| \geq \delta = 1$ and hence

$$|\gamma| + |\beta| = 1, \text{ for } (\gamma, \beta) \in \mathcal{D}_2 \cap B. \tag{3.19}$$

Finally, the inequality (3.14) and above identity(3.19) give $|\gamma| + |\beta| \geq 1$ for every Kleinian group $\langle f, g\rangle$ with the parameters $\gamma = \gamma(f, g)$ and $\beta = \beta(f)$.

Moreover, by the identities (3.18) the equality $|\gamma_0| + |\beta_0| = 1$ gives $|\gamma_0 -$

$\beta_0| = 1$ and hence $|p_w(\gamma_0)| = |\gamma_0||\gamma_0 - \beta_0| = |\gamma_0|$.

(3) Exceptional variety:

Here one must examine the supposition that $\langle f_0, w\rangle$ is Kleinian. As seen above, this can only fail in two circumstances.

(i) $(\gamma_0, \beta_0, -4)$ lies in the exceptional list of Table 1. In this case f is elliptic of order 2, 3, 4 or 5. In all cases $|\beta_0| \geq 1$ and hence $|\gamma_0| + |\beta_0| \geq 1$.

(ii) $p_w(\gamma_0) = \gamma_0(\gamma_0 - \beta_0) = 0$. The set where this polynomial vanishes forms an exceptional variety. Since $\gamma_0 \neq 0$, $\gamma_0 = \beta_0$. Thus, the point (γ_0, β_0) must lie on the variety $\{\gamma = \beta\} \subseteq \mathbb{C}^2$ and hence $\langle f_0, w\rangle$ must be cyclic or dihedral – and either of these contradicts the hypothesis that $\langle f_0,\, g_0\rangle$ is Kleinian.

(4) Sharpness:

The proof suggests that sharpness is achieved in a group with parameters $(\gamma_0, \beta_0, -4)$ with $|p_w(\gamma_0)| = |\gamma_0|$. Under these circumstances, $(p_w(\gamma_0), \beta_0, -4)$ would also be the triple of parameters for a group for which equality holds - and similarly for the iterates of $p_w = z(z - \beta)$. In particular, one should expect that γ_0 is a periodic point of p_w as it is unlikely that there are infinitely many different groups for which sharpness occurs for the same generator f. Let $p_w(z) = z$, then $z(z - \beta) - z = 0$, i.e., $z\left((z - \beta) - 1\right) = 0$ gives $z = 0$, and $z = 1 + \beta$. Hence, the fixed points of p_w are 0 and $1 + \beta$. $\square$

Chapter 4: New Approach to Inequalities for Kleinian Groups

The Chebyshev polynomials were developed by Chebyshev in the mid-19th century for a completely different purpose and that they form an orthogonal system of polynomials which makes them of great use in Numerical Analysis and Approximation Theory that are very different fields from that of the current field Geometric Analysis. The dissertation discovers infinitely many trace polynomials that can be expressed simply in terms of the Chebyshev polynomials in Theorem 4.1.1 and Theorem 4.1.2. These trace polynomials will be useful for obtaining geometric information about Kleinian groups.

The identification of precise inequalities for discrete groups of Möbius transformations started with Jørgensen's famous inequality [35] from 1976. Such inequalities typically give necessary conditions to force a group to be a Kleinian group, i.e., give sufficient condition to force a group to be a non-Kleinian group. In this chapter a new approach is provided to establish new types of inequalities by using some sorts of trace polynomials discovered in the first section. These inequalities generalize the Jørgensen's inequality so that one can learn more about Kleinian groups by studying the isolation of elementary discrete groups.

The novel approach here to establish the universal constraints for Kleinian groups is to use the closedness of $\mathcal{D}_2$ in $\mathbb{C}^2$ (Theorem 3.3.4) that is an essential tool for the scheme of establishing the quantifiable inequalities. It follows that the complement set $\mathbb{C}^2\backslash\mathcal{D}_2$ is open in $\mathbb{C}^2$. Now one can consider the following kind of distance

$$d = |z-a| + |w-b|$$

between the subspace $\mathcal{D}_2$ in $\mathbb{C}^2$ and the point (γ_0, β_0) in $\mathbb{C}^2\backslash\mathcal{D}_2$, where (γ_0, β_0) is the pair of parameters for an elementary group (i.e., non-Kleinian group) in

Tables 1, 2, and 3. Thus, there is a radius $r > 0$ such that the following open ball:

$$B = \left\{ (z, w) \in \mathbb{C}^2 : |z - \gamma_0| + |w - \beta_0| < r \right\}$$

in the open set $\mathbb{C}^2 \backslash \mathcal{D}_2$ and inside the following Euclidean ball:

$$B \subseteq \left\{ (z, w) \in \mathbb{C}^2 : |z - \gamma_0|^2 + |w - \beta_0|^2 < r^2 \right\} \subseteq \mathbb{C}^2 \backslash \mathcal{D}_2.$$

Accordingly, one can implement the scheme of establishing the following sorts of quantifiable inequalities for two-generator Kleinian groups $\langle f,\, g \rangle$ in this chapter:

$$|\gamma\,(f,\, g) - \gamma_0| + |\beta\,(f) - \beta_0| \geq r, \tag{4.1}$$

where $\gamma_0 = \gamma\,(\phi, \psi)$ and $\beta_0 = \beta\,(\phi)$ are the parameters for a discrete elementary two-generator group $\langle \phi, \psi \rangle$. However, the challenge here is how to find the various greatest lower bounds and to choose suitable trace polynomials.

4.1 Chebychev polynomials

The calculation of the trace polynomial p_w from a word w can be a little tricky except for some short words. For good words $w\,(f, g)$ such as $(gf)^n g$, $(gf)^n$, and $[g,\, f]^n$ in a Kleinian group $\langle f,\, g \rangle$ with the triple of parameters (γ, β, β'), the trace polynomial p_w of two complex variables $\gamma = \gamma\,(f,\, g)$ and $\beta = \beta\,(f)$ is the following (see Theorem 3.1.17):

$$p_w\,(\gamma,\, \beta) = \gamma\,(w\,(g,\, f)\,, f)$$

and infinitely many useful trace polynomials can be discovered and expressed simply in terms of the Chebyshev polynomials in Theorem 4.1.1 and Theorem

4.1.2.

It is well known that *Chebychev polynomials* are defined by the recursion formula:

$$T_0(z) = 1,\ T_1(z) = z,\ T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z),\ \text{for } n \in \mathbb{N}. \tag{4.2}$$

or by the explicit formula:

$$T_n(z) = \frac{1}{2}\left(\left(z - \sqrt{z^2-1}\right)^n + \left(z + \sqrt{z^2-1}\right)^n\right),\ \text{for } n \in \mathbb{N}. \tag{4.3}$$

For example, the first few Chebychev polynomials are

$$\begin{aligned}
T_0(z) &= 1, \\
T_1(z) &= z, \\
T_2(z) &= 2z^2 - 1, \\
T_3(z) &= 4z^3 - 3z, \\
T_4(z) &= 8z^4 - 8z^2 + 1, \\
T_5(z) &= 16z^5 - 20z^3 + 5z, \\
T_6(z) &= 32z^6 - 48z^4 + 18z^2 - 1, \\
T_7(z) &= 64z^7 - 112z^5 + 56z^3 - 7z, \\
T_8(z) &= 128z^8 - 256z^6 + 160z^4 - 32z^2 + 1.
\end{aligned} \tag{4.4}$$

Also, recall the identity

$$T_{2n}(z) = (-1)^n T_n(1 - 2z^2),\ \text{for } n \in \mathbb{N}. \tag{4.5}$$

and the Chebychev polynomials of the first kind T_n with the defining property

$$T_n(\cosh(z)) = \cosh(nz),\ \text{for } n \in \mathbb{N}. \tag{4.6}$$

Theorem 4.1.1 *Let $\langle f, g\rangle$ be a Kleinian group with the triple of parameters (γ, β, β'), where f is elliptic or loxodromic. Then,*

$$\beta(f^n) = 2T_n\left(1+\frac{\beta}{2}\right) - 2,\ \textit{for } n \in \mathbb{N}, \tag{4.7}$$

$$\gamma(f^n, g) = \frac{\beta(f^n)}{\beta}\gamma,\ \textit{for } n \in \mathbb{N}. \tag{4.8}$$

In particular,

$$\begin{aligned}
&\beta(f) = \beta,\\
&\beta(f^2) = \beta(\beta+4),\\
&\beta(f^3) = \beta(\beta+3)^2,\\
&\beta(f^4) = \beta(\beta+4)(\beta+2)^2,\\
&\beta(f^5) = \beta\left(\beta^2+5\beta+5\right)^2,\\
&\beta(f^6) = \beta(\beta+4)(\beta+3)^2(\beta+1)^2.
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&\gamma(f, g) = \gamma,\\
&\gamma(f^2, g) = \gamma(\beta+4),\\
&\gamma(f^3, g) = \gamma(\beta+3)^2,\\
&\gamma(f^4, g) = \gamma(\beta+4)(\beta+2)^2,\\
&\gamma(f^5, g) = \gamma\left(\beta^2+5\beta+5\right)^2,\\
&\gamma(f^6, g) = \gamma(\beta+4)(\beta+3)^2(\beta+1)^2.
\end{aligned} \tag{4.10}$$

Proof. (1) Since $\cosh^2\left(\frac{z}{2}\right) - \sinh^2\left(\frac{z}{2}\right) = 1$ and $\cosh(z) = \cosh^2\left(\frac{z}{2}\right) + \sinh^2\left(\frac{z}{2}\right)$ give $\cosh(z) = 2\sinh^2\left(\frac{z}{2}\right) + 1$, therefore,

$$\cosh(nz) = 1 + 2\sinh^2\left(\frac{nz}{2}\right),\ n \in \mathbb{N}. \tag{4.11}$$

Assuming that f has two fixed points 0 and ∞, up to conjugacy. By Corollary 2.1.16, f can be represented by $f = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C})$ and hence $f^n = \begin{pmatrix} \lambda^n & 0 \\ 0 & \frac{1}{\lambda^n} \end{pmatrix}$, where λ can be expressed as $e^{\frac{\tau}{2}}$ for a suitable $\tau = \tau_f + i\theta_f$. Thus, $\beta(f^n) = (\lambda^n - \frac{1}{\lambda^n})^2 = 4\left(\frac{e^{\frac{n\tau}{2}} - e^{-\frac{n\tau}{2}}}{2}\right)^2 = 4\sinh^2\left(\frac{n\tau}{2}\right)$. That is,

$$\beta(f^n) = 4\sinh^2\left(\frac{n\tau}{2}\right), \text{ for } n \in \mathbb{N}, \tau = \tau_f + i\theta_f. \tag{4.12}$$

where τ_f and θ_f are the translation length and the holonomy of f, respectively.

It follows from the identities (4.11) and (4.12) that

$$\cosh(nz) = 1 + \frac{\beta(f^n)}{2}, \text{ for } n \in \mathbb{N}. \tag{4.13}$$

Applying for the defining property (4.6) and the previous identity (4.13), that give $T_n\left(1 + \frac{\beta}{2}\right) = 1 + \frac{\beta(f^n)}{2}$ and hence

$$\beta(f^n) = 2T_n\left(1 + \frac{\beta}{2}\right) - 2, \text{ for } n \in \mathbb{N}.$$

In particular, the first few Chebychev polynomials (4.4) give the following:

$$\beta\left(f^{2}\right)=2T_{2}\left(1+\frac{\beta}{2}\right)-2=2\left(2\left(1+\frac{\beta}{2}\right)^{2}-1\right)-2=\beta\left(\beta+4\right),$$

$$\beta\left(f^{3}\right)=2T_{3}\left(1+\frac{\beta}{2}\right)-2=2\left(4\left(1+\frac{\beta}{2}\right)^{3}-3\left(1+\frac{\beta}{2}\right)\right)-2=\beta\left(\beta+3\right)^{2},$$

$$\begin{aligned}\beta\left(f^{4}\right)&=2T_{4}\left(1+\frac{\beta}{2}\right)-2\\&=2\left(8\left(1+\frac{\beta}{2}\right)^{4}-8\left(1+\frac{\beta}{2}\right)^{2}+1\right)-2=\beta\left(\beta+4\right)\left(\beta+2\right)^{2},\end{aligned}$$

$$\begin{aligned}\beta\left(f^{5}\right)&=2T_{5}\left(1+\frac{\beta}{2}\right)-2\\&=2\left(16\left(1+\frac{\beta}{2}\right)^{5}-20\left(1+\frac{\beta}{2}\right)^{3}+5\left(1+\frac{\beta}{2}\right)\right)-2=\beta\left(\beta^{2}+5\beta+5\right)^{2},\end{aligned}$$

$$\begin{aligned}\beta\left(f^{6}\right)&=2T_{6}\left(1+\frac{\beta}{2}\right)-2\\&=2\left(32\left(1+\frac{\beta}{2}\right)^{6}-48\left(1+\frac{\beta}{2}\right)^{4}+18\left(1+\frac{\beta}{2}\right)^{2}-1\right)-2\\&=\beta\left(\beta+4\right)\left(\beta^{2}+4\beta+3\right)^{2}.\end{aligned}$$

(2) Notice that one can represent f^n and g as the following:

$$f^{n}=\begin{pmatrix}\lambda^{n} & 0\\ 0 & \frac{1}{\lambda^{n}}\end{pmatrix}\ \text{and}\ g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{PSL}(2,\mathbb{C}),$$

where $bc \neq 0$ because that $\langle f, g\rangle$ is Kleinian. It follows from Case 2 of the proof of Theorem 2.3.12 that

$$\gamma(f^{n},g)=-bc(\lambda^{n}-\frac{1}{\lambda^{n}})^{2},\ \text{for}\ n\in\mathbb{N}.$$

Since f is non-parabolic, $\beta\left(f\right)=(\lambda-\frac{1}{\lambda})^{2}\neq 0$. Thus,

$$\gamma(f^{n},\,g)=-\frac{(\lambda^{n}-\frac{1}{\lambda^{n}})^{2}}{(\lambda-\frac{1}{\lambda})^{2}}bc(\lambda-\frac{1}{\lambda})^{2}=\frac{\beta(f^{n})}{\beta\left(f\right)}\,\gamma,\ \text{for}\ n\in\mathbb{N}.$$

It is easy to see the following by replacing β with γ in the previous formulas (4.9):

$$\begin{aligned}
\gamma(f,g) &= \gamma, \\
\gamma(f^2,g) &= \gamma\left(\beta+4\right), \\
\gamma(f^3,g) &= \gamma\left(\beta+3\right)^2, \\
\gamma(f^4,g) &= \gamma\left(\beta+4\right)\left(\beta+2\right)^2, \\
\gamma(f^5,g) &= \gamma\left(\beta^2+5\beta+5\right)^2, \\
\gamma(f^6,g) &= \gamma\left(\beta+4\right)\left(\beta+3\right)^2\left(\beta+1\right)^2.
\end{aligned}$$

□

Recall the identity $\gamma(f, f^m w f^n) = \gamma(f,w)$ for $m,n \in \mathbb{Z}$ from Lemma 3.1.18, and take $(m,n) = (0,1)$ and $w = g$ then

$$\gamma(f,gf) = \gamma\left(f,\, g\right). \tag{4.14}$$

In case that g is elliptic of order 2, the triple of the parameters is

$$\left(\gamma\left(f,\, g\right), \beta\left(f\right), \beta(g)\right) = (\gamma, \beta, -4).$$

Now Proposition 2.3.9 and the identity (4.12) give that

$$\beta(gf) = \gamma - \beta - 4 = 4\sinh^2(\frac{\tau}{2}), \tag{4.15}$$

where $\frac{\tau}{2} = \frac{\tau_f + i\theta_f}{2}$ for suitable τ.

$$(\gamma(f,\, gf), \beta\left(f\right), \beta(gf)) = \left(\gamma, \beta, \gamma-\beta-4\right). \tag{4.16}$$

Theorem 4.1.2 *Suppose that $\langle f, g\rangle$ is a Kleinian group with the triple of parameters (γ, β, β'), where gf is elliptic or loxodromic. Let $\gamma_{n+1} = \gamma\left((gf)^{n+1}, f\right)$ then*

$$\gamma_{n+1} = \frac{T_{n+1}(\cosh(\tau)) - 1}{\cosh(\tau) - 1}\gamma, \ \textit{for } n \in \mathbb{N}. \tag{4.17}$$

Further, the following recursion formulas are held:

$$\gamma_0 = 0, \ \gamma_1 = \gamma, \ \gamma_{n+1} = (\gamma_1 - \beta - 2)\gamma_n - \gamma_{n-1} + 2\gamma_1, \ \textit{for } n \in \mathbb{N}.$$

In particular,

$$\gamma_0 = 0, \ \gamma_1 = \gamma$$

$$\gamma_2 = \gamma(\gamma - \beta)$$

$$\gamma_3 = \gamma(\gamma - \beta - 1)^2$$

$$\gamma_4 = \gamma(\gamma - \beta)(\gamma - \beta - 2)^2$$

$$\gamma_5 = \gamma(1 + 3\beta + \beta^2 - 3\gamma - 2\beta\gamma + \gamma^2)^2$$

$$\gamma_6 = \gamma(\gamma - \beta)(\gamma - \beta - 1)^2(\gamma - \beta - 3)^2$$

$$\gamma_7 = \gamma(-1 - 6\beta - 5\beta^2 - \beta^3 + 6\gamma + 10\beta\gamma + 3\beta^2\gamma - 5\gamma^2 - 3\beta\gamma^2 + \gamma^3)^2$$

$$\gamma_8 = \gamma(\gamma - \beta)(\gamma - \beta - 2)^2(2 + 4\beta + \beta^2 - 4\gamma - 2\beta\gamma + \gamma^2)^2$$

$$\gamma_9 = \gamma(\gamma - \beta - 1)^2(-1 - 9\beta - 6\beta^2 - \beta^3 + 9\gamma + 12\beta\gamma + 3\beta^2\gamma - 6\gamma^2 - 3\beta\gamma^2 + \gamma^3)^2$$

$$\gamma_{10} = \gamma(\gamma - \beta)(5 + 5\beta + \beta^2 - 5\gamma - 2\beta\gamma + \gamma^2)^2(1 + 3\beta + \beta^2 - 3\gamma - 2\beta\gamma + \gamma^2)^2$$

Proof. The identity (4.8) gives $\gamma(f^n, g) = \frac{\beta(f^n)}{\beta}\gamma$, and the Chebychev polynomials of the first kind (4.6) gives $T_n(\cosh(z)) = \cosh(nz)$, it follows that

$$\begin{aligned}\gamma_{n+1} &= \frac{\beta((gf)^{n+1})}{\beta\,(gf)}\gamma\,(gf, f)\\ &= \frac{4\sinh^2(\frac{(n+1)\,\tau}{2})}{4\sinh^2(\frac{\tau}{2})}\,\gamma\,(gf, f)\\ &= \frac{\cosh((n+1)\,\tau) - 1}{\cosh(\tau) - 1}\gamma\,(gf, f)\\ &= \frac{T_{n+1}(\cosh(\tau)) - 1}{\cosh(\tau) - 1}\,\gamma\,(gf, f)\,.\end{aligned}$$

That is, $\gamma_{n+1} = \frac{T_{n+1}(\cosh(\tau))-1}{\cosh(\tau)-1}\gamma\,(gf, f)$, for $n \in \mathbb{N}$, where $\gamma\,(gf, f) = \gamma$ by the identity (4.14). Thus,

$$\gamma_{n+1} = \frac{T_{n+1}(\cosh(\tau)) - 1}{\cosh(\tau) - 1}\gamma, \text{ for } n \in \mathbb{N}.$$

One can express $\cosh(\tau)$ in terms of the parameters γ and β. From the identity (4.15),

$$\begin{aligned}\gamma - \beta &= 4 + 4\sinh^2(\frac{\tau}{2})\\ &= 4\cosh^2(\frac{\tau}{2})\\ &= 2\,(1 + \cosh(\tau))\,.\end{aligned}$$

It can be deduced that

$$\cosh(\tau) = \frac{1}{2}(\gamma - \beta - 2). \tag{4.18}$$

Now using the recursion formula of Chebychev polynomials (4.2) and the previous identity (4.18), the identity (4.17) becomes,

$\gamma\left((gf)^{n+1}, f\right) = \frac{2\cosh(\tau)\,T_n(\cosh(\tau)) - T_{n-1}(\cosh(\tau)) - 1}{\cosh(\tau)-1}\gamma\left(gf, f\right)$

$= \frac{2\cosh(\tau)\,T_n(\cosh(\tau)) - 2\cosh(\tau) + 2\cosh(\tau) - T_{n-1}(\cosh(\tau)) - 1}{\cosh(\tau)-1}\gamma\left(gf, f\right)$

$= 2\cosh(\tau)\,\gamma((gf)^n, f) + \frac{2\cosh(\tau)-2}{\cosh(\tau)-1}\gamma\left(gf, f\right) - \frac{T_{n-1}(\cosh(\tau))-1}{\cosh(\tau)-1}\gamma\left(gf, f\right)$

$= 2\cosh(\tau)\,\gamma((gf)^n, f) + 2\gamma - \gamma((gf)^{n-1}, f)$

$= (\gamma - \beta - 2)\gamma_n + 2\gamma - \gamma_{n-1},$ for $n \in \mathbb{N}$.

Notice that $\gamma_{n+1} = \gamma\left((gf)^{n+1}, f\right)$, it reaches the required identity:

$$\gamma_{n+1} = (\gamma_1 - \beta - 2)\gamma_n - \gamma_{n-1} + 2\gamma_1, \text{ for } n \in \mathbb{N}.$$

Where $\gamma_1 = \gamma$ by the identity (4.14), and the formulas of $\gamma_0, \gamma_2, \cdots, \gamma_{10}$ can be verified easily. □

4.2 Inequalities for Chebychev polynomials

In this section, one can apply the inequalities generated in Lemma 4.2.5 for the Chebychev polynomial T_{n+1} when $n = 1, 2, 3, 4,$ and 7 to establish the sorts of inequalities for two-generator Kleinian groups $\langle f,\, g\rangle$:

$$|\gamma\,(f,\, g)| + |\beta\,(f) - \beta_0| \geq r,$$

where $\gamma\,(\phi, \psi) = 0$ and $\beta_0 = \beta\,(\phi)$ are the parameters for a discrete elementary two-generator group $\langle \phi, \psi \rangle$. In what follows one will typically cancel the term $|\gamma - \beta - 4|$ and so to avoid division to be zero one should ensure $\gamma \neq \beta + 4$. For example, under the assumption that g has order 2, $\beta(gf) = \gamma - \beta - 4$ by the identity (4.15). If $\gamma - \beta - 4 = 0$ then $\beta(gf) = 0$ and hence gf is parabolic. Thus, if gf is loxodromic or elliptic then one can cancel these terms. However, it follows from Lemma 4.2.2 and Lemma 4.2.4 that there are no exceptions for inequalities generated by the particular words $(gf)^n\, g$.

Lemma 4.2.1 *Let $\langle f,\, g\rangle$ be a Kleinian group. If f^n is not the identity for $n \in \mathbb{N}$, then $\langle f^n,\, g\rangle$ is a Kleinian subgroup of $\langle f,\, g\rangle$.*

Proof. Clearly, $\langle f^n,\, g\rangle$ is a subgroup of $\langle f,\, g\rangle$. Since $\langle f,\, g\rangle$ is a Kleinian group, by Theorem 3.2.14, $\langle f^n,\, g\rangle$ is discrete. The only issue is if it is Kleinian. If f is parabolic, loxodromic, or elliptic of order $p \geq 7$, this is true as long as f^n is not the identity. Otherwise, f is elliptic of order $p \leq 6$. If f^n and g have a common fixed point for some $n \in \mathbb{N}$, then so does f and g. If f^n and g have no common fixed points for all $n \in \mathbb{N}$, by the classification of elementary groups Theorem 2.4.9, then $\langle f^n,\, g\rangle$ is one of the finite spherical triangle groups A_4, S_4, and A_5 for each $n \in \mathbb{N}$. Thus, either case $\langle f,\, g\rangle$ can't be Kleinian, it is a contradiction. □

Lemma 4.2.2 *Let f and g be Möbius transformations, then*

$$\gamma\left((gf)^n g, f\right) = \gamma\left((gf)^{n+1}, f\right), \;\; \textit{for } n \in \mathbb{N} \cup \{0\}.$$

Proof. Let $w = (gf)^n g$, then $\gamma(w, f) = \gamma(wf, f)$ for $n \in \mathbb{N} \cup \{0\}$. By using Proposition 2.3.9 and Lemma 3.1.18, $\gamma(f, f^m w f^n) = \gamma(f^m w f^n, f) = \gamma(w, f)$. If $m = 0$ and $n = 1$, then $\gamma(wf, f) = \gamma(w, f)$ and hence $\gamma((gf)^n g, f) = \gamma((gf)^n gf, f)$. □

Corollary 4.2.3 *If $\langle f,\, g\rangle$ is a Kleinian group, then*

$$\gamma((gf)^n g, f) \neq 0, \;\; \textit{for } n \in \mathbb{N}.$$

Proof. Suppose that $\gamma((gf)^n g,\, f) = 0$, for some n. Then by Lemma 4.2.2, $\gamma((gf)^{n+1},\, f) = 0$, for some n. It follows from Theorem 2.3.12 that $(gf)^{n+1}$ and f share a fixed point for some n. Thus, $\langle (gf)^{n+1},\, f\rangle$ is elementary, a contradiction to Lemma 4.2.4. □

In fact, by the following lemma, $\left\langle (gf)^{n+1}, f\right\rangle$ is a Kleinian group and hence $\gamma\left((gf)^n g, f\right) = \gamma\left((gf)^{n+1}, f\right) \neq 0$.

Lemma 4.2.4 *Let $\langle f, g\rangle$ be a Kleinian group. If $(gf)^n$ is not the identity for $n \in \mathbb{N}$, then $\langle (gf)^n, f\rangle$ is a Kleinian subgroup of $\langle f, g\rangle$.*

Proof. It is clear that $\langle (gf)^n, f\rangle$ is a subgroup of $\langle gf, f\rangle$. Since $\langle gf, f\rangle = \langle f, g\rangle$, $\langle gf, f\rangle$ is a Kleinian group. Applying for Lemma 4.2.1, $\langle (gf)^n, f\rangle$ is a Kleinian subgroup of $\langle gf, f\rangle$ and hence $\langle f, g\rangle$. □

Lemma 4.2.5 *Let $\langle f, g\rangle$ be a Kleinian group with the triple of parameters (γ, β, β') and suppose that gf is loxodromic or elliptic. Then for all x at a minimum of the sum $|\gamma| + |\beta + x|$:*

$$|\gamma - \beta - 4| \leq 2\left|T_{n+1}\left(\frac{1}{2}(\gamma - \beta - 2)\right) - 1\right|, \text{ for } n \in \mathbb{N}.$$

Proof. Since $\langle f, g\rangle$ is a Kleinian group, by Lemma 4.2.4, $\left\langle (gf)^{n+1}, f\right\rangle$ is a Kleinian group.

Since the minimum of the sum $|\gamma| + |\beta + x|$ is attained,

$$|\gamma| + |\beta + x| \leq \left|\gamma\left((gf)^{n+1}, f\right)\right| + |\beta + x|,$$

which gives $|\gamma| \leq |\gamma\left((gf)^{n+1}, f\right)|$. Since Theorem 4.1.2 gives $\gamma((gf)^{n+1}, f) = \frac{T_{n+1}(\cosh(\tau))-1}{\cosh(\tau)-1}\gamma$,

$$|\gamma| \leq \left|\frac{T_{n+1}(\cosh(\tau)) - 1}{\cosh(\tau) - 1}\right| |\gamma|.$$

Since $\gamma \neq 0$, dividing by $|\gamma|$ gives $\left|\frac{T_{n+1}(\cosh(\tau))-1}{\cosh(\tau)-1}\right| \geq 1$. Thus,

$$|\cosh(\tau) - 1| \leq |T_{n+1}(\cosh(\tau)) - 1|.$$

By using identity (4.18), $\left|\frac{1}{2}(\gamma - \beta - 2) - 1\right| \leq \left|T_{n+1}(\frac{1}{2}(\gamma - \beta - 2)) - 1\right|$.

Thus, it gives the following inequality at a minimum,

$$|\gamma - \beta - 4| \leq 2 \left| T_{n+1}(\frac{1}{2}(\gamma - \beta - 2)) - 1 \right|.$$

□

First, apply Lemma 4.2.5 for the Chebychev polynomial T_2, it provides a new approach of proving Jørgensen's inequality. In fact, $T_2 = 2z^2 - 1$ and hence

$$\begin{aligned}
|\gamma - \beta - 4| &\leq 2 \left| T_2(\frac{1}{2}(\gamma - \beta - 2)) - 1 \right| \\
&= 2 \left| 2 \left(\frac{1}{4}(\gamma - \beta - 2)^2 \right) - 2 \right| \\
&= \left| (\gamma - \beta - 2)^2 - 4 \right| \\
&\leq |\gamma - \beta - 4| \, |\gamma - \beta|.
\end{aligned}$$

Suppose that $\gamma \neq \beta + 4$, dividing by $|\gamma - \beta - 4| \neq 0$ gives $1 \leq |\gamma - \beta| \leq |\gamma| + |\beta|$. It is the Jørgensen's inequality:

$$|\gamma| + |\beta| \geq 1.$$

Theorem 4.2.6 *Let $\langle f, g \rangle$ be a Kleinian group with the parameters $\beta = \beta(f)$ and $\gamma = \gamma(f, g) \neq \beta + 4$. Then for a minimum of the sum $|\gamma| + |\beta + 1|$,*

$$|\gamma| + |\beta + 1| \geq 1.$$

Proof. Consider the Chebychev polynomial $T_3(z) = 4z^3 - 3z$, by Lemma 4.2.5,

$$\begin{aligned}
|\gamma - \beta - 4| &\leq \left|T_3(\frac{1}{2}(\gamma - \beta - 2)) - 1\right| \\
&= 2\left|\frac{4}{8}(\gamma - \beta - 2)^3 - \frac{3}{2}(\gamma - \beta - 2) - 1\right| \\
&= \left|(\gamma - \beta - 2)^3 - 3(\gamma - \beta - 2) - 2\right| \\
&= \left|(\gamma - \beta - 2 - 2)\,(\gamma - \beta - 2 + 1)^2\right| \\
&= \left|(\gamma - \beta - 4)\,(\gamma - \beta - 1)^2\right| \\
&= \left|(\gamma - \beta - 4)\,(\gamma - (\beta + 1))^2\right| \\
&\leq |\gamma - \beta - 4|\,|\gamma - (\beta + 1)\,|^2.
\end{aligned}$$

Since $\gamma \neq \beta + 4, \gamma - \beta - 4 \neq 0$. Dividing by $|\gamma - \beta - 4|$,

$$1 \leq |\gamma - (\beta + 1)\,|^2.$$

It follows that $1 \leq |\gamma - (\beta + 1)\,| \leq |\gamma| + |\beta + 1|$. Thus,

$$1 \leq |\gamma| + |\beta + 1|.$$

□

Theorem 4.2.7 *Let $\langle f,\, g\rangle$ be a Kleinian group with the parameters $\beta = \beta\,(f)$ and $\gamma = \gamma\,(f,\, g) \neq \beta + 4$. Then for a minimum of the sum $|\gamma| + |\beta + 2|$:*

$$|\gamma| + |\beta + 2| \geq \frac{\sqrt{5} - 1}{2}.$$

This inequality is sharp for the $(2, 4, 5)$ hyperbolic triangle group with the parameters $\left(\frac{\sqrt{5}-1}{2}, -2, -4\right)$.

Proof. Consider the Chebychev polynomial $T_4 = 8z^4 - 8z^2 + 1$, by Lemma 4.2.5,

$$\begin{aligned}
|\gamma-\beta-4| &\le 2|T_4(\frac{1}{2}(\gamma-\beta-2))-1| \\
&= 2\left|\frac{8}{16}(\gamma-\beta-2)^4 - \frac{8}{4}(\gamma-\beta-2)^2 + 1 - 1\right| \\
&= \left|(\gamma-\beta-2)^4 - 4(\gamma-\beta-2)^2\right| \\
&= \left|(\gamma-\beta-2)^2\left((\gamma-\beta-2)^2-4\right)\right| \\
&= \left|(\gamma-\beta-2)^2(\gamma-\beta)\,(\gamma-\beta-4)\right| \\
&\le |\gamma-\beta-2|^2|\gamma-\beta|\,|\gamma-\beta-4|\,.
\end{aligned}$$

Since $\gamma \neq \beta + 4, \gamma - \beta - 4 \neq 0$. Dividing by $|\gamma - \beta - 4|$ gives

$$\begin{aligned}
1 &\le |\gamma-\beta-2|^2|\gamma-\beta| \\
&= |\gamma-\beta-2|^2|\gamma-\beta-2+2| \\
&\le |\gamma-\beta-2|^2\left(|\gamma-\beta-2|+2\right).
\end{aligned}$$

Let $x = |\gamma - \beta - 2|$, then $x \le |\gamma| + |\beta + 2|$ and $1 \le x^2\,(x+2)$. Solving the latter inequality gives $x \ge \frac{\sqrt{5}-1}{2}$. Hence, $\frac{\sqrt{5}-1}{2} \le x \le |\gamma| + |\beta+2|$. So it is concluded that

$$|\gamma| + |\beta+2| \ge \frac{\sqrt{5}-1}{2}.$$

By Example 3.1.2, $(2,4,5)$ hyperbolic triangle group is the Kleinian group $\langle f, g\rangle$ with the triple of parameters $(\gamma\,(f,\,g)\,,\beta\,(f)\,,\beta(g)) = \left(\frac{\sqrt{5}-1}{2}, -2, -4\right)$. Thus, $|\beta+2|$ vanishes and hence $|\gamma| + |\beta+2| = \frac{\sqrt{5}-1}{2}$, this is the verification of the sharpness. □

Theorem 4.2.8 *Let $\langle f,\,g\rangle$ be a Kleinian group with the parameters $\beta = \beta\,(f)$*

and $\gamma = \gamma(f, g) \neq \beta + 4$. *Then for a minimum of the sum* $|\gamma| + \left|\beta + \frac{3\pm\sqrt{5}}{2}\right|$,

$$|\gamma| + \left|\beta + \frac{3 \pm \sqrt{5}}{2}\right| \geq \frac{3-\sqrt{5}}{2}.$$

Proof. Consider the Chebychev polynomial $T_5 = 16z^5 - 20z^3 + 5z$. Referring to the inequality in Lemma 4.2.5

$$\begin{aligned}|\gamma - \beta - 4| &\leq 2|T_5(\frac{1}{2}(\gamma - \beta - 2)) - 1| \\ &\leq 2\left|\frac{16}{32}(\gamma - \beta - 2)^5 - \frac{20}{8}(\gamma - \beta - 2)^3 + \frac{5}{2}(\gamma - \beta - 2) - 1\right|,\end{aligned}$$

Thus,

$$|\gamma - \beta - 4| \leq \left|(\gamma - \beta - 2)^5 - 5(\gamma - \beta - 2)^3 + 5(\gamma - \beta - 2) - 2\right|. \qquad (4.19)$$

Notice that $\gamma \neq \beta + 4$ and hence $\gamma - \beta - 4 \neq 0$. Let $x = \gamma - \beta - 2$, then $\gamma - \beta - 4 = x - 2 \neq 0$, and hence the previous inequality (4.19) gives

$$\begin{aligned}|x - 2| &\leq \left|x^5 - 5x^3 + 5x - 2\right| \\ &= \left|(x-2)\left(x^2 + x - 1\right)^2\right| \\ &\leq |x - 2|\left|x^2 + x - 1\right|^2\end{aligned}$$

Dividing $|x-2|$, which gives $1 \leq |x^2+x-1|^2$, i.e., $|x^2+x-1| \geq 1$. Furthermore,

$$\begin{aligned}
1 &\leq |x^2+x-1| \\
&= \left|\left(x - \frac{-1+\sqrt{5}}{2}\right)\left(x - \frac{-1-\sqrt{5}}{2}\right)\right| \\
&= \left|(\gamma-\beta-2) - \frac{-1+\sqrt{5}}{2}\right|\left|(\gamma-\beta-2) - \frac{-1-\sqrt{5}}{2}\right| \\
&= \left|\gamma-\beta-\frac{3+\sqrt{5}}{2}\right|\left|\gamma-\beta-\frac{3-\sqrt{5}}{2}\right|,
\end{aligned}$$

which gives

$$\left|\gamma-\beta-\frac{3+\sqrt{5}}{2}\right|\left|\gamma-\beta-\frac{3-\sqrt{5}}{2}\right| \geq 1. \tag{4.20}$$

There are two different ways of argument in the following:

The first way of argument starts with rearranging the first factor of the previous inequality (4.20:

$$\begin{aligned}
1 &\leq \left|\gamma-\beta-\frac{3+\sqrt{5}}{2}\right|\left|\gamma-\beta-\frac{3-\sqrt{5}}{2}\right| \\
&= \left|\gamma-\beta-\frac{3-\sqrt{5}}{2} - \sqrt{5}\right|\left|\gamma-\beta-\frac{3-\sqrt{5}}{2}\right| \\
&\leq \left(\left|\gamma-\beta-\frac{3-\sqrt{5}}{2}\right| + \sqrt{5}\right)\left|\gamma-\beta-\frac{3-\sqrt{5}}{2}\right| \\
&= \left|\gamma-\beta-\frac{3-\sqrt{5}}{2}\right|^2 + \sqrt{5}\left|\gamma-\beta-\frac{3-\sqrt{5}}{2}\right|.
\end{aligned}$$

Let $s = \left|\gamma-\beta-\frac{3-\sqrt{5}}{2}\right|$ in the inequality above, then the solution of $1 \leq s^2+\sqrt{5}s$ is $s \geq \frac{3-\sqrt{5}}{2} \geq 0$. Similarly,

$$\frac{3-\sqrt{5}}{2} \leq s = \left|\gamma-\beta-\frac{3-\sqrt{5}}{2}\right| \leq |\gamma| + \left|\beta+\frac{3-\sqrt{5}}{2}\right|.$$

Hence,

$$|\gamma| + \left|\beta + \frac{3-\sqrt{5}}{2}\right| \geq \frac{3-\sqrt{5}}{2}. \tag{4.21}$$

The second way of argument is to rearrange the second factor of the inequality on (4.20):

$$\begin{aligned}
1 &\leq \left|\gamma - \beta - \frac{3+\sqrt{5}}{2}\right| \left|\gamma - \beta - \frac{3-\sqrt{5}}{2}\right| \\
&= \left|\gamma - \beta - \frac{3+\sqrt{5}}{2}\right| \left|\gamma - \beta - \frac{3+\sqrt{5}}{2} + \sqrt{5}\right| \\
&\leq \left|\gamma - \beta - \frac{3+\sqrt{5}}{2}\right| \left(\left|\gamma - \beta - \frac{3+\sqrt{5}}{2}\right| + \sqrt{5}\right) \\
&= \left|\gamma - \beta - \frac{3+\sqrt{5}}{2}\right|^2 + \sqrt{5}\left|\gamma - \beta - \frac{3+\sqrt{5}}{2}\right|.
\end{aligned}$$

Let $y = \left|\gamma - \beta - \frac{3+\sqrt{5}}{2}\right|$, then the inequality above becomes $1 \leq y^2 + \sqrt{5}y$, solving it gives $y \geq \frac{3-\sqrt{5}}{2} \geq 0$. Hence,

$$\frac{3-\sqrt{5}}{2} \leq y = \left|\gamma - \beta - \frac{3+\sqrt{5}}{2}\right| \leq |\gamma| + \left|\beta + \frac{3+\sqrt{5}}{2}\right|.$$

Thus,

$$|\gamma| + \left|\beta + \frac{3+\sqrt{5}}{2}\right| \geq \frac{3-\sqrt{5}}{2}. \tag{4.22}$$

Finally, one can conclude from the inequalities (4.22) and (4.21) that

$$|\gamma| + \left|\beta + \frac{3\pm\sqrt{5}}{2}\right| \geq \frac{3-\sqrt{5}}{2}.$$

□

Theorem 4.2.9 *Let $\langle f, g\rangle$ be a Kleinian group with the parameters $\beta = \beta(f)$*

and $\gamma = \gamma(f, g) \neq \beta + 4$. *Then for a minimum of the sum* $|\gamma| + \left|\beta + 2 + \sqrt{2}\right|$,

$$|\gamma| + \left|\beta + 2 + \sqrt{2}\right| \geq 0.117875.$$

Proof. Now consider the Chebychev polynomial $T_8(z) = 128z^8 - 256z^6 + 160z^4 - 32z^2 + 1$. Applying for Lemma 4.2.5,

$$\begin{aligned}|\gamma - \beta - 4| &\leq 2|T_8(\tfrac{1}{2}(\gamma - \beta - 2)) - 1| \\ &= 2\left|\tfrac{128}{256}(\gamma - \beta - 2)^8 - \tfrac{256}{64}(\gamma - \beta - 2)^6 + \tfrac{160}{16}(\gamma - \beta - 2)^4 - \tfrac{32}{4}(\gamma - \beta - 2)^2\right|.\end{aligned}$$

Thus,

$$|\gamma - \beta - 4| \leq \left|(\gamma - \beta - 2)^8 - 8(\gamma - \beta - 2)^6 + 20(\gamma - \beta - 2)^4 - 16(\gamma - \beta - 2)^2\right|$$

Let $x = \gamma - \beta - 2$, then $\gamma - \beta - 4 = x - 2 \neq 0$ as $\gamma \neq \beta + 4$ and hence the previous inequality becomes

$$\begin{aligned}|x - 2| &\leq \left|x^8 - 8x^6 + 20x^4 - 16x^2\right| \\ &= \left|x^2(x-2)(x+2)\left(x^2 - 2\right)^2\right| \\ &\leq |x|^2 |x - 2| |x + 2| \left|\left(x^2 - 2\right)\right|^2,\end{aligned}$$

which gives $1 \leq |x|^2 |x + 2| |(x^2 - 2)|^2$. Furthermore,

$$\begin{aligned}1 &\leq |x|^2 |x + 2| \left|\left(x^2 - 2\right)\right|^2 \\ &= |\gamma - \beta - 2|^2 |\gamma - \beta| \left|(\gamma - \beta - 2)^2 - 2\right|^2 \\ &= |\gamma - \beta - 2|^2 |\gamma - \beta| \left|\left((\gamma - \beta - 2) - \sqrt{2}\right)\left((\gamma - \beta - 2) + \sqrt{2}\right)\right|^2 \\ &\leq \left(\left|\gamma - \beta - 2 - \sqrt{2}\right| + \sqrt{2}\right)^2 \left(\left|\gamma - \beta - 2 - \sqrt{2}\right| + 2 + \sqrt{2}\right) \\ &\cdot \left|\gamma - \beta - 2 - \sqrt{2}\right|^2 \left(\left|\gamma - \beta - 2 - \sqrt{2}\right| + 2\sqrt{2}\right)^2.\end{aligned}$$

Let $y = \left|\gamma - \beta - 2 - \sqrt{2}\right|$ then

$$1 \leq \left(y + \sqrt{2}\right)^2 \left(y + 2 + \sqrt{2}\right) y^2 \left(y + 2\sqrt{2}\right)^2 .$$

Solving the inequality gives $0.117875 \leq y$ and hence

$$\begin{aligned} 0.117875 \leq y &= \left|\gamma - \beta - 2 - \sqrt{2}\right| \\ &\leq |\gamma| + \left|\beta + 2 + \sqrt{2}\right| . \end{aligned}$$

Therefore,

$$|\gamma| + \left|\beta + 2 + \sqrt{2}\right| \geq 0.117875.$$

□

4.3 Trace polynomials linear in β

In this final section, an infinite family of the trace polynomials of two complex variables γ and β, which are linear in β, is given in Theorem 4.3.2, and then using these polynomials and the established inequalities in the previous section complete the quantifiable universal constraints by inequalities in the scheme including Theorems 4.3.4, 4.3.5, 4.3.7, 4.3.8 and 4.3.10.

Starting with the following lemma show that the subgroups $\langle [g,\, f]^n ,\, f\rangle$ of Kleinian group $\langle f,\, g\rangle$ is Kleinian for $n \in \mathbb{N}$.

Lemma 4.3.1 *Let $\langle f,\, g\rangle$ be a Kleinian group. If f is loxodromic or parabolic or elliptic of order $p \geq 6$, and $[g,\, f]^n$ is not the identity, then $\langle [g,\, f]^n ,\, f\rangle$ is a Kleinian subgroup of $\langle f,\, g\rangle$ for $n \in \mathbb{N}$.*

Proof. Obviously, $\langle [g,\, f]^n ,\, f\rangle$ is a subgroup of $\langle [g,\, f] ,\, f\rangle$. Since f is loxodromic or parabolic or elliptic of order $p \geq 7$, by Theorem 3.2.6, then $\langle f,\, gfg^{-1}\rangle$

is a Kleinian group. Notice that

$$\langle [g,\, f]\,,\, f\rangle = \langle f,\, [g,\, f]\rangle = \left\langle f, gfg^{-1}f^{-1}\right\rangle = \left\langle f,\, gfg^{-1}\right\rangle,$$

so $\langle f,\, [g,\, f]\rangle$ is a Kleinian group. Applying for Lemma 4.2.1, $\langle [g,\, f]^n\,,\, f\rangle$ is a Kleinian subgroup of $\langle f,\, gfg^{-1}\rangle$ and hence $\langle f,\, g\rangle$. □

There are infinitely many trace polynomials of two complex variables γ and β which are linear in β. For example, if con considers the word $w_1\,(g,\, f) = g$, then $p_{w_1}\,(\gamma,\, \beta) = \gamma\,(f,\, g) = \gamma$ is of course, a polynomial linear in β. If the subgroup $\langle f,\, gfg^{-1}\rangle$ of $\langle f,\, g\rangle$ is Kleinian, then $\langle f,\, [g,\, f]\rangle$ is Kleinian subgroup of $\langle f,\, g\rangle$ as well, because $\langle f,\, gfg^{-1}\rangle = \langle f, gfg^{-1}f^{-1}\rangle = \langle f,\, [g,\, f]\rangle$. The identity (3.4) gives the following trace polynomial from the word $w_2\,(g,\, f) = gfg^{-1}$:

$$p_{w_2}\,(\gamma,\, \beta) = \gamma(f, gfg^{-1}) = \gamma\,(\gamma - \beta)\,.$$

By taking $w = gfg^{-1}$ and $(m, n) = (0, -1)$, Lemma 3.1.18 gives the trace polynomial of the word $w_3\,(g,\, f) = [g,\, f]$:

$$p_{w_3}\,(\gamma,\, \beta) = \gamma\,(f, [g,\, f]) = \gamma\left(f, gfg^{-1}\right) = \gamma\,(\gamma - \beta)\,. \tag{4.23}$$

In addition, $\beta([g,\, f]) = \mathrm{tr}^2\,([g,\, f]) - 4 = (\gamma\,(g,\, f) + 2)^2 - 4 = \gamma^2 + 4\gamma$, that is

$$\beta([g,\, f]) = \gamma(\gamma + 4). \tag{4.24}$$

Thus, trace polynomials $p_{w_2}\,(\gamma,\, \beta)$ and $p_{w_3}\,(\gamma,\, \beta)$ are linear in β as well. Moreover, if the subgroup $\langle f,\, gfg^{-1}\rangle$ of $\langle f,\, g\rangle$ is Kleinian, then $\langle f, [g,\, f]^{n+1}\rangle$ is Kleinian. It is natural to ask whether the infinite family of trace polynomials of the word $w\,(g,\, f) = [g,\, f]^{n+1}$ for each $n \in \mathbb{N}$ is linear in β, the following theorem shows the confirmative answer.

Theorem 4.3.2 *Suppose that* $\langle f, g\rangle$ *is a Kleinian group with the triple of parameters* (γ, β, β'), *where* $[g, f]$ *is elliptic or loxodromic. Then the following recursion formulas for the trace polynomials* $\gamma_{n+1} = \gamma(f, [g, f]^{n+1})$ *are held,*

$$\gamma_0 = 0, \quad \gamma_1 = \gamma(\gamma-\beta), \quad \gamma_{n+1} = (\gamma^2+4\gamma+2)\gamma_n - \gamma_{n-1} + 2\gamma(\gamma-\beta), \text{ for } n \in \mathbb{N}.$$

In particular,

$$\gamma_2 = \gamma(\gamma-\beta)(\gamma+2)^2, \tag{4.25}$$

$$\gamma_3 = \gamma(\gamma-\beta)(\gamma+1)^2(\gamma+3)^2, \tag{4.26}$$

$$\gamma_4 = \gamma\,(\gamma-\beta)\,(\gamma+2)^2\left(\gamma^2+4\gamma+2\right)^2, \tag{4.27}$$

$$\gamma_5 = \gamma\,(\gamma-\beta)\left(\gamma^2+3\gamma+1\right)^2\left(\gamma^2+5\gamma+5\right)^2. \tag{4.28}$$

Proof. According to the identity (4.12),

$$\beta([g, f]^n) = 4\sinh^2(\frac{n\tau}{2}) = 2\cosh(n\tau) - 2, \text{ for } n \in \mathbb{N}. \tag{4.29}$$

In particular, $\beta([g, f]) = 4\sinh^2(\frac{\tau}{2}) = 2\cosh(\tau) - 2$. Thus the identity (4.24) implies

$$2\cosh(\tau) - 2 = \gamma(\gamma+4) \text{ and } \cosh(\tau) = \frac{\gamma^2+4\gamma+2}{2}. \tag{4.30}$$

Applying for the identities (4.8), (4.29), (4.6), and (4.30),

$$\begin{aligned}
\gamma(f,[g,\,f]^n) &= \frac{\beta\left([g,\,f]^n\right)}{\beta\left([g,\,f]\right)}\gamma\left(f,[g,\,f]\right) \\
&= \frac{2\cosh(n\tau)-2}{2\cosh(\tau)-2}\gamma(\gamma-\beta) \\
&= \left(2T_n(\cosh(\tau))-2\right)\frac{\gamma-\beta}{\gamma+4} \\
&= \left(2T_n\left(\frac{\gamma^2+4\gamma+2}{2}\right)-2\right)\frac{\gamma-\beta}{\gamma+4}, \text{ for } n\in\mathbb{N}.
\end{aligned}$$

Thus, by Chebychev polynomials recursion formula (4.2),

$$\begin{aligned}
\gamma_{n+1} &= \left(2T_{n+1}\left(\frac{\gamma^2+4\gamma+2}{2}\right)-2\right)\frac{\gamma-\beta}{\gamma+4} \\
&= 4\left(\frac{\gamma^2+4\gamma+2}{2}\right)T_n\left(\frac{\gamma^2+4\gamma+2}{2}\right)\,\frac{\gamma-\beta}{\gamma+4} \\
&- \left(2T_{n-1}\left(\frac{\gamma^2+4\gamma+2}{2}\right)+4\right)\frac{\gamma-\beta}{\gamma+4} \\
&= \left(2\left(\gamma^2+4\gamma+2\right)T_n\left(\frac{\gamma^2+4\gamma+2}{2}\right)-4\right)\,\frac{\gamma-\beta}{\gamma+4} \\
&- \left(2T_{n-1}\left(\frac{\gamma^2+4\gamma+2}{2}\right)-2\right)\frac{\gamma-\beta}{\gamma+4} \\
&= \left(\gamma^2+4\gamma+2\right)\left(2T_n\left(\frac{\gamma^2+4\gamma+2}{2}\right)-2\right)\,\frac{\gamma-\beta}{\gamma+4} \\
&+ \left(2\left(\gamma^2+4\gamma+2\right)-4\,\right)\frac{\gamma-\beta}{\gamma+4}-\gamma_{n-1} \\
&= (\gamma^2+4\gamma+2)\gamma_n-\gamma_{n-1}+2\gamma(\gamma-\beta).
\end{aligned}$$

It is clear that $\gamma_0 = 0$, and the trace polynomial (4.23) gives

$$\gamma_1 = \gamma(f,\,[g,\,f]) = \gamma(\gamma-\beta).$$

Applying for the recursion formulas $\gamma_{n+1} = (\gamma^2+4\gamma+2)\gamma_n - \gamma_{n-1} + 2\gamma(\gamma-\beta)$

for $n = 2, 3, 4$, and 5, the following trace polynomials are given,

$$
\begin{aligned}
\gamma_2 &= (\gamma^2 + 4\gamma + 2)\gamma_1 - \gamma_0 + 2\gamma(\gamma - \beta) \\
&= (\gamma^2 + 4\gamma + 2)\gamma(\gamma - \beta) + 2\gamma(\gamma - \beta) \\
&= \gamma(\gamma - \beta)(\gamma + 2)^2, \\
\gamma_3 &= (\gamma^2 + 4\gamma + 2)\gamma_2 - \gamma_1 + 2\gamma(\gamma - \beta) \\
&= (\gamma^2 + 4\gamma + 2)\gamma(\gamma - \beta)(\gamma + 2)^2 - \gamma(\gamma - \beta) + 2\gamma(\gamma - \beta) \\
&= \gamma(\gamma - \beta)(\gamma + 1)^2(\gamma + 3)^2, \\
\gamma_4 &= (\gamma^2 + 4\gamma + 2)\gamma_3 - \gamma_2 + 2\gamma(\gamma - \beta) \\
&= (\gamma^2 + 4\gamma + 2)\gamma(\gamma - \beta)(\gamma + 1)^2(\gamma + 3)^2 - \gamma(\gamma - \beta)(\gamma + 2)^2 + 2\gamma(\gamma - \beta) \\
&= \gamma\left(\gamma - \beta\right)\left(\gamma + 2\right)^2\left(\gamma^2 + 4\gamma + 2\right)^2, \\
\gamma_5 &= (\gamma^2 + 4\gamma + 2)\gamma_4 - \gamma_3 + 2\gamma(\gamma - \beta) \\
&= (\gamma^2 + 4\gamma + 2)\gamma\left(\gamma - \beta\right)\left(\gamma + 2\right)^2\left(\gamma^2 + 4\gamma + 2\right)^2 \\
&- \gamma(\gamma - \beta)(\gamma + 1)^2(\gamma + 3)^2 + 2\gamma(\gamma - \beta) \\
&= \gamma\left(\gamma - \beta\right)\left(\gamma^2 + 3\gamma + 1\right)^2\left(\gamma^2 + 5\gamma + 5\right)^2.
\end{aligned}
$$

□

Next turn the attention to create the inequalities in the following theorems by using the trace polynomial $\gamma(f, [g,\ f]^2) = \gamma(\gamma - \beta)(\gamma + 2)^2$ or $\gamma(f, [g,\ f]^3) = \gamma(\gamma - \beta)(\gamma + 1)^2(\gamma + 3)^2$.

Theorem 4.3.3 *Let $\langle f,\ g\rangle$ be a Kleinian group, and let $\gamma = \gamma\left(f,\ g\right)$ and $\beta = \beta\left(f\right)$. If f is loxodromic or parabolic or elliptic of order $p \geq 6$, then*

$$|\gamma + 2| + |\beta| \geq \sqrt{2} - 1.$$

Proof. Suppose that $|\gamma + 2| + |\beta| < r$ for some $r > 0$, then $|\gamma + 2| < r$.

By Lemma 4.3.1, $\left\langle f,\ [g,\ f]^2\right\rangle$ is a Kleinian group and the identity (4.25) gives the parameters $\gamma_2 = \gamma(\gamma-\beta)(\gamma+2)^2$, then γ_2 and β fit the Jørgensen's inequality $|\gamma|+|\beta| \geq 1$:

$$\begin{aligned} 1 &\leq |\gamma_2| + |\beta| \\ &\leq |\gamma(\gamma-\beta)(\gamma+2)^2| + |\beta| \\ &\leq (|\gamma+2|+2)\,(|\gamma+2|+|\beta|+2)\,|\gamma+2|\,|\gamma+2| + |\beta|. \end{aligned}$$

By the assumption at the beginning,

$$(r+2)^2 r|\gamma+2| + |\beta| > 1. \tag{4.31}$$

If $(r+2)^2 r \leq 1$, then (4.31) becomes $1 < (r+2)^2 r|\gamma+2|+|\beta| \leq |\gamma+2|+|\beta|$, i.e., $|\gamma+2|+|\beta| > 1$. On the other hand, solving the inequality $(r+2)^2 r \leq 1$ gives $r \leq 0.205\,57 < 1$ and infers $|\gamma+2|+|\beta| < r < 1$. It contradicts to $|\gamma+2|+|\beta| > 1$.

Thus, $(r+2)^2 r > 1$, then (4.31) arrives at

$$\begin{aligned} 1 &< (r+2)^2 r|\gamma+2| + |\beta| < (r+2)^2 r|\gamma+2| + (r+2)^2 r|\beta| \\ &= (r+2)^2 r\,(|\gamma+2|+|\beta|) < (r+2)^2 r^2. \end{aligned}$$

i.e., $(r+2)^2 r^2 > 1$, solving the inequality gives $r > \sqrt{2}-1$.

So if one assumes $|\gamma+2|+|\beta+1| < r$ then $r > \sqrt{2}-1$. Equivalently, if some $r \leq \sqrt{2}-1$ then $|\gamma+2|+|\beta| \geq r$. So one can take the largest lower bound $r = \sqrt{2}-1$, then this yields $|\gamma+2|+|\beta| \geq \sqrt{2}-1$. The proof is completed. □

Theorem 4.3.4 *Let $\langle f, g\rangle$ be a Kleinian group, and let $\gamma = \gamma(f, g)$ and $\beta = \beta(f)$. If f is loxodromic or parabolic or elliptic of order $p \geq 6$, then*

$$|\gamma+2| + |\beta+2| \geq \frac{\sqrt{5}-1}{2} = 0.618\,034\cdots.$$

This inequality is sharp for the $\mathbb{Z}_2$*-extension of* $(4,4,5)$ *hyperbolic triangle group with the triple of parameters* $\left(\frac{-\sqrt{5}-3}{2}, -2, -4\right)$.

Proof. (1) Assume that $|\gamma+2|+|\beta+2| < \frac{\sqrt{5}-1}{2}$ then there is a contradiction.

Recall the inequality from Theorem 4.2.7,

$$|\gamma|+|\beta+2| \geq \frac{\sqrt{5}-1}{2}.$$

By Lemma 4.3.1, $\langle f, [g,\, f]^2\rangle$ is a Kleinian group and hence β and $\gamma_2 = \gamma(\gamma-\beta)(\gamma+2)^2$ giving by the identity (4.25) fit the inequality:

$$|\gamma_2|+|\beta+2| \geq \frac{\sqrt{5}-1}{2}.$$

Since $|\gamma+2|+|\beta+2| < \frac{\sqrt{5}-1}{2}$, $|\gamma+2| < |\gamma+2|+|\beta+2| < \frac{\sqrt{5}-1}{2}$ and $|\beta+2| < |\gamma+2|+|\beta+2| < \frac{\sqrt{5}-1}{2}$.

It follows that

$$\begin{aligned}
\frac{\sqrt{5}-1}{2} &\leq |\gamma(\gamma-\beta)(\gamma+2)^2|+|\beta+2| \\
&\leq (|\gamma+2|+2)(|\gamma+2|+|\beta+2|)|\gamma+2||\gamma+2|+|\beta+2| \\
&< (2+\frac{\sqrt{5}-1}{2})\left(\frac{\sqrt{5}-1}{2}\right)^2|\gamma+2|+|\beta+2| \\
&= |\gamma+2|+|\beta+2|.
\end{aligned}$$

Thus, $|\gamma+2|+|\beta+2| > \frac{\sqrt{5}-1}{2}$ that contradicts to the assumption $|\gamma+2|+|\beta+2| < \frac{\sqrt{5}-1}{2}$, which gives the inequality

$$|\gamma+2|+|\beta+2| \geq \frac{\sqrt{5}-1}{2} = 0.618\,034\cdots.$$

(2) Let f be of order 4, then $\beta=-2$ and hence $|\beta+2|=0$ and $|\gamma+2|=$

$\frac{\sqrt{5}-1}{2}$, so $\gamma = \frac{-\sqrt{5}-3}{2}$ or $\gamma = \frac{\sqrt{5}-5}{2}$.

Now choose the first case of γ :

$$\gamma = \frac{-\sqrt{5}-3}{2}.$$

Since $\operatorname{tr}[f,g] = \gamma + 2 = \frac{1-\sqrt{5}}{2} = -2\cos\left(\frac{2\pi}{5}\right)$,

$$\begin{aligned}\beta([f,g]) &= \operatorname{tr}^2[f,g] - 4\\ &= \left(-2\cos\left(\frac{2\pi}{5}\right)\right)^2 - 4\\ &= -4\sin^2(\frac{2\pi}{5}).\end{aligned}$$

By Theorem 2.3.5, $[f,g]$ is elliptic of order 5. Take f and $h = gf^{-1}g^{-1}$ elliptics of order 4 whose product $fh = [f,g]$ is elliptic of order 5. Since $\langle f,\, gfg^{-1}\rangle = \langle f,\, gf^{-1}g^{-1}\rangle$ and the identity 2.11, $\langle f,\, gfg^{-1}\rangle$ is the $(4,4,5)$ hyperbolic triangle group that is a Kleinian group.

Now choose g of order 2, this gives a $\mathbb{Z}_2$-extension Γ of the group $\langle f,\, gfg^{-1}\rangle$ and hence it is a Kleinian group by Lemma 3.2.5 and the triple of parameters for Γ is

$$\left(\frac{-\sqrt{5}-3}{2}, -2, -4\right),$$

which gives the sharpness,

$$\begin{aligned}|\gamma+2| + |\beta+2| &= \left|\frac{-\sqrt{5}-3}{2} + 2\right| + |-2+2|\\ &= \left|\frac{-\sqrt{5}+1}{2}\right| = \frac{\sqrt{5}-1}{2}.\end{aligned}$$

□

Theorem 4.3.5 *Let $\langle f,\, g\rangle$ be a Kleinian group, and let $\gamma = \gamma(f,\, g)$ and $\beta =$*

$\beta(f)$. *If f is loxodromic or parabolic or elliptic of order $p \geq 6$, then*

$$|\gamma+2|+|\beta+1| \geq 0.512876\cdots$$

Proof. The assumption $|\gamma+2|+|\beta+1| < r$ for some $r > 0$ implies $|\gamma+2| < |\gamma+2|+|\beta+1| < r$. i.e., $|\gamma+2| < r$.

Recall the inequality from Theorem 4.2.6

$$|\gamma|+|\beta+1| \geq 1$$

By Lemma 4.3.1, $\langle f, [g,\ f]^2 \rangle$ is a Kleinian group and identity (4.25) gives the parameters $\gamma_2 = \gamma(\gamma-\beta)(\gamma+2)^2$, then γ_2 and β fit the inequality:

$$|\gamma_2|+|\beta+1| \geq 1.$$

Since $|\gamma+2|+|\beta+1| < r$, $|\gamma+2| < r$ and $|\beta+1| < r$ gives

$$\begin{aligned} 1 &\leq |\gamma(\gamma-\beta)(\gamma+2)^2|+|\beta+1| \\ &= |\,(\gamma+2-2)\,(\gamma+2-1-\beta-1)(\gamma+2)^2|+|\beta+1| \\ &\leq (|\gamma+2|+2)\,[(|\gamma+2|+1+|\beta+1|)|\gamma+2|]\,|\gamma+2|+|\beta+1| \\ &< (r+2)(r+1)r|\gamma+2|+|\beta+1|. \end{aligned}$$

i.e., $1 < (r+2)(r+1)r|\gamma+2|+|\beta+1|$.

If $(r+1)(r+2)r \leq 1$, then the above inequality becomes

$$\begin{aligned} 1 &< (r+2)(r+1)r|\gamma+2|+|\beta+1| \\ &\leq |\gamma+2|+|\beta+1|. \end{aligned}$$

That is,

$$|\gamma + 2| + |\beta + 1| > 1.$$

On the other hand side, solving the inequality $(r+2)(r+1)r \leq 1$ gives $r \leq 0.324\,72$ and hence $r < 1$. It follows that $|\gamma+2| + |\beta+1| < r < 1$ and this gives the contradiction to $|\gamma + 2| + |\beta + 1| > 1$.

Otherwise, if $(r+2)(r+1)r > 1$, then using the assumption $|\gamma+2|+|\beta+1| < r$ gives

$$\begin{aligned}1 &< (r+2)(r+1)r|\gamma+2| + |\beta+1| \\ &< (r+2)(r+1)r\left(|\gamma+2| + |\beta+1|\right) \\ &< (2+r)(1+r)r^2.\end{aligned}$$

Solving $(r+2)(r+1)r^2 > 1$ implies $r > 0.512876\cdots$. Thus, if $r \leq 0.512876\cdots$, then $|\gamma+3| + |\beta+2| \geq r$. Take the largest lower bound $r = 0.512876\cdots$, then

$$|\gamma + 2| + |\beta + 1| \geq 0.512876\cdots.$$

□

Theorem 4.3.6 *Let* $\langle f, g\rangle$ *be a Kleinian group and let* $\gamma = \gamma(f, g)$ *and* $\beta = \beta(f)$. *If* $\beta(f) \neq -3$, *then*

$$|\gamma| + |\beta + 3| \geq 3\left(\frac{1-\cos\frac{\pi}{7}}{2}\right)^{\frac{1}{3}} = 1.1016\cdots. \tag{4.32}$$

Proof. If $\beta = -4$ there is nothing to prove. So fixing $\beta = \beta_0 \neq -4$ seek the minimum value for γ, call it γ_0. Using the trace polynomial $g[3, \gamma, \beta] = (\beta+3)^2\gamma(\gamma-\beta)$ in [43] one can see that at the minimum

$$1 \leq |\beta_0 + 3|^2|\gamma_0 - \beta_0|. \tag{4.33}$$

This estimate alone quickly gives a reasonable bound. But it can be improved as follows. An extremal group $\langle f, g\rangle$ is Kleinian and may be assumed to have parameters $(\gamma_0, \beta_0, -4)$. Then, by Theorem 4.1.1, $\langle f^3, g\rangle$ has parameters

$$\left(\gamma\left(f^3,\, g\right), \beta\left(f^3\right), \beta\left(g\right)\right) = (\gamma_0(\beta_0+3)^2,\, \beta_0(\beta+3)^2,\, -4).$$

Thus, by the identity (3.4),

$$\begin{aligned}\gamma(f^3, gf^3g^{-1}) &= \gamma_0(\beta_0+3)^2\left(\gamma_0(\beta_0+3)^2 - \beta_0(\beta_0+3)^2\right)\\ &= \gamma_0(\beta_0+3)^4(\gamma_0-\beta_0)\end{aligned}$$

and $\langle f^3,\, gf^3g^{-1}\rangle$ is a group generated by two elements with the same trace. Thus, see [6],

$$|\gamma(f^3, gf^3g^{-1})| \geq 2 - 2\cos\frac{\pi}{7} = 0.198\cdots,$$

$$|\gamma_0(\beta_0+3)^4(\gamma_0-\beta_0)| \geq 0.198\cdots. \tag{4.34}$$

Using the inequalities (4.33) and (4.34), it is deduced that

$$|\gamma_0||\beta_0+3|^2 \geq 0.198\cdots.$$

The function $(k-t)t^2$ has minimum value $\frac{4k^3}{27}$ on the interval $t \in [0, k]$.

Thus, with $k = |\gamma_0| + |\beta_0+3|$ and $t = |\beta+3|$, it is seen that

$$|\gamma_0| + |\beta_0+3| \geq 3\left(\frac{1-\cos\frac{\pi}{7}}{2}\right)^{\frac{1}{3}} = 1.1016\cdots.$$

□

Theorem 4.3.7 *Let $\langle f,\, g\rangle$ be a Kleinian group, and let $\gamma = \gamma\left(f,\, g\right)$ and $\beta =$*

$\beta(f)$. *If* f *is loxodromic or parabolic or elliptic of order* $p \geq 6$, *then*

$$|\gamma + 2| + |\beta + 3| \geq 0.512876\cdots .$$

Proof. Suppose $|\gamma + 2| + |\beta + 3| < r$, for some $r > 0$. Since the order $p \geq 6$, by Theorem 4.3.6,

$$|\gamma| + |\beta + 3| \geq 1.1016\cdots ,$$

which gives

$$|\gamma| + |\beta + 3| > 1.$$

By Lemma 4.3.1, $\left\langle f,\, [g,\, f]^2 \right\rangle$ is a Kleinian group and identity (4.25) gives the parameters $\gamma_2 = \gamma(\gamma - \beta)(\gamma + 2)^2$, then γ_2 and β fit the inequality:

$$|\gamma_2| + |\beta + 3| > 1.$$

The assumption $|\gamma + 2| + |\beta + 3| < r$ gives $|\gamma + 2| < r$, so

$$\begin{aligned} 1 &< |\gamma(\gamma - \beta)(\gamma + 2)^2| + |\beta + 3| \\ &= |\,(\gamma + 2 - 2)\,(\gamma + 2 + 1 - \beta - 3)(\gamma + 2)^2| + |\beta + 3| \\ &\leq (|\gamma + 2| + 2)\,[(|\gamma + 2| + 1 + |\beta + 3|)|\gamma + 2|]\,|\gamma + 2| + |\beta + 3| \\ &< (r + 2)(r + 1)r|\gamma + 2| + |\beta + 3|. \end{aligned}$$

Thus,

$$1 < (r + 2)(r + 1)r|\gamma + 2| + |\beta + 3|. \tag{4.35}$$

If $(r+2)(r+1)r \leq 1$, then the identity (4.35) becomes

$$\begin{aligned}1 &< (r+2)(r+1)r|\gamma+2| + |\beta+3| \\ &\leq |\gamma+2| + |\beta+3|,\end{aligned}$$

That is

$$|\gamma+2| + |\beta+3| > 1.$$

Solving the inequality $(r+2)(r+1)r \leq 1$ gives $r \leq 0.324\,72$ and hence $r < 1$. It follows that $|\gamma+2| + |\beta+3| < r < 1$ and this contradicts to $|\gamma+2| + |\beta+3| > 1$.

Otherwise, $(r+2)(r+1)r > 1$, then the assumption $|\gamma+2| + |\beta+3| < r$ becomes

$$\begin{aligned}1 &< (r+2)(r+1)r|\gamma+2| + |\beta+3| \\ &< (r+2)(r+1)r|\gamma+2| + (r+2)(r+1)r|\beta+3| \\ &= (r+2)(r+1)r\left(|\gamma+2| + |\beta+3|\right) \\ &< (2+r)(1+r)r^2\end{aligned}$$

Solving $(2+r)(1+r)r^2 > 1$ implies $r > 0.512876\cdots$. Equivalently, if some $r \leq 0.512876\cdots$ then $|\gamma+2| + |\beta| \geq r$. The largest lower bound $r = 0.512876\cdots$ gives the result:

$$|\gamma+2| + |\beta+3| \geq 0.512876\cdots.$$

□

Now taking the trace polynomial $\gamma_3 = \gamma(f, [g, f]^3)$ to establish more inequalities in the following.

Theorem 4.3.8 *Let $\langle f, g\rangle$ be a Kleinian group, and let $\gamma = \gamma(f, g)$ and $\beta =$*

$\beta(f)$. *If* f *is loxodromic or parabolic or elliptic of order* $p \geq 6$, *then*

$$|\gamma + 1| + |\beta| \geq 0.324718.$$

Proof. To get a lower bound for $|\gamma + 1| + |\beta|$:

$$|\gamma + 1| + |\beta| \geq r.$$

Assume $|\gamma + 1| + |\beta| < r$, for some $r > 0$, then $|\gamma + 1| < r$ and $|\beta| < r$.

By Lemma 4.3.1, $\left\langle f, [g, f]^3 \right\rangle$ is a Kleinian group and identity (4.26) provides the parameters $\gamma_3 = \gamma(\gamma - \beta)(\gamma + 1)^2(\gamma + 3)^2$, then γ_3 and β fit the Jørgensen's inequality $|\gamma| + |\beta| \geq 1$:

$$\begin{aligned} 1 &\leq |\gamma_3| + |\beta| \\ &= |\gamma(\gamma - \beta)(\gamma + 1)^2(\gamma + 3)^2| + |\beta| \\ &\leq (|\gamma + 1| + 1)(|\gamma + 1| + |\beta| + 1)(|\gamma + 1| + 2)^2|\gamma + 1||\gamma + 1| + |\beta| \\ &< (r + 1)^2(r + 2)^2 r|\gamma + 1| + |\beta|. \end{aligned}$$

Hence, $1 < (r + 1)^2(r + 2)^2 r|\gamma + 1| + |\beta|$.

If $(r + 1)^2(r + 2)^2 r \leq 1$, then

$$\begin{aligned} 1 &< (r + 1)^2(r + 2)^2 r|\gamma + 1| + |\beta| \\ &\leq |\gamma + 1| + |\beta|. \end{aligned}$$

On the other hand, by solving the inequality $(r + 1)^2(r + 2)^2 r \leq 1$ gives $r \leq 0.159\,5$ and hence$|\gamma + 1| + |\beta| < r < 1$. It is a contradiction to $|\gamma + 1| + |\beta| > 1$.

Now $(r+1)^2(r+2)^2 r > 1$, so

$$\begin{aligned} 1 &< (r+1)^2(r+2)^2 r|\gamma+1| + |\beta| \\ &< (r+1)^2(r+2)^2 r(|\gamma+1| + |\beta|) \\ &< (r+1)^2(r+2)^2 r^2 \end{aligned}$$

By solving $(r+1)^2(r+2)^2 r^2 > 1$ gives $r > 0.324718$. Therefore, if $r \leq 0.324718$, then $|\gamma+3| + |\beta+2| \geq r$. Take the largest lower bound $r = 0.324718$, then

$$|\gamma+1| + |\beta| \geq 0.324718.$$

□

Theorem 4.3.9 *Let $\langle f, g\rangle$ be a Kleinian group, and let $\gamma = \gamma(f, g)$ and $\beta = \beta(f)$. If f is loxodromic or parabolic or elliptic of order $p \geq 6$, then*

$$|\gamma+3| + |\beta| \geq 0.147899.$$

Proof. Suppose that $|\gamma+3| + |\beta| < r$, for some $r > 0$, then $|\gamma+3| < r$ and $|\beta| < r$.

By Lemma 4.3.1, $\langle f, [g, f]^3\rangle$ is a Kleinian group and identity (4.26) provides the parameters $\gamma_3 = \gamma(\gamma-\beta)(\gamma+1)^2(\gamma+3)^2$, then γ_3 and β fit the Jørgensen's inequality $(|\gamma| + |\beta| \geq 1)$:

$$\begin{aligned} 1 &\leq |\gamma(\gamma-\beta)(\gamma+1)^2(\gamma+3)^2| + |\beta| \\ &\leq (|\gamma+3|+3)(|\gamma+3|+|\beta|+3)(|\gamma+3|+2)^2|\gamma+3||\gamma+3| + |\beta| \\ &< (r+3)^2(r+2)^2 r|\gamma+3| + |\beta|. \end{aligned}$$

If $(r+3)^2(r+2)^2 r \leq 1$, then

$$\begin{aligned} 1 &< (r+3)^2(r+2)^2 r|\gamma+3| + |\beta| \\ &\leq |\gamma+3| + |\beta|. \end{aligned}$$

Solving $(r+3)^2(r+2)^2 r \leq 1$ gives $r < 0.02658$ and hence $|\gamma+3|+|\beta| < r$ < 1 and this gives a contradiction to $|\gamma+3|+|\beta| > 1$.

Otherwise, $(r+3)^2(r+2)^2 r > 1$ and hence

$$\begin{aligned} 1 &< (r+3)^2(r+2)^2 r|\gamma+3| + |\beta| \\ &< (r+3)^2(r+2)^2 r\left(|\gamma+3| + |\beta|\right) \\ &< (r+3)^2(r+2)^2 r^2. \end{aligned}$$

By solving $(r+3)^2(r+2)^2 r^2 > 1$ gives $r > 0.147899$. Hence, if $r \leq 0.147899$, then $|\gamma+3| + |\beta+2| \geq r$. Now take the largest lower bound $r = 0.147899$, then

$$|\gamma+3| + |\beta| \geq 0.147899.$$

□

Theorem 4.3.10 *Let $\langle f, g\rangle$ be a Kleinian group, and let $\gamma = \gamma(f, g)$ and $\beta = \beta(f)$. If f is loxodromic or parabolic or elliptic of order $p \geq 6$, then*

$$|\gamma+3| + |\beta+3| \geq 2\cos\left(\frac{2\pi}{7}\right) - 1 = 0.246\,9\cdots.$$

This inequality is sharp for the $\mathbb{Z}_2$-extension of the $(3,3,7)$ hyperbolic triangle group with the triple of parameters $\left(-2-2\cos\left(\frac{2\pi}{7}\right), -3, -4\right)$.

Proof. (1) Suppose that the inequality is not held, i.e., it becomes the following inequality

$$|\gamma+3|+|\beta+3|<2\cos\left(\frac{2\pi}{7}\right)-1.$$

It follows that $|\gamma+3|<2\cos\left(\frac{2\pi}{7}\right)-1$ and $|\beta+3|<2\cos\left(\frac{2\pi}{7}\right)-1$.

Since the order $p\geq 6$, applying for the inequality in Theorem 4.3.6:

$$|\gamma|+|\beta+3|\geq 1.1016\cdots,$$

and hence

$$|\gamma|+|\beta+3|>2\cos\left(\frac{2\pi}{7}\right)-1=0.246\,9\cdots. \tag{4.36}$$

By Lemma 4.3.1, $\langle f,\,[g,\,f]^3\rangle$ is a Kleinian group and the corresponding polynomial trace identity (4.26) provides $\gamma_3=\gamma(\gamma-\beta)(\gamma+1)^2(\gamma+3)^2$, then γ_3 and β fit the inequality (4.36):

$$|\gamma_3|+|\beta+3|>2\cos\left(\frac{2\pi}{7}\right)-1.$$

Therefore,

$$\begin{aligned}
&2\cos\left(\tfrac{2\pi}{7}\right)-1<|\gamma(\gamma-\beta)(\gamma+1)^2(\gamma+3)^2|+|\beta+3|\\
&=|\,(\gamma+3-3)\left((\gamma+3)-(\beta+3)\right)(\gamma+3-2)^2(\gamma+3)^2|+|\beta+3|\\
&\leq(|\gamma+3|+3)(|\gamma+3|+|\beta+3|)(|\gamma+3|+2)^2|\gamma+3|^2+|\beta+3|\\
&\leq(|\gamma+3|+3)(|\gamma+3|+|\beta+3|)(|\gamma+3|+2)^2|\gamma+3||\gamma+3|+|\beta+3|\\
&<\left(2\cos\left(\tfrac{2\pi}{7}\right)+2\right)\left(2\cos\left(\tfrac{2\pi}{7}\right)-1\right)^2\left(2\cos\left(\tfrac{2\pi}{7}\right)+1\right)^2|\gamma+3|+|\beta+3|\\
&=|\gamma+3|+|\beta+3|.
\end{aligned}$$

That is,

$$|\gamma+3|+|\beta+3|>2\cos\left(\frac{2\pi}{7}\right)-1. \tag{4.37}$$

It contradicts the assumption at the beginning:

$$|\gamma + 3| + |\beta + 3| < 2\cos\left(\frac{2\pi}{7}\right) - 1.$$

Thus, $|\gamma + 3| + |\beta + 3| \geq 2\cos\left(\frac{2\pi}{7}\right) - 1 = 0.246\,9\cdots$.

(2) Consider f of order 3, then $\beta = -3$ and hence the term $|\beta+3|$ vanishes. It follows that $|\gamma+3| = 2\cos\left(\frac{2\pi}{7}\right) - 1$, i.e., $\gamma = -2\cos\left(\frac{2\pi}{7}\right) - 2$ or $\gamma = 2\cos\left(\frac{2\pi}{7}\right) - 4$. Choose the first case of γ :

$$\gamma = -2\cos\left(\frac{2\pi}{7}\right) - 2.$$

Since $\text{tr}\,[f, g] = \gamma + 2 = -2\cos\left(\frac{2\pi}{7}\right)$,

$$\begin{aligned}\beta([f,g]) &= \text{tr}^2\,[f,g] - 4 \\ &= \left(-2\cos\left(\frac{2\pi}{7}\right)\right)^2 - 4 \\ &= -4\sin^2\left(\frac{2\pi}{7}\right).\end{aligned}$$

By Theorem 2.3.5, $[f, g]$ is elliptic of order 7. Take f and $h = gf^{-1}g^{-1}$ elliptics of order 3 whose product $fh = [f, g]$ is elliptic of order 7. Since $\langle f,\, gfg^{-1}\rangle = \langle f,\, gf^{-1}g^{-1}\rangle$ and the identity 2.11, $\langle f,\, gfg^{-1}\rangle$ is the $(3, 3, 7)$ hyperbolic triangle group and hence it is a Kleinian group.

Now choose g of order 2, this gives a $\mathbb{Z}_2$-extension Γ of the group $\langle f,\, gfg^{-1}\rangle$ and hence it is a Kleinian group by Lemma 3.2.5 and the triple of parameters for Γ is

$$\left(-2 - 2\cos\left(\frac{2\pi}{7}\right), -3, -4\right),$$

which gives the sharpness,

$$|\gamma+3|+|\beta+3| = \left|-2-2\cos\left(\frac{2\pi}{7}\right)+3\right| + |-3+3|$$
$$= 2\cos\left(\frac{2\pi}{7}\right) - 1.$$

□

Theorem 4.3.11 *Let $\langle f,\, g\rangle$ be a Kleinian group, and let $\gamma = \gamma\,(f,\, g)$ and $\beta = \beta\,(f)$. If f is loxodromic or parabolic or elliptic of order $p \geq 6$, then*

$$|\gamma+3|+|\beta+2| \geq 0.185168\cdots.$$

Proof. Assume the following inequality for some $r > 0$:

$$|\gamma+3|+|\beta+2| < r.$$

Recall the inequality from Theorem 4.2.7,

$$|\gamma|+|\beta+2| \geq \frac{\sqrt{5}-1}{2}.$$

By Lemma 4.3.1, $\left\langle f,\, [g,\, f]^3\right\rangle$ is a Kleinian group with the parameters $\gamma_3 = \gamma(\gamma-\beta)(\gamma+1)^2(\gamma+3)^2$, then γ_3 and β fit the inequality:

$$|\gamma_3|+|\beta+2| \geq \frac{\sqrt{5}-1}{2}.$$

Since $|\gamma+3|+|\beta+2|<r$, then $|\gamma+3|<r$ and $|\beta+2|<r$. Thus,

$$\begin{aligned}\frac{\sqrt{5}-1}{2} &\le |\gamma(\gamma-\beta)(\gamma+1)^2(\gamma+3)^2|+|\beta+2|| \\ &= |\,(\gamma+3-3)\,(\gamma+3-(\beta+2)-1)\,(\gamma+3-2)^2(\gamma+3)^2|+|\beta+2| \\ &\le (|\gamma+3|+3)\,(|\gamma+3|+|\beta+2|+1)\,(|\gamma+3|+2)^2\,|\gamma+3||\gamma+3|+|\beta+2| \\ &< (r+3)(r+1)(r+2)^2 r|\gamma+3|+|\beta+2|.\end{aligned}$$

If $(r+3)(r+1)(r+2)^2r\le 1$, then

$$\frac{\sqrt{5}-1}{2}<(r+3)(r+1)(r+2)^2r|\gamma+3|+|\beta+2|\le|\gamma+3|+|\beta+2|. \tag{4.38}$$

i.e., $\dfrac{\sqrt{5}-1}{2}<|\gamma+3|+|\beta+2|$.

On the other hand, solving $(r+3)(r+1)(r+2)^2r\le 1$ gives $r\le 0.07\,090\,3<\frac{\sqrt{5}-1}{2}$, and hence $|\gamma+3|+|\beta+2|<r<\frac{\sqrt{5}-1}{2}$. It is a contradiction to $|\gamma+3|+|\beta+2|>\frac{\sqrt{5}-1}{2}$

Now $(r+3)(r+1)(r+2)^2r>1$, so the inequality (4.38) becomes

$$\begin{aligned}\frac{\sqrt{5}-1}{2} &< (r+3)(r+1)(r+2)^2r|\gamma+3|+|\beta+2| \\ &< (r+3)(r+1)(r+2)^2r|\,(\gamma+3|+|\beta+2|) \\ &< (r+3)(r+1)(r+2)^2r^2.\end{aligned}$$

Solving $\frac{\sqrt{5}-1}{2}<(r+3)(r+1)(r+2)^2r^2$ gives $r>0.185168$. Thus, if $r\le 0.185168$, then $|\gamma+3|+|\beta+2|\ge r$. So take the largest lower bound $r=0.185168\cdots$, then

$$|\gamma+3|+|\beta+2|\ge 0.185168\cdots.$$

$\square$

The dissertation will end here, the further research work related to the project is to create more sharpness and to implement applications in low-dimensional topology and geometry, especially in 3-manifold theory.

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Appendix