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Study of Improved Sorting Weighting CFAR Detectors for Gaussian Environment

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Abstract

The goal of this paper is to improve the detection performance and the false alarm regulation of the conventional order statistics Constant False Alarm Rate (OS-CFAR) detectors in a non-homogeneous Gaussian environment. To this end, we design and study the New Sorting Weighting (NSW-) and the Modified Sorting Weighting (MSW-) CFAR detectors. We find closed forms of the detection (P_d) and the false alarm (P_{fa}) probabilities for both detectors. Moreover, we identify the optimum pairs of weights that maximize the P_d and ensure a constant P_{fa}. Finally, we prove through Monte Carlo simulations that these detectors provide better detection performance and false alarm regulation than the order statistics conventional ones in various clutter situations with the NSW-CFAR detector being the best one.

Keywords: New Sorting Weighting, Modified Sorting Weighting, Constant False Alarm Rate, Interferences, Clutter edge, Gaussian environment.

1. Introduction

The automatic radar detection system has been created to detect the presence of targets in its surveillance space, without a human operator's intervention. This has been possible with the development of computers and signal-processing techniques. This system stores the sampled signal of the received echo in shift registers that correspond to the reference cells surrounding the Cell Under Test (CUT) being tested for a possible target. This sampling makes it possible to generate, for each cell, a decision characterizing the presence or absence of a target echo is accompanied by noise or clutter [1-3].

The target echo is accompanied by noise or clutter. Clutter is a term used to describe any object located arbitrarily in the space monitored by radar, Figure 1. It can generate unwanted echoes that can disrupt normal radar operations and make target detection very difficult. In many cases, the clutter power is much higher than that of the noise. The clutter can be homogeneous or non-homogeneous. The homogeneity describes a stationary clutter situation in the reference window. In this case, the CUT is in a clutter that has a uniform statistic. That is, the reference window samples are assumed to be from the same random variable and are therefore statistically Independent and Identically Distributed (IID). The non-homogeneity is due to the presence of a clutter edge and/or interferences in the reference window. In this case, the reference samples are no longer identically distributed. The clutter edge is a transition between two environments of different natures; which produces an abrupt change in the clutter power. It appears in the reference window as two adjacent regions; a lower-powered clutter region and a higher-powered clutter region. The interferences may appear in one or more reference cells as spikes.

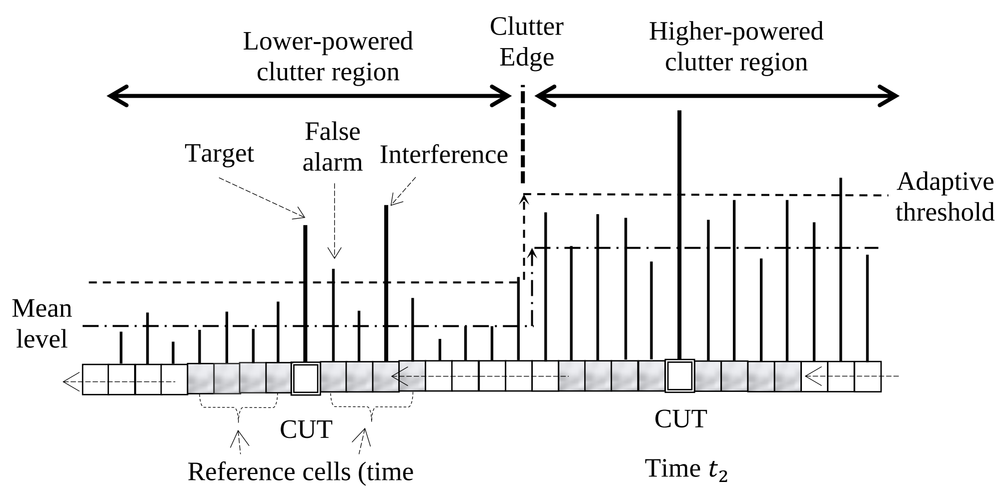

Figure 1 Adaptive threshold in non homogeneous environment

Since the unwanted echo power is unknown, detection is done by the comparison of the received signal to an adaptive threshold that varies depending on the estimated power level from the reference cells, located before (leading cells) and after (lagging cells) the CUT. Adaptive detection uses this estimator to maintain a constant false alarm probability (P_{fa}); these are Constant False Alarm Rate (CFAR) detectors [1-3].

All the CFAR detectors proposed in the literature each deal with a problem inherent to the particular conditions of use of the radar. Nevertheless, the main objective of each detector remains the improvement of the detection probability (P_d). In this paper, we fell on some CFAR detectors operating in Gaussian clutter. We recall the class of mean-level CFAR detectors, the first that appears in the radar literature, that best suits homogeneous clutter. Subsequently, we focus on the class based on order statistics that best suits non-homogeneous clutter.

In [4], Finn and Johnson have proposed the Cell-Averaging (CA-) CFAR detector that sums all the reference samples to estimate the local background level. It performs perfectly well in homogeneous clutter but in non-homogeneous clutter, the detection performance and the false alarm regulation are significantly affected.

In the presence of a clutter edge, the CUT can be in the higher-powered clutter region. If all the reference samples are used, the local background level decreases, increasing the P_{fa}. This is known as the capture effect. To minimize this effect, while maintaining an almost constant P_{fa}, in [5], Hansen has proposed the Greatest-Of (GO-) CFAR detector in which the sums of the samples in the leading and lagging windows are calculated, and the greatest one is used to estimate the local background level. However, it can be that the CUT is in the lower-powered clutter region. If all the reference samples are used, the local background level may increase, decreasing the P_d. This is known as the masking effect.

In the presence of interferences and target-dense environments, it is possible to come across instances where targets are very close together. To improve the resolution of close targets and therefore improve the detection performance, Trunk [6] has developed the Smallest-Of (SO-) CFAR detector wherein the sums of the samples in the leading and lagging windows are calculated and the smallest one is used to estimate the local background level. This detector is less sensitive to detection loss than the CA-CFAR unless the number of reference cells is relatively large. However, its detection performance deteriorates considerably if the interferences are located simultaneously before and after the CUT.

In [7, 8], Barkat et *al.* have designed the Weighted CA (WCA-) CFAR detector. In this detector, the sums of the leading and lagging samples are weighted and added so that it gives better detection performance than those of the modified CA-, GO-, and SO-CFAR detectors when a Swerling I (SW I) interference is in the leading window. Under this condition, they have resorted to closed forms of the P_d

and P_{fa}. They have also found the optimum pair of weights that maximize the P_d while keeping the P_{fa} constant in a homogeneous case.

To improve the robustness of mean-level detectors and guarantee CFAR detection in non-homogeneous environments, a class of detectors based on order statistics has been developed in the literature. In each detector, the reference samples are first classified in ascending order according to their powers. Then, the samples assumed to contain unwanted echoes are eliminated. The set of the remaining samples, assumed to be homogeneous, is then used to estimate the background level.

In multiple target situations, Rohling [9] has proposed the Order Statistic (OS-) CFAR detector whose k^{th} greatest sample is used to estimate the local background level. In homogeneous clutter, the P_d has shown a slight degradation. It is certain that in interfering clutter, it has exhibited better detection performance than that of the CA-CFAR detector. The fact remains that it is expensive in computing time and that the CA-CFAR detector transcends it in a homogeneous clutter.

In [10], Elias-Fuste et *al.* have proposed the OSGO-CFAR and OSSO-CFAR detectors that require only half the processing time of the OS-CFAR detector. In both detectors, the samples in the leading and lagging windows are sorted in an ascending order separately. Then, the k^{th} greatest samples in each window are obtained and the greatest one is used by the OSGO-CFAR while the smallest one is used by the OSSO-CFAR to estimate the background level. They have shown that the OSGO-CFAR detector exhibits the advantages of the OS-CFAR detector with a negligible increase in the CFAR loss.

In [11], Saeed et *al.* have proposed and implemented with FPGA the Sorting Weighting (SW-) CFAR detector to improve the detection performance of the OS-, OSGO-, and OSSO-CFAR detectors. In this detector, after sorting the samples of the leading and lagging windows separately in ascending order, the k^{th} greatest samples in each window are weighted and added to estimate the local background level. The pair of weights have been chosen so that a constant P_{fa} is guaranteed but closed forms of the P_d and P_{fa} were not obtained.

In [12], Mansouri et *al.* have inspired the Weighted MAXimum (WMAX-) CFAR detector from the OS-CFAR one. In this detector, the greatest reference sample is weighted and used to estimate the local background level. In homogenous clutter, the WMAX-CFAR detector has presented better detection performance than the OS-CFAR one under special conditions. They have also inspired the Greatest-Of WMAX (GOWMAX-) CFAR detector from the GO-CFAR and WMAX-CFAR ones. In this detector, the greatest reference samples in the leading and lagging windows are weighted by different coefficients, and the greatest one is used to estimate the local background level. In non-homogenous clutter, Monte Carlo simulations and implementation on a DSP processor of the GOWMAX-CFAR detector were carried out. The study has shown that whatever the conditions this detector performs better than the OS-CFAR one.

In [13], Magaz et *al.* have proposed the Forward Automatic Order Selection Ordered Statistics Detector (FAOSOSD-) CFAR detector. This detector does not require any prior information about the number of interferences which is determined merely by minimizing the Information-Theoretic Criteria (ITC). The obtained number is exploited to determine the optimal sample order to estimate the local background level. This detector has shown a much better performance than the OS-CFAR one in severe interference situations.

In light of the state of the art, it is clear that none of the existing detectors is the best and is suitable for all clutter situations. In this paper, we want to overcome the shortcomings of the existing detectors and improve the CFAR detection performance in homogeneous and non-homogeneous Gaussian environments and also improve the false alarm regulation by the proposition of simple and inexpensive detectors. To do this, we design and study the New Sorting Weighting (NSW-) CFAR detector and the Modified Sorting Weighting (MSW-) CFAR detector. We derive closed forms of the detection (P_d) and the false alarm (P_{fa}) probabilities for both detectors. We also obtain, for each detector, the optimum pairs of weights that maximize the P_d and guarantee a constant P_{fa}. To demonstrate the superiority of these detectors, we compare through Monte Carlo simulations its performance with those of the convenable conventional order statistics ones.

The rest of this paper is organized as follows. In Section 2, we study the NSW-CFAR detector. we present the statistical model, we found closed forms of the detection and false alarm probabilities and the optimum pairs of weights. In Section 3, we study the MSW-CFAR detector. We lay out the statistical model, closed forms of the detection and false alarm probabilities, and the optimum pairs of weights. In Section 4, we evaluate and compare, using Monte Carlo simulations, the robustness of the studied detectors and the corresponding conventional order statistics ones in different clutter situations along with discussions. Finally, a summary of the results along with our conclusions is given in Section 5.

2. New Sorting Weighting CFAR detector

Figure 2 shows the general structure of the New Sorting Weighting (NSW-) CFAR detector. The received signal is first passed through a Square-Law Detector (SLD) to restore the signal envelope. Then, the matched filter outputs q_i, $i = 1, 2, \ldots, N$, are stored serially into a tapped delay line of length $N + 1$, corresponding to the N reference cells surrounding the CUT. The samples in the leading and lagging windows are then ranked in ascending order to obtain:

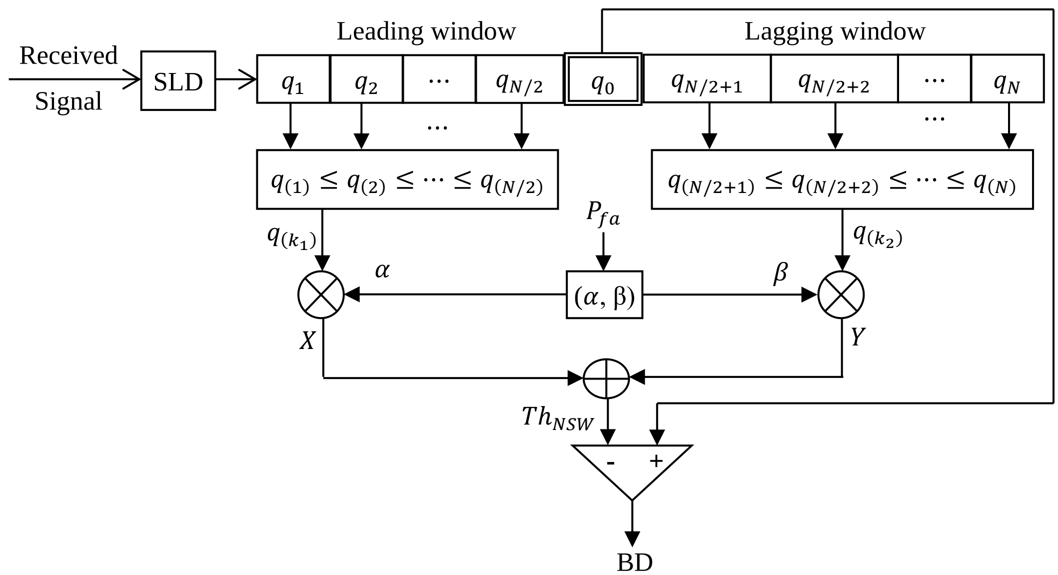

Figure 2 Block diagram of the NSW-CFAR detector

$$\begin{cases} q_{(1)} \le \cdots \le q_{(k)} \le \cdots \le q_{\left(\frac{N}{2}\right)} & k = 1, \dots, \dfrac{N}{2} \\ q_{\left(\frac{N}{2}+1\right)} \le \cdots \le q_{(k)} \le \cdots \le q_{(N)} & k = \dfrac{N}{2} + 1, \dots, N \end{cases} \quad ; \quad q_{(k)} \ge 0 \tag{1}$$

The probability density function (pdf) and the cumulative distribution function (cdf) of the k^{th} order statistic, $q_{(k)}$, are defined by [14]:

$$f_{Q_{(k)}}(q) = k \binom{\frac{N}{2}}{k} \left(1 - F_Q(q)\right)^{\frac{N}{2}-k} \left(F_Q(q)\right)^{k-1} f_Q(q) \tag{2}$$

and

$$F_{Q_{(k)}}(q) = \sum_{i=k}^{N/2} \binom{\frac{N}{2}}{i} \left(1 - F_Q(q)\right)^{\frac{N}{2}-i} \left(F_Q(q)\right)^{i} \tag{3}$$

where $f_Q(q)$ and $F_Q(q)$ are the pdf and cdf, respectively, of each random variable, q, in the reference window before the ranking. Since the clutter has a Gaussian quadrature component, each random variable, q, can follow the normalized Exponential distribution with pdf and cdf given by:

$$f_Q(q) = e^{-q} \quad ; \quad q \ge 0 \tag{4}$$

and

$$F_Q(q) = 1 - e^{-q} \quad ; \quad q \ge 0 \tag{5}$$

Substituting equations (4) and (5) into equations (2) and (3), we obtain:

$$f_{Q_{(k)}}(q) = k \binom{\frac{N}{2}}{k} e^{-q\left(\frac{N}{2}-k+1\right)} (1-e^{-q})^{k-1} \tag{6}$$

and

$$F_{Q_{(k)}}(q) = \sum_{i=k}^{N/2} \binom{\frac{N}{2}}{i} e^{-q\left(\frac{N}{2}-i\right)} (1-e^{-q})^{i} \tag{7}$$

The sample (q_0) in the CUT, located in the middle of the reference window and assumed to be independent of the N reference samples, is finally compared to the detection threshold (Th_{NSW}) to make a Binary Decision (BD) about the presence, H_1 Hypothesis, or the absence, H_0 Hypothesis, of a target in the CUT based on the following statistical test:

$$q_0 \underset{H_0}{\overset{H_1}{\gtrless}} Th_{NSW} \tag{8}$$

The target in the CUT is of SW I type and has a Gaussian quadrature component. Then, the normalized conditional Exponential pdf of the random variable q_0 is given by:

$$f_{Q_0|H_i}(q_0|H_i) = \begin{cases} e^{-q_0} & H_0 \\ \dfrac{1}{1+S} e^{-\frac{q_0}{1+S}} & H_1 \end{cases} \quad ; \quad q_0 \geq 0 \tag{9}$$

where S means the Signal-to-Noise Ratio (SNR). The adaptive detection threshold (Th_{NSW}) is given by:

$$Th_{NSW} = \alpha q_{(k_1)} + \beta q_{(k_2)} \tag{10}$$

where (α, β) is the pair of weights that guarantee a constant P_{fa} in a homogeneous clutter, i.e., the CFAR property is guaranteed for any values of the distribution parameters. On the other hand, $q_{(k_1)}$ and $q_{(k_2)}$ are the k_1^{th} and k_2^{th} greatest samples in the leading and lagging windows, respectively.

The pdf of the adaptive detection threshold, denoted by Q, is done by (see Appendix A for details):

$$f_Q(q) = \frac{k_1 k_2}{|\alpha||\beta|} \binom{\frac{N}{2}}{k_1} \binom{\frac{N}{2}}{k_2} \int_0^q e^{-\frac{y}{\beta}\left(\frac{N}{2}-k_2+1\right)} \left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1} e^{-\frac{q-y}{\alpha}\left(\frac{N}{2}-k_1+1\right)} \left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_1-1} dy \tag{11}$$

This equation has a solution based on the hypergeometric function, but this form does not lead to a closed form of the detection probability (P_d) defined by:

$$P_d = \int_0^\infty \left[\int_q^\infty f_{Q_0|H_1}(q_0|H_1)\, dq_0 \right] f_Q(q)\, dq \tag{12}$$

The false alarm probability (P_{fa}) can be obtained from the equation (12) by setting $S = 0$. Thus, it is defined by:

$$P_{fa} = \int_0^\infty \left[\int_q^\infty f_{Q_0|H_0}(q_0|H_0)\, dq_0 \right] f_Q(q)\, dq \tag{13}$$

The closed form of the P_d is given by (see Appendix B for details):

$$P_d = \binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2} \frac{\Gamma(k_1+1)\,\Gamma(k_2+1)\,\Gamma\left(\frac{N}{2}-k_1+1+\frac{\alpha}{1+S}\right)\Gamma\left(\frac{N}{2}-k_2+1+\frac{\beta}{1+S}\right)}{\Gamma\left(\frac{N}{2}+1+\frac{\alpha}{1+S}\right)\Gamma\left(\frac{N}{2}+1+\frac{\beta}{1+S}\right)} \tag{14}$$

The closed form of the P_{fa} can be deduced directly from equation (14) by setting $S = 0$. It is given by:

$$P_{fa} = \binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2} \frac{\Gamma(k_1+1)\,\Gamma(k_2+1)\,\Gamma\left(\frac{N}{2}-k_1+1+\alpha\right)\Gamma\left(\frac{N}{2}-k_2+1+\beta\right)}{\Gamma\left(\frac{N}{2}+1+\alpha\right)\Gamma\left(\frac{N}{2}+1+\beta\right)} \tag{15}$$

The clutter power does not appear in the expression of the P_{fa} given by equation (15). This means that instantaneous changes in the environmental conditions do not influence the desired P_{fa} value. Thus, this detector has the CFAR property. Note that, in the case where $k_1 = k_2 = k$, the P_d and P_{fa} become:

$$P_d = \frac{\Gamma^2\left(\frac{N}{2}+1\right)\Gamma\left(\frac{N}{2}-k+1+\frac{\alpha}{1+S}\right)\Gamma\left(\frac{N}{2}-k+1+\frac{\beta}{1+S}\right)}{\Gamma^2\left(\frac{N}{2}-k+1\right)\Gamma\left(\frac{N}{2}+1+\frac{\alpha}{1+S}\right)\Gamma\left(\frac{N}{2}+1+\frac{\beta}{1+S}\right)} \tag{16}$$

and

$$P_{fa} = \frac{\Gamma^2\left(\frac{N}{2}+1\right)\Gamma\left(\frac{N}{2}-k+1+\alpha\right)\Gamma\left(\frac{N}{2}-k+1+\beta\right)}{\Gamma^2\left(\frac{N}{2}-k+1\right)\Gamma\left(\frac{N}{2}+1+\alpha\right)\Gamma\left(\frac{N}{2}+1+\beta\right)} \tag{17}$$

Now, we aim to obtain the optimum pair of weights (α, β) to obtain the maximum P_d while keeping the P_{fa} to be constant in homogeneous case. This can be done by the use of the objective function defined by [7]:

$$J(\alpha,\beta) = P_d(\alpha,\beta) + \xi\left[P_{fa}(\alpha,\beta) - \nu\right] \tag{18}$$

where ν is the desired P_{fa} value and ξ is the Lagrange multiplier. Substituting equations (16) and (17) into (18), we find:

$$J(\alpha,\beta) = \left[\begin{array}{l} \dfrac{\Gamma^2\left(\frac{N}{2}+1\right)\Gamma\left(\frac{N}{2}-k+1+\frac{\alpha}{1+S}\right)\Gamma\left(\frac{N}{2}-k+1+\frac{\beta}{1+S}\right)}{\Gamma^2\left(\frac{N}{2}-k+1\right)\Gamma\left(\frac{N}{2}+1+\frac{\alpha}{1+S}\right)\Gamma\left(\frac{N}{2}+1+\frac{\beta}{1+S}\right)} \\ +\xi\dfrac{\Gamma^2\left(\frac{N}{2}+1\right)\Gamma\left(\frac{N}{2}-k+1+\alpha\right)\Gamma\left(\frac{N}{2}-k+1+\beta\right)}{\Gamma^2\left(\frac{N}{2}-k+1\right)\Gamma\left(\frac{N}{2}+1+\alpha\right)\Gamma\left(\frac{N}{2}+1+\beta\right)} - \xi\nu \end{array}\right] \tag{19}$$

Now, we take the derivatives of the equation (19) concerning α and β and we set them equal to zero. We obtain, respectively:

$$\frac{\partial J(\alpha,\beta)}{\partial \alpha} = \frac{\Gamma^2\left(\frac{N}{2}+1\right)}{\Gamma^2\left(\frac{N}{2}-k+1\right)} \left\{ \begin{array}{c} \dfrac{\Gamma\left(\frac{N}{2}-k+1+\frac{\alpha}{1+S}\right)\Gamma\left(\frac{N}{2}-k+1+\frac{\beta}{1+S}\right)}{(1+S)\Gamma\left(\frac{N}{2}+1+\frac{\alpha}{1+S}\right)\Gamma\left(\frac{N}{2}+1+\frac{\beta}{1+S}\right)} \\ \times\left[\psi^{(0)}\left(\frac{N}{2}-k+1+\frac{\alpha}{1+S}\right)-\psi^{(0)}\left(\frac{N}{2}+1+\frac{\alpha}{1+S}\right)\right] \\ +\xi\dfrac{\Gamma\left(\frac{N}{2}-k+1+\alpha\right)\Gamma\left(\frac{N}{2}-k+1+\beta\right)}{\Gamma\left(\frac{N}{2}+1+\alpha\right)\Gamma\left(\frac{N}{2}+1+\beta\right)} \\ \times\left[\psi^{(0)}\left(\frac{N}{2}-k+1+\alpha\right)-\psi^{(0)}\left(\frac{N}{2}+1+\alpha\right)\right] \end{array} \right\} = 0 \quad (20)$$

and

$$\frac{\partial J(\alpha,\beta)}{\partial \beta} = \frac{\Gamma^2\left(\frac{N}{2}+1\right)}{\Gamma^2\left(\frac{N}{2}-k+1\right)} \left\{ \begin{array}{c} \dfrac{\Gamma\left(\frac{N}{2}-k+1+\frac{\alpha}{1+S}\right)\Gamma\left(\frac{N}{2}-k+1+\frac{\beta}{1+S}\right)}{(1+S)\Gamma\left(\frac{N}{2}+1+\frac{\alpha}{1+S}\right)\Gamma\left(\frac{N}{2}+1+\frac{\beta}{1+S}\right)} \\ \times\left[\psi^{(0)}\left(\frac{N}{2}-k+1+\frac{\beta}{1+S}\right)-\psi^{(0)}\left(\frac{N}{2}+1+\frac{\beta}{1+S}\right)\right] \\ +\xi\dfrac{\Gamma\left(\frac{N}{2}-k+1+\alpha\right)\Gamma\left(\frac{N}{2}-k+1+\beta\right)}{\Gamma\left(\frac{N}{2}+1+\alpha\right)\Gamma\left(\frac{N}{2}+1+\beta\right)} \\ \times\left[\psi^{(0)}\left(\frac{N}{2}-k+1+\beta\right)-\psi^{(0)}\left(\frac{N}{2}+1+\beta\right)\right] \end{array} \right\} = 0 \quad (21)$$

where $\psi^{(0)}(\cdot)$ is the Digamma function defined by:

$$\psi^{(0)}(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \frac{1}{x} - \sum_{n=1}^{\infty}\left(\frac{1}{x+n} - \frac{1}{n}\right) \quad (22)$$

with γ is the Euler-Mascheroni constant. Now, we must get ξ from equation (20) and substitute it into equation (21). We find directedly that in homogeneous case, the optimum pair of weights is $\alpha = \beta$. Note that, in non-homogeneous case, if α tends to zero, there is two possibilities. If $q_{(k_1)} \gg q_{(k_2)}$, the NSW-CFAR is like the OSSO-CFAR. Else, it is like the OSGO-CFAR. On the other hand, if β tends to zero, there is also two scenarios. If $q_{(k_1)} \gg q_{(k_2)}$, the NSW-CFAR is like the OSGO-CFAR. Else, it is like the OSSO-CFAR.

3. Modified Sorting Weighting CFAR detector

Figure 3 shows the general structure of the Modified Sorting Weighting (MSW-) CFAR detector. This detector is based on the following statistical test:

$$q_0 \underset{H_0}{\overset{H_1}{\gtrless}} Th_{MSW} \quad (23)$$

where Th_{MSW} is the adaptive detection threshold given by:

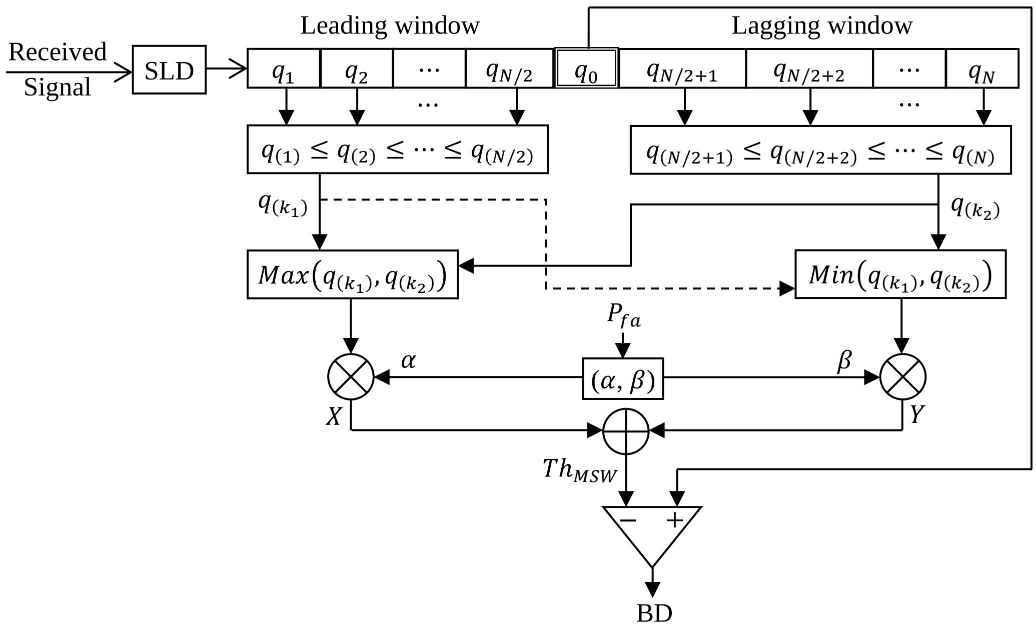

Figure 3 Block diagram of the MSW-CFAR detector

$$Th_{MSW} = \alpha q_{OSGO} + \beta q_{OSSO} \tag{24}$$

and (α, β) is the pair of weights that guarantee a constant P_{fa} in a homogeneous clutter. The random variables q_{OSGO} and q_{OSSO} are defined, respectively, by:

$$q_{OSGO} = \max\{q_{(k_1)}, q_{(k_2)}\} \tag{25}$$

and

$$q_{OSSO} = \min\{q_{(k_1)}, q_{(k_2)}\} \tag{26}$$

The pdf of the adaptive detection threshold, denoted by Q, is done by (see Appendix C for details):

$$\begin{aligned} f_Q(q) = \frac{1}{|\alpha||\beta|} \left\{ k_1^2 \binom{\frac{N}{2}}{k_1}^2 \sum_{i=k_2}^{N/2} \binom{\frac{N}{2}}{i} \int_0^q \begin{bmatrix} e^{-\frac{y}{\beta}\left(\frac{N}{2}-k_1+1\right)} \left(1-e^{-\frac{y}{\beta}}\right)^{k_1-1} \\ \times\, e^{-\frac{q-y}{\alpha}(N-k_1+1-i)} \left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_1-1+i} \end{bmatrix} dy \right. \\ + k_1 k_2 \binom{\frac{N}{2}}{k_1} \binom{\frac{N}{2}}{k_2} \sum_{i=k_1}^{N/2} \binom{\frac{N}{2}}{i} \int_0^q \begin{bmatrix} e^{-\frac{y}{\beta}\left(\frac{N}{2}-k_1+1\right)} \left(1-e^{-\frac{y}{\beta}}\right)^{k_1-1} \\ \times\, e^{-\frac{q-y}{\alpha}(N-k_2+1-i)} \left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_2-1+i} \end{bmatrix} dy \\ + k_1 k_2 \binom{\frac{N}{2}}{k_1} \binom{\frac{N}{2}}{k_2} \sum_{i=k_2}^{N/2} \binom{\frac{N}{2}}{i} \int_0^q \begin{bmatrix} e^{-\frac{y}{\beta}\left(\frac{N}{2}-k_2+1\right)} \left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1} \\ \times\, e^{-\frac{q-y}{\alpha}(N-k_1+1-i)} \left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_1-1+i} \end{bmatrix} dy \\ + k_2^2 \binom{\frac{N}{2}}{k_2}^2 \sum_{i=k_1}^{N/2} \binom{\frac{N}{2}}{i} \int_0^q \begin{bmatrix} e^{-\frac{y}{\beta}\left(\frac{N}{2}-k_2+1\right)} \left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1} \\ \times\, e^{-\frac{q-y}{\alpha}(N-k_2+1-i)} \left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_2-1+i} \end{bmatrix} dy \end{aligned}$$

$$-k_1^2 \binom{\frac{N}{2}}{k_1}^2 \sum_{i=k_2}^{N/2} \binom{\frac{N}{2}}{i} \sum_{j=k_2}^{N/2} \binom{\frac{N}{2}}{j} \int_0^q \left[\begin{array}{c} e^{-\frac{y}{\beta}(N-k_1+1-j)} \left(1-e^{-\frac{y}{\beta}}\right)^{k_1-1+j} \\ \times e^{-\frac{q-y}{\alpha}(N-k_1+1-i)} \left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_1-1+i} \end{array} \right] dy$$

$$-k_1 k_2 \binom{\frac{N}{2}}{k_1} \binom{\frac{N}{2}}{k_2} \sum_{i=k_1}^{N/2} \binom{\frac{N}{2}}{i} \sum_{j=k_2}^{N/2} \binom{\frac{N}{2}}{j} \int_0^q \left[\begin{array}{c} e^{-\frac{y}{\beta}(N-k_1+1-j)} \left(1-e^{-\frac{y}{\beta}}\right)^{k_1-1+j} \\ \times e^{-\frac{q-y}{\alpha}(N-k_2+1-i)} \left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_2-1+i} \end{array} \right] dy$$

$$-k_1 k_2 \binom{\frac{N}{2}}{k_1} \binom{\frac{N}{2}}{k_2} \sum_{i=k_2}^{N/2} \binom{\frac{N}{2}}{i} \sum_{j=k_1}^{N/2} \binom{\frac{N}{2}}{j} \int_0^q \left[\begin{array}{c} e^{-\frac{y}{\beta}(N-k_2+1-j)} \left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1+j} \\ \times e^{-\frac{q-y}{\alpha}(N-k_1+1-i)} \left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_1-1+i} \end{array} \right] dy$$

$$\left. -k_2^2 \binom{\frac{N}{2}}{k_2}^2 \sum_{i=k_1}^{N/2} \binom{\frac{N}{2}}{i} \sum_{j=k_1}^{N/2} \binom{\frac{N}{2}}{j} \int_0^q \left[\begin{array}{c} e^{-\frac{y}{\beta}(N-k_2+1-j)} \left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1+j} \\ \times e^{-\frac{q-y}{\alpha}(N-k_2+1-i)} \left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_2-1+i} \end{array} \right] dy \right\} \tag{27}$$

Recall that in the SW-CFAR detector [11], closed forms of the P_d and P_{fa} are not obtained and also $k_1 = k_2 = k$ which is not the case in the MSW-CFAR detector where the closed form of the P_d is given by (see Appendix D for details):

$$P_d = \binom{\frac{N}{2}}{k_1}^2 \frac{k_1^2\, \Gamma(k_1)\, \Gamma\left(\frac{N}{2}-k_1+1+\frac{\beta}{1+S}\right)}{\Gamma\left(\frac{N}{2}+1+\frac{\beta}{1+S}\right)} \sum_{i=k_2}^{N/2} \binom{\frac{N}{2}}{i} \frac{\Gamma(k_1+i)\, \Gamma\left(N-k_1+1-i+\frac{\alpha}{1+S}\right)}{\Gamma\left(N+1+\frac{\alpha}{1+S}\right)}$$

$$+\binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2} \frac{k_1 k_2\, \Gamma(k_1)\, \Gamma\left(\frac{N}{2}-k_1+1+\frac{\beta}{1+S}\right)}{\Gamma\left(\frac{N}{2}+1+\frac{\beta}{1+S}\right)} \sum_{i=k_1}^{N/2} \binom{\frac{N}{2}}{i} \frac{\Gamma(k_2+i)\, \Gamma\left(N-k_2+1-i+\frac{\alpha}{1+S}\right)}{\Gamma\left(N+1+\frac{\alpha}{1+S}\right)}$$

$$+\binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2} \frac{k_1 k_2\, \Gamma(k_2)\, \Gamma\left(\frac{N}{2}-k_2+1+\frac{\beta}{1+S}\right)}{\Gamma\left(\frac{N}{2}+1+\frac{\beta}{1+S}\right)} \sum_{i=k_2}^{N/2} \binom{\frac{N}{2}}{i} \frac{\Gamma(k_1+i)\, \Gamma\left(N-k_1+1-i+\frac{\alpha}{1+S}\right)}{\Gamma\left(N+1+\frac{\alpha}{1+S}\right)}$$

$$+\binom{\frac{N}{2}}{k_2}^2 \frac{k_2^2\, \Gamma(k_2)\, \Gamma\left(\frac{N}{2}-k_2+1+\frac{\beta}{1+S}\right)}{\Gamma\left(\frac{N}{2}+1+\frac{\beta}{1+S}\right)} \sum_{i=k_1}^{N/2} \binom{\frac{N}{2}}{i} \frac{\Gamma(k_2+i)\, \Gamma\left(N-k_2+1-i+\frac{\alpha}{1+S}\right)}{\Gamma\left(N+1+\frac{\alpha}{1+S}\right)}$$

$$-\binom{\frac{N}{2}}{k_1}^2 \sum_{i=k_2}^{N/2} \binom{\frac{N}{2}}{i} \frac{k_1^2\, \Gamma(k_1+i)\, \Gamma\left(N-k_1+1-i+\frac{\alpha}{1+S}\right)}{\Gamma\left(N+1+\frac{\alpha}{1+S}\right)}$$

$$\times \sum_{j=k_2}^{N/2} \binom{\frac{N}{2}}{j} \frac{\Gamma(k_1+j)\, \Gamma\left(N-k_1+1-j+\frac{\beta}{1+S}\right)}{\Gamma\left(N+1+\frac{\beta}{1+S}\right)}$$

$$-\binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2} \sum_{i=k_1}^{N/2} \binom{\frac{N}{2}}{i} \frac{k_1 k_2\, \Gamma(k_2+i)\, \Gamma\left(N-k_2+1-i+\frac{\alpha}{1+S}\right)}{\Gamma\left(N+1+\frac{\alpha}{1+S}\right)}$$

$$\times \sum_{j=k_2}^{N/2} \binom{\frac{N}{2}}{j} \frac{\Gamma(k_1+j)\, \Gamma\left(N-k_1+1-j+\frac{\beta}{1+S}\right)}{\Gamma\left(N+1+\frac{\beta}{1+S}\right)}$$

$$
\begin{aligned}
&-\binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2}\sum_{i=k_2}^{N/2}\binom{\frac{N}{2}}{i}\frac{k_1k_2\,\Gamma(k_1+i)\,\Gamma\left(N-k_1+1-i+\frac{\alpha}{1+S}\right)}{\Gamma\left(N+1+\frac{\alpha}{1+S}\right)}\\
&\times\sum_{j=k_1}^{N/2}\binom{\frac{N}{2}}{j}\frac{\Gamma(k_2+j)\,\Gamma\left(N-k_2+1-j+\frac{\beta}{1+S}\right)}{\Gamma\left(N+1+\frac{\beta}{1+S}\right)}\\
&-\binom{\frac{N}{2}}{k_2}^2\sum_{i=k_1}^{N/2}\binom{\frac{N}{2}}{i}\frac{k_2^2\,\Gamma(k_2+i)\,\Gamma\left(N-k_2+1-i+\frac{\alpha}{1+S}\right)}{\Gamma\left(N+1+\frac{\alpha}{1+S}\right)}\\
&\times\sum_{j=k_1}^{N/2}\binom{\frac{N}{2}}{j}\frac{\Gamma(k_2+j)\,\Gamma\left(N-k_2+1-j+\frac{\beta}{1+S}\right)}{\Gamma\left(N+1+\frac{\beta}{1+S}\right)} \qquad (28)
\end{aligned}
$$

The P_{fa} is deduced directly from equation (28) by setting $S=0$ by:

$$
\begin{aligned}
P_{fa}=&\binom{\frac{N}{2}}{k_1}^2\frac{k_1^2\,\Gamma(k_1)\,\Gamma\left(\frac{N}{2}-k_1+1+\beta\right)}{\Gamma\left(\frac{N}{2}+1+\beta\right)}\sum_{i=k_2}^{N/2}\binom{\frac{N}{2}}{i}\frac{\Gamma(k_1+i)\,\Gamma(N-k_1+1-i+\alpha)}{\Gamma(N+1+\alpha)}\\
&+\binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2}\frac{k_1k_2\,\Gamma(k_1)\,\Gamma\left(\frac{N}{2}-k_1+1+\beta\right)}{\Gamma\left(\frac{N}{2}+1+\beta\right)}\sum_{i=k_1}^{N/2}\binom{\frac{N}{2}}{i}\frac{\Gamma(k_2+i)\,\Gamma(N-k_2+1-i+\alpha)}{\Gamma(N+1+\alpha)}\\
&+\binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2}\frac{k_1k_2\,\Gamma(k_2)\,\Gamma\left(\frac{N}{2}-k_2+1+\beta\right)}{\Gamma\left(\frac{N}{2}+1+\beta\right)}\sum_{i=k_2}^{N/2}\binom{\frac{N}{2}}{i}\frac{\Gamma(k_1+i)\,\Gamma(N-k_1+1-i+\alpha)}{\Gamma(N+1+\alpha)}\\
&+\binom{\frac{N}{2}}{k_2}^2\frac{k_2^2\,\Gamma(k_2)\,\Gamma\left(\frac{N}{2}-k_2+1+\beta\right)}{\Gamma\left(\frac{N}{2}+1+\beta\right)}\sum_{i=k_1}^{N/2}\binom{\frac{N}{2}}{i}\frac{\Gamma(k_2+i)\,\Gamma(N-k_2+1-i+\alpha)}{\Gamma(N+1+\alpha)}\\
&-\binom{\frac{N}{2}}{k_1}^2\sum_{i=k_2}^{N/2}\binom{\frac{N}{2}}{i}\frac{k_1^2\,\Gamma(k_1+i)\,\Gamma(N-k_1+1-i+\alpha)}{\Gamma(N+1+\alpha)}\sum_{j=k_2}^{N/2}\binom{\frac{N}{2}}{j}\frac{\Gamma(k_1+j)\,\Gamma(N-k_1+1-j+\beta)}{\Gamma(N+1+\beta)}\\
&-\binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2}\sum_{i=k_1}^{N/2}\binom{\frac{N}{2}}{i}\frac{k_1k_2\,\Gamma(k_2+i)\,\Gamma(N-k_2+1-i+\alpha)}{\Gamma(N+1+\alpha)}\\
&\times\sum_{j=k_2}^{N/2}\binom{\frac{N}{2}}{j}\frac{\Gamma(k_1+j)\,\Gamma(N-k_1+1-j+\beta)}{\Gamma(N+1+\beta)}\\
&-\binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2}\sum_{i=k_2}^{N/2}\binom{\frac{N}{2}}{i}\frac{k_1k_2\,\Gamma(k_1+i)\,\Gamma(N-k_1+1-i+\alpha)}{\Gamma(N+1+\alpha)}\\
&\times\sum_{j=k_1}^{N/2}\binom{\frac{N}{2}}{j}\frac{\Gamma(k_2+j)\,\Gamma(N-k_2+1-j+\beta)}{\Gamma(N+1+\beta)}\\
&-\binom{\frac{N}{2}}{k_2}^2\sum_{i=k_1}^{N/2}\binom{\frac{N}{2}}{i}\frac{k_2^2\,\Gamma(k_2+i)\,\Gamma(N-k_2+1-i+\alpha)}{\Gamma(N+1+\alpha)}\sum_{j=k_1}^{N/2}\binom{\frac{N}{2}}{j}\frac{\Gamma(k_2+j)\,\Gamma(N-k_2+1-j+\beta)}{\Gamma(N+1+\beta)} \qquad (29)
\end{aligned}
$$

In this detector as well, the clutter power does not appear in the expression of the P_{fa} given by equation (29). This means that the MSW- detector has the CFAR property. On the other hand, if we consider that $k_1=k_2=k$, the closed forms of the P_d and P_{fa} become:

$$P_d = 4k^2 \binom{\frac{N}{2}}{k}^2 \sum_{i=k}^{N/2} \binom{\frac{N}{2}}{i} \frac{\Gamma(k+i)\,\Gamma\left(N-k+1-i+\frac{\alpha}{1+S}\right)}{\Gamma\left(N+1+\frac{\alpha}{1+S}\right)} \times \left[\frac{\Gamma(k)\,\Gamma\left(\frac{N}{2}-k+1+\frac{\beta}{1+S}\right)}{\Gamma\left(\frac{N}{2}+1+\frac{\beta}{1+S}\right)} - \sum_{j=k}^{N/2} \binom{\frac{N}{2}}{j} \frac{\Gamma(k+j)\,\Gamma\left(N-k+1-j+\frac{\beta}{1+S}\right)}{\Gamma\left(N+1+\frac{\beta}{1+S}\right)}\right] \tag{30}$$

and

$$P_{fa} = 4k^2 \binom{\frac{N}{2}}{k}^2 \sum_{i=k}^{N/2} \binom{\frac{N}{2}}{i} \frac{\Gamma(k+i)\,\Gamma(N-k+1-i+\alpha)}{\Gamma(N+1+\alpha)} \times \left[\frac{\Gamma(k)\,\Gamma\left(\frac{N}{2}-k+1+\beta\right)}{\Gamma\left(\frac{N}{2}+1+\beta\right)} - \sum_{j=k}^{N/2} \binom{\frac{N}{2}}{j} \frac{\Gamma(k+j)\,\Gamma(N-k+1-j+\beta)}{\Gamma(N+1+\beta)}\right] \tag{31}$$

The optimum pair of weights (α, β) that maximise the P_d while keeping the P_{fa} to be constant in homogeneous case is exactly like the NSW-CFAR detector, i.e., $\alpha = \beta$. In non-homogeneous case, if β tends to zero, the MSW-CFAR tends to the OSGO-CFAR. Also, if α tends to zero, the MSW-CFAR tends to the OSSO-CFAR.

4. Simulation results and discussions

In this section, we evaluate and compare the performance of the CFAR detectors studied in Sections 2 and 3 in Gaussian clutter through a series of Monte Carlo simulations. We focus on single-pulse detection, which corresponds to the SW I fluctuating model. We assume that the clutter may be homogeneous or non-homogeneous, i.e., the presence of interferences or a clutter edge in the reference window. Before we proceed, let us first mention the simulation hypotheses that are used throughout the experiments. We assume that the size of the reference window is $N = 32$. We want a desired false alarm probability $P_{fa} = 10^{-4}$. This means that the number of Monte Carlo Runs must be $MCR = 100/P_{fa}$. To normalize the Gaussian clutter power, we assume that the position parameter ($\mu = 0$) and the scale parameter ($\sigma = 1/\sqrt{2}$). We are interested in $S \in [0, 40]\ dB$. We test the cases of the presence of two interferences after (positions 20 and 25) the CUT. We assume that the interferences have the same nature as the target, i.e., the Interference-to-Noise Ratio (INR) noted by I is equal to S. We also test the case of the presence of a clutter edge before the CUT with 10 reference cells in the higher-powered (clutter) region. In this case, the CUT is in the lower-powered (noise) region. To do this, we assume that the Clutter-to-Noise ratio (CNR) noted by C is equal to $20\ dB$.

Table 1 summarizes the values of the weights of the NSW-, MSW-, OSGO-, and OSSO-CFAR detectors with $k = 3N/8$ and the OS-CFAR detector with

Table 1: Values of the weights of the NSW-, MSW-, OSGO-, and OSSO-CFAR detectors with $k = 3N/8$ and OS-CFAR detector with $k = 3N/4$ for $N = 32$ and $P_{fa} = 10^{-4}$.

<table>
<tr><th>Clutter situation</th><th>NSW-</th><th>MSW-</th><th>OS-</th><th>OSSO-</th><th>OSGO-</th></tr>
<tr><td>Homogeneous clutter</td><td>(4.4990, 4.4990)</td><td>(4.4990, 4.4990)</td><td rowspan="3">8.7510</td><td rowspan="3">12.4500</td><td rowspan="3">7.9880</td></tr>
<tr><td>Two interferences after the CUT</td><td>(7.3700, 2.0000)</td><td>(3.2000, 6.3350)</td></tr>
<tr><td>Clutter edge before the CUT (CUT in noise)</td><td>(0, 11.5000)</td><td>(0, 12.4500)</td></tr>
</table>

$k = 3N/4$ that guarantee a desired $P_{fa} = 10^{-4}$ and maximize the P_d for $N = 32$ in different clutter situations.

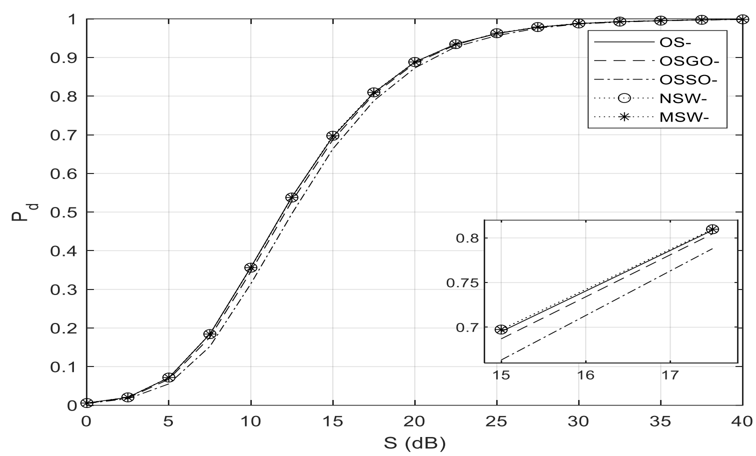

Figure 4 P_d versus S for the OS-, OSGO-, OSSO-, NSW- and MSW-CFAR detectors in a homogeneous clutter.

4.1. Homogeneous clutter

Figure 4 compares the detection probabilities of various detectors in homogeneous clutter. We observe that the NSW-CFAR and MSW-CFAR detectors have similar detection probabilities to the OS-CFAR detector. This is because the weights of these detectors are equal and the use of one or two cells does not make a significant difference when the number of reference cells is high. The OSGO-CFAR detector has a negligeable loss compared to the previous detectors, while the OSSO-CFAR detector has more loss.

4.2. Presence of interferences

In Figure 5, we can see a comparison of the detection probabilities of the detectors being studied in the presence of two interferences after (cells 20 and 25) the CUT. We observe that the NSW-, MSW-, and OS-CFAR detectors perform almost

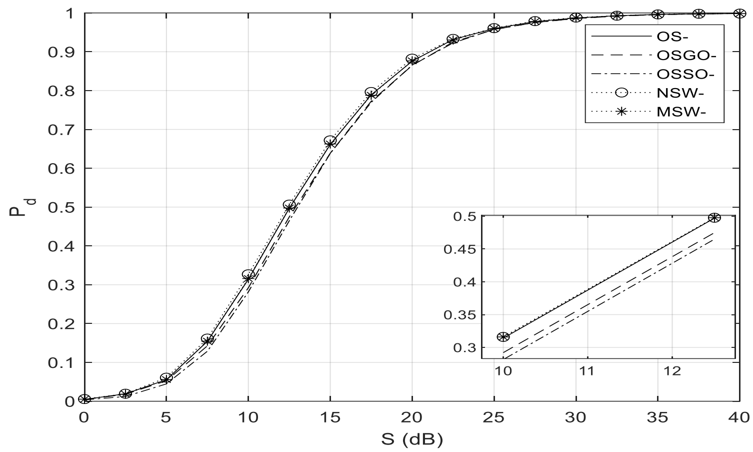

Figure 5 P_d versus S for the OS-, OSGO-, OSSO-, NSW-, and MSW-CFAR detectors in the presence of two interferences after (cells 20 and 25) the CUT.

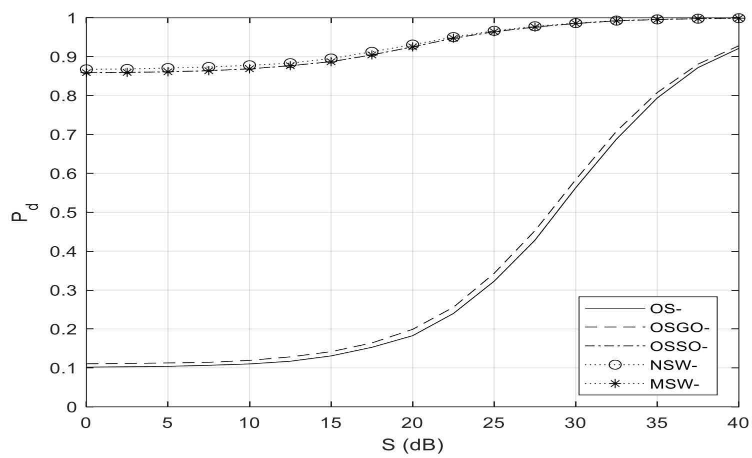

Figure 6 P_d versus S for the OS-, OSGO-, OSSO-, NSW-, and MSW-CFAR detectors in the presence of a clutter edge before the CUT (CUT in noise).

equally. It's worth noting that the interfering cells do not contribute to the clutter level calculation, and thus, the masking effect is absent. As for the OSGO-CFAR detector, it has a slightly lower performance. Finally, the OSSO-CFAR detector shows more loss in detection compared to the previous detectors.

4.3. Presence of a clutter edge

In Figure 6, we can see a comparison of the detection probabilities of the previous detectors in the presence of a clutter edge before the CUT with the CUT in the lower-powered region. The NSW-CFAR detector shows the best detection performance, followed by the MSW- and OSSO-CFAR detectors with a negligible loss. However, the OSGO-CFAR detector has a very poor detection probability and

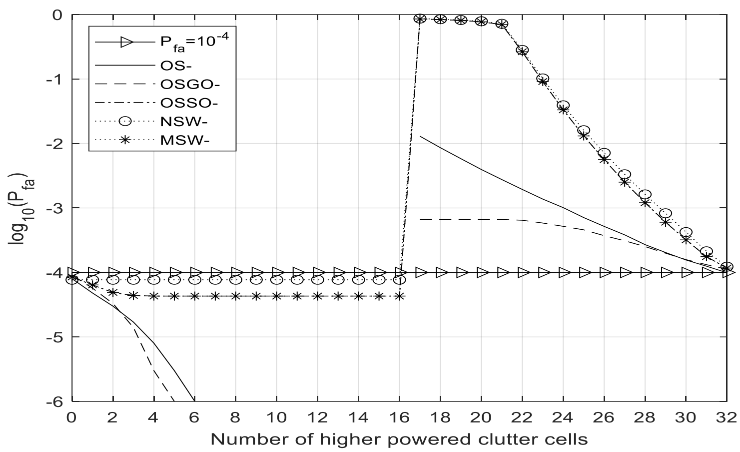

Figure 7 P_{fa} versus the number of the higher-powered clutter cells of the OS-, OSGO-, OSSO-, NSW-, and MSW-CFAR detectors in the presence of a clutter edge before the CUT (CUT in clear).

the OS-CFAR is the worst. This is because, in this situation, the masking effect is very high in the OSGO- and OS-CFAR detectors due to the higher-powered samples entering the calculation of the clutter level. The new detectors have been designed to overcome this shortcoming and their weights have proven to be useful in such situations.

In the presence of a clutter edge, it is important to analyze the false alarm regulation in addition to the detection performance. So, when a clutter edge is present before the CUT (CUT in noise), Figure 7 illustrates the regulation of the P_{fa} of the previous detectors as they cross a clutter edge with $C = 20\,dB$. The NSW-CFAR detector is the only one that can regulate the P_{fa} correctly when the CUT is in the lower-powered region. It gives almost constant and very closed values to the desired P_{fa}. It is followed by the MSW- and OSSO-CFAR detectors that exhibit good regulation, while the OS-CFAR detector has a significant loss and the OSGO-CFAR detector performs the worst.

5. Summary and Conclusion

Throughout this research, we have developed and evaluated two new mono-pulse CFAR detectors, the New Sorting Weighting (NSW-) CFAR and the Modified Sorting Weighting (MSW-) CFAR detectors. Our objective is to enhance the detection performance and the false alarm regulation in both homogeneous and non-homogeneous Gaussian environments. We have obtained closed forms of the detection (P_d) and false alarm (P_{fa}) probabilities of these detectors. We have also found the optimum pairs of weights that guarantee a constant P_{fa} and maximize the P_d. Monte Carlo simulation results have shown that these detectors have improved the detection performance and the regulation of the P_{fa} of the conventional ones, in any clutter situation. We have concluded that the NSW-CFAR detector has outperformed all the previous CFAR detectors and have retained it as our recommended option.

Appendix A

Let have two new independent random variables $X = \alpha\, q_{(k_1)}$ and $Y = \beta\, q_{(k_2)}$. Having $q_{(k_1)} = X/\alpha$ and $q_{(k_2)} = Y/\beta$, the pdfs of X and Y are given by [7]:

$$f_X(x) = \frac{1}{|\alpha|} f_{Q_{(k_1)}}\left(\frac{x}{\alpha}\right); \quad \alpha > 0 \tag{A.1}$$

and

$$f_Y(y) = \frac{1}{|\beta|} f_{Q_{(k_2)}}\left(\frac{y}{\beta}\right); \quad \beta > 0 \tag{A.2}$$

Substituting equation (6) into equation (A.1) with k_1 instead of k and into equation (A.2) with k_2 instead of k. We have the pdfs of X and Y to be:

$$f_X(x) = \frac{k_1}{|\alpha|}\binom{\frac{N}{2}}{k_1} e^{-\frac{x}{\alpha}\left(\frac{N}{2}-k_1+1\right)}\left(1 - e^{-\frac{x}{\alpha}}\right)^{k_1-1} \tag{A.3}$$

and

$$f_Y(y) = \frac{k_2}{|\beta|}\binom{\frac{N}{2}}{k_2} e^{-\frac{y}{\beta}\left(\frac{N}{2}-k_2+1\right)}\left(1 - e^{-\frac{y}{\beta}}\right)^{k_2-1} \tag{A.4}$$

The adaptive detection threshold, denoted by Q, is the sum of X and Y. Using the fact that $X \geq 0$ and $Y \geq 0$, the pdf of Q is defined by [7]:

$$f_Q(q) = f_X(x) * f_Y(y) = \int_0^q f_Y(y)\, f_X(q-y)\, dy \tag{A.5}$$

where $*$ denotes the convolution. Now, substituting equations (A.3) and (A.4) into equation (A.5), we find the equation (11).

Appendix B

Substituting equations (9) and (11) into equation (12), the P_d becomes:

$$P_d = \frac{k_1 k_2}{|\alpha||\beta|}\binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2}\int_0^\infty\int_0^q \left[\begin{array}{l} e^{-\frac{q}{1+S}}\, e^{-\frac{y}{\beta}\left(\frac{N}{2}-k_2+1\right)}\left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1} \\ \times\, e^{-\frac{q-y}{\alpha}\left(\frac{N}{2}-k_1+1\right)}\left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_1-1} \end{array}\right] dy\, dq \tag{A.6}$$

Using the following identity [7]:

$$\int_0^\infty f_3(q)\int_0^q f_1(y)\, f_2(q-y) dy\, dq = \int_0^\infty f_1(y)\int_y^\infty f_3(q)\, f_2(q-y) dq\, dy \tag{A.7}$$

The equation (A.6) becomes:

$$P_d = \frac{k_1 k_2}{|\alpha||\beta|}\binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2}\int_0^\infty \left[\begin{array}{c} e^{-\frac{y}{\beta}\left(\frac{N}{2}-k_2+1\right)}\left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1} \times \\ \int_y^\infty e^{-\frac{q}{1+S}} e^{-\frac{q-y}{\alpha}\left(\frac{N}{2}-k_1+1\right)}\left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_1-1} dq \end{array}\right] dy \tag{A.8}$$

Now, applying the change of variable $q = v + y$, the equation (A.8) can then be simplified to:

$$P_d = \frac{k_1 k_2}{|\alpha||\beta|}\binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2}\left[\int_0^\infty e^{-y\left[\frac{1}{\beta}\left(\frac{N}{2}-k_2+1\right)+\frac{1}{1+S}\right]}\left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1} dy\right] \times \left[\int_0^\infty e^{-v\left[\frac{1}{\alpha}\left(\frac{N}{2}-k_1+1\right)+\frac{1}{1+S}\right]}\left(1-e^{-\frac{v}{\alpha}}\right)^{k_1-1} dv\right] \tag{A.9}$$

Now, applying again the change of variable $y = -\ln(w)$ in the first integral and $v = -\ln(w)$ in the second one, the equation (A.9) can then be rewritten as:

$$P_d = \frac{k_1 k_2}{|\alpha||\beta|}\binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2}\left[\int_0^1 w^{\left[\frac{1}{\beta}\left(\frac{N}{2}-k_2+1\right)+\frac{1}{1+S}-1\right]}\left(1-w^{\frac{1}{\beta}}\right)^{k_2-1} dw\right] \times \left[\int_0^1 w^{\left[\frac{1}{\alpha}\left(\frac{N}{2}-k_1+1\right)+\frac{1}{1+S}-1\right]}\left(1-w^{\frac{1}{\alpha}}\right)^{k_1-1} dw\right] \tag{A.10}$$

To evaluate the two integrals of equation (A.10), we use:

$$\int_0^1 w^{a-1}(1-w^b)^c dw = \frac{\Gamma(c+1)\,\Gamma\left(\frac{a}{b}\right)}{b\,\Gamma\left(\frac{a}{b}+c+1\right)} \tag{A.11}$$

Finally, we find the closed form of the P_d mentioned previously by the equation (14).

Appendix C

The pdf $f_{Q_{OSGO}}(q)$ and $f_{Q_{OSSO}}(q)$ are defined by [5, 6]:

$$f_{Q_{OSGO}}(q) = f_{Q_{(k_1)}}(q)F_{Q_{(k_2)}}(q) + f_{Q_{(k_2)}}(q)F_{Q_{(k_1)}}(q) \tag{A.12}$$

and

$$f_{Q_{OSSO}}(q) = f_{Q_{(k_1)}}(q)\left(1-F_{Q_{(k_2)}}(q)\right) + f_{Q_{(k_2)}}(q)\left(1-F_{Q_{(k_1)}}(q)\right) \tag{A.13}$$

where $f_{Q_{(k_1)}}(q)$ and $f_{Q_{(k_2)}}(q)$ are obtained from the equation (6). Also, $F_{Q_{(k_1)}}(q)$ and $F_{Q_{(k_2)}}(q)$ are obtained from the equation (7).

Rescaling q_{OSGO} and q_{OSSO} by $\alpha > 0$ and $\beta > 0$, respectively, we obtain two new independent random variables $X = \alpha\, q_{OSGO}$ and $Y = \beta\, q_{OSSO}$. Having $q_{OSGO} = X/\alpha$ and $q_{OSSO} = Y/\beta$, the pdfs of X and Y are given, based on the equations (A.1) and (A.2) with Q_{OSGO} and Q_{OSSO} instead of $Q_{(k_1)}$ and $Q_{(k_2)}$, respectively, by:

$$f_X(x) = \frac{1}{|\alpha|}\left[\begin{array}{l} k_1 \binom{\frac{N}{2}}{k_1} \displaystyle\sum_{i=k_2}^{N/2} \binom{\frac{N}{2}}{i}\, e^{-\frac{x}{\alpha}(N-k_1+1-i)} \left(1-e^{-\frac{x}{\alpha}}\right)^{k_1-1+i} \\ +k_2 \binom{\frac{N}{2}}{k_2} \displaystyle\sum_{i=k_1}^{N/2} \binom{\frac{N}{2}}{i}\, e^{-\frac{x}{\alpha}(N-k_2+1-i)} \left(1-e^{-\frac{x}{\alpha}}\right)^{k_2-1+i} \end{array}\right] \tag{A.14}$$

and

$$f_Y(y) = \frac{1}{|\beta|}\left[\begin{array}{l} k_1 \binom{\frac{N}{2}}{k_1}\, e^{-\frac{y}{\beta}\left(\frac{N}{2}-k_1+1\right)} \left(1-e^{-\frac{y}{\beta}}\right)^{k_1-1} \left(1-\displaystyle\sum_{j=k_2}^{N/2} \binom{\frac{N}{2}}{j}\, e^{-\frac{y}{\beta}(N/2-j)} \left(1-e^{-\frac{y}{\beta}}\right)^{j}\right) \\ +k_2 \binom{\frac{N}{2}}{k_2}\, e^{-\frac{y}{\beta}\left(\frac{N}{2}-k_2+1\right)} \left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1} \left(1-\displaystyle\sum_{j=k_1}^{N/2} \binom{\frac{N}{2}}{j}\, e^{-\frac{y}{\beta}\left(\frac{N}{2}-j\right)} \left(1-e^{-\frac{y}{\beta}}\right)^{j}\right) \end{array}\right] \tag{A.15}$$

The adaptive detection threshold, denoted by Q, is the sum of X and Y. Using the fact that $X \geq 0$ and $Y \geq 0$, substituting equations (A.14) and (A.15) into equation (A.5), the pdf of Q is given by the equation (27).

Appendix D

Substituting equations (9) and (27) into equation (12), the P_d becomes:

$$\begin{aligned} P_d = \frac{1}{|\alpha||\beta|}\Bigg\{ & k_1^2 \binom{\frac{N}{2}}{k_1}^2 \sum_{i=k_2}^{N/2} \binom{\frac{N}{2}}{i} \int_0^\infty e^{-\frac{q}{1+S}} \int_0^q \left[\begin{array}{c} e^{-\frac{y}{\beta}\left(\frac{N}{2}-k_1+1\right)} \left(1-e^{-\frac{y}{\beta}}\right)^{k_1-1} \times \\ e^{-\frac{q-y}{\alpha}(N-k_1+1-i)} \left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_1-1+i} \end{array}\right] dy\, dq \\ & +k_1 k_2 \binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2} \sum_{i=k_1}^{N/2} \binom{\frac{N}{2}}{i} \int_0^\infty e^{-\frac{q}{1+S}} \int_0^q \left[\begin{array}{c} e^{-\frac{y}{\beta}\left(\frac{N}{2}-k_1+1\right)} \left(1-e^{-\frac{y}{\beta}}\right)^{k_1-1} \times \\ e^{-\frac{q-y}{\alpha}(N-k_2+1-i)} \left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_2-1+i} \end{array}\right] dy\, dq \\ & +k_1 k_2 \binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2} \sum_{i=k_2}^{N/2} \binom{\frac{N}{2}}{i} \int_0^\infty e^{-\frac{q}{1+S}} \int_0^q \left[\begin{array}{c} e^{-\frac{y}{\beta}\left(\frac{N}{2}-k_2+1\right)} \left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1} \times \\ e^{-\frac{q-y}{\alpha}(N-k_1+1-i)} \left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_1-1+i} \end{array}\right] dy\, dq \\ & +k_2^2 \binom{\frac{N}{2}}{k_2}^2 \sum_{i=k_1}^{N/2} \binom{\frac{N}{2}}{i} \int_0^\infty e^{-\frac{q}{1+S}} \int_0^q \left[\begin{array}{c} e^{-\frac{y}{\beta}\left(\frac{N}{2}-k_2+1\right)} \left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1} \times \\ e^{-\frac{q-y}{\alpha}(N-k_2+1-i)} \left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_2-1+i} \end{array}\right] dy\, dq \\ & -k_1^2 \binom{\frac{N}{2}}{k_1}^2 \sum_{i=k_2}^{N/2} \binom{\frac{N}{2}}{i} \sum_{j=k_2}^{N/2} \binom{\frac{N}{2}}{j} \int_0^\infty e^{-\frac{q}{1+S}} \int_0^q \left[\begin{array}{c} e^{-\frac{y}{\beta}(N-k_1+1-j)} \left(1-e^{-\frac{y}{\beta}}\right)^{k_1-1+j} \times \\ e^{-\frac{q-y}{\alpha}(N-k_1+1-i)} \left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_1-1+i} \end{array}\right] dy\, dq \\ & -k_1 k_2 \binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2} \sum_{i=k_1}^{N/2} \binom{\frac{N}{2}}{i} \sum_{j=k_2}^{N/2} \binom{\frac{N}{2}}{j} \int_0^\infty e^{-\frac{q}{1+S}} \int_0^q \left[\begin{array}{c} e^{-\frac{y}{\beta}(N-k_1+1-j)} \left(1-e^{-\frac{y}{\beta}}\right)^{k_1-1+j} \times \\ e^{-\frac{q-y}{\alpha}(N-k_2+1-i)} \left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_2-1+i} \end{array}\right] dy\, dq \\ & -k_1 k_2 \binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2} \sum_{i=k_2}^{N/2} \binom{\frac{N}{2}}{i} \sum_{j=k_1}^{N/2} \binom{\frac{N}{2}}{j} \int_0^\infty e^{-\frac{q}{1+S}} \int_0^q \left[\begin{array}{c} e^{-\frac{y}{\beta}(N-k_2+1-j)} \left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1+j} \times \\ e^{-\frac{q-y}{\alpha}(N-k_1+1-i)} \left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_1-1+i} \end{array}\right] dy\, dq \end{aligned}$$

$$-k_2^2\binom{\frac{N}{2}}{k_2}^2\sum_{i=k_1}^{N/2}\binom{\frac{N}{2}}{i}\sum_{j=k_1}^{N/2}\binom{\frac{N}{2}}{j}\int_0^{\infty}e^{-\frac{q}{1+S}}\int_0^{q}\left[\begin{array}{c}e^{-\frac{y}{\beta}(N-k_2+1-j)}\left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1+j}\times\\ e^{-\frac{q-y}{\alpha}(N-k_2+1-i)}\left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_2-1+i}\end{array}\right]dy\,dq\Bigg\}\quad (A.16)$$

Using the identity (A.7), the equation (A.16) becomes:

$$P_d=\frac{1}{|\alpha||\beta|}\Bigg\{k_1^2\binom{\frac{N}{2}}{k_1}^2\sum_{i=k_2}^{N/2}\binom{\frac{N}{2}}{i}\int_0^{\infty}\left[\begin{array}{c}e^{-\frac{y}{\beta}\left(\frac{N}{2}-k_1+1\right)}\left(1-e^{-\frac{y}{\beta}}\right)^{k_1-1}\times\\ \int_y^{\infty}e^{-\frac{q}{1+S}}e^{-\frac{q-y}{\alpha}(N-k_1+1-i)}\left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_1-1+i}dq\end{array}\right]dy$$

$$+k_1k_2\binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2}\sum_{i=k_1}^{N/2}\binom{\frac{N}{2}}{i}\int_0^{\infty}\left[\begin{array}{c}e^{-\frac{y}{\beta}\left(\frac{N}{2}-k_1+1\right)}\left(1-e^{-\frac{y}{\beta}}\right)^{k_1-1}\times\\ \int_y^{\infty}e^{-\frac{q}{1+S}}e^{-\frac{q-y}{\alpha}(N-k_2+1-i)}\left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_2-1+i}dq\end{array}\right]dy$$

$$+k_1k_2\binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2}\sum_{i=k_2}^{N/2}\binom{\frac{N}{2}}{i}\int_0^{\infty}\left[\begin{array}{c}e^{-\frac{y}{\beta}\left(\frac{N}{2}-k_2+1\right)}\left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1}\times\\ \int_y^{\infty}e^{-\frac{q}{1+S}}e^{-\frac{q-y}{\alpha}(N-k_1+1-i)}\left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_1-1+i}dq\end{array}\right]dy$$

$$+k_2^2\binom{\frac{N}{2}}{k_2}^2\sum_{i=k_1}^{N/2}\binom{\frac{N}{2}}{i}\int_0^{\infty}\left[\begin{array}{c}e^{-\frac{y}{\beta}\left(\frac{N}{2}-k_2+1\right)}\left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1}\times\\ \int_y^{\infty}e^{-\frac{q}{1+S}}e^{-\frac{q-y}{\alpha}(N-k_2+1-i)}\left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_2-1+i}dq\end{array}\right]dy$$

$$-k_1^2\binom{\frac{N}{2}}{k_1}^2\sum_{i=k_2}^{N/2}\binom{\frac{N}{2}}{i}\sum_{j=k_2}^{N/2}\binom{\frac{N}{2}}{j}\int_0^{\infty}\left[\begin{array}{c}e^{-\frac{y}{\beta}(N-k_1+1-j)}\left(1-e^{-\frac{y}{\beta}}\right)^{k_1-1+j}\times\\ \int_y^{\infty}e^{-\frac{q}{1+S}}e^{-\frac{q-y}{\alpha}(N-k_1+1-i)}\left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_1-1+i}dq\end{array}\right]dy$$

$$-k_1k_2\binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2}\sum_{i=k_1}^{N/2}\binom{\frac{N}{2}}{i}\sum_{j=k_2}^{N/2}\binom{\frac{N}{2}}{j}\int_0^{\infty}\left[\begin{array}{c}e^{-\frac{y}{\beta}(N-k_1+1-j)}\left(1-e^{-\frac{y}{\beta}}\right)^{k_1-1+j}\times\\ \int_y^{\infty}e^{-\frac{q}{1+S}}e^{-\frac{q-y}{\alpha}(N-k_2+1-i)}\left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_2-1+i}dq\end{array}\right]dy$$

$$-k_1k_2\binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2}\sum_{i=k_2}^{N/2}\binom{\frac{N}{2}}{i}\sum_{j=k_1}^{N/2}\binom{\frac{N}{2}}{j}\int_0^{\infty}\left[\begin{array}{c}e^{-\frac{y}{\beta}(N-k_2+1-j)}\left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1+j}\times\\ \int_y^{\infty}e^{-\frac{q}{1+S}}e^{-\frac{q-y}{\alpha}(N-k_1+1-i)}\left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_1-1+i}dq\end{array}\right]dy$$

$$-k_2^2\binom{\frac{N}{2}}{k_2}^2\sum_{i=k_1}^{N/2}\binom{\frac{N}{2}}{i}\sum_{j=k_1}^{N/2}\binom{\frac{N}{2}}{j}\int_0^{\infty}\left[\begin{array}{c}e^{-\frac{y}{\beta}(N-k_2+1-j)}\left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1+j}\times\\ \int_y^{\infty}e^{-\frac{q}{1+S}}e^{-\frac{q-y}{\alpha}(N-k_2+1-i)}\left(1-e^{-\frac{q-y}{\alpha}}\right)^{k_2-1+i}dq\end{array}\right]dy\quad (A.17)$$

Now, applying the change of variable $q=v+y$, the equation (A.17) can then be simplified to:

$$
\begin{aligned}
P_d = \frac{1}{|\alpha||\beta|}\Bigg\{ & k_1^2 \binom{\frac{N}{2}}{k_1}^2 \sum_{i=k_2}^{N/2} \binom{\frac{N}{2}}{i} \left[\begin{array}{l} \int_0^\infty e^{-y\left[\frac{1}{\beta}\left(\frac{N}{2}-k_1+1\right)+\frac{1}{1+S}\right]} \left(1-e^{-\frac{y}{\beta}}\right)^{k_1-1} dy \\ \times \int_0^\infty e^{-v\left[\frac{1}{\alpha}(N-k_1+1-i)+\frac{1}{1+S}\right]} \left(1-e^{-\frac{v}{\alpha}}\right)^{k_1-1+i} dv \end{array} \right] \\
& + k_1 k_2 \binom{\frac{N}{2}}{k_1} \binom{\frac{N}{2}}{k_2} \sum_{i=k_1}^{N/2} \binom{\frac{N}{2}}{i} \left[\begin{array}{l} \int_0^\infty e^{-y\left[\frac{1}{\beta}\left(\frac{N}{2}-k_1+1\right)+\frac{1}{1+S}\right]} \left(1-e^{-\frac{y}{\beta}}\right)^{k_1-1} dy \\ \int_0^\infty e^{-v\left[\frac{1}{\alpha}(N-k_2+1-i)+\frac{1}{1+S}\right]} \left(1-e^{-\frac{v}{\alpha}}\right)^{k_2-1+i} dv \end{array} \right] \\
& + k_1 k_2 \binom{\frac{N}{2}}{k_1} \binom{\frac{N}{2}}{k_2} \sum_{i=k_2}^{N/2} \binom{\frac{N}{2}}{i} \left[\begin{array}{l} \int_0^\infty e^{-y\left[\frac{1}{\beta}\left(\frac{N}{2}-k_2+1\right)+\frac{1}{1+S}\right]} \left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1} dy \\ \int_0^\infty e^{-v\left[\frac{1}{\alpha}(N-k_1+1-i)+\frac{1}{1+S}\right]} \left(1-e^{-\frac{v}{\alpha}}\right)^{k_1-1+i} dv \end{array} \right] \\
& + k_2^2 \binom{\frac{N}{2}}{k_2}^2 \sum_{i=k_1}^{N/2} \binom{\frac{N}{2}}{i} \left[\begin{array}{l} \int_0^\infty e^{-y\left[\frac{1}{\beta}\left(\frac{N}{2}-k_2+1\right)+\frac{1}{1+S}\right]} \left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1} dy \\ \int_0^\infty e^{-v\left[\frac{1}{\alpha}(N-k_2+1-i)+\frac{1}{1+S}\right]} \left(1-e^{-\frac{v}{\alpha}}\right)^{k_2-1+i} dv \end{array} \right] \\
& - k_1^2 \binom{\frac{N}{2}}{k_1}^2 \sum_{i=k_2}^{N/2} \binom{\frac{N}{2}}{i} \sum_{j=k_2}^{N/2} \binom{\frac{N}{2}}{j} \left[\begin{array}{l} \int_0^\infty e^{-y\left[\frac{1}{\beta}(N-k_1+1-j)+\frac{1}{1+S}\right]} \left(1-e^{-\frac{y}{\beta}}\right)^{k_1-1+j} dy \\ \int_0^\infty e^{-v\left[\frac{1}{\alpha}(N-k_1+1-i)+\frac{1}{1+S}\right]} \left(1-e^{-\frac{v}{\alpha}}\right)^{k_1-1+i} dv \end{array} \right] \\
& - k_1 k_2 \binom{\frac{N}{2}}{k_1} \binom{\frac{N}{2}}{k_2} \sum_{i=k_1}^{N/2} \binom{\frac{N}{2}}{i} \sum_{j=k_2}^{N/2} \binom{\frac{N}{2}}{j} \left[\begin{array}{l} \int_0^\infty e^{-y\left[\frac{1}{\beta}(N-k_1+1-j)+\frac{1}{1+S}\right]} \left(1-e^{-\frac{y}{\beta}}\right)^{k_1-1+j} dy \\ \int_0^\infty e^{-v\left[\frac{1}{\alpha}(N-k_2+1-i)+\frac{1}{1+S}\right]} \left(1-e^{-\frac{v}{\alpha}}\right)^{k_2-1+i} dv \end{array} \right] \\
& - k_1 k_2 \binom{\frac{N}{2}}{k_1} \binom{\frac{N}{2}}{k_2} \sum_{i=k_2}^{N/2} \binom{\frac{N}{2}}{i} \sum_{j=k_1}^{N/2} \binom{\frac{N}{2}}{j} \left[\begin{array}{l} \int_0^\infty e^{-y\left[\frac{1}{\beta}(N-k_2+1-j)+\frac{1}{1+S}\right]} \left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1+j} dy \\ \int_0^\infty e^{-v\left[\frac{1}{\alpha}(N-k_1+1-i)+\frac{1}{1+S}\right]} \left(1-e^{-\frac{v}{\alpha}}\right)^{k_1-1+i} dv \end{array} \right] \\
& - k_2^2 \binom{\frac{N}{2}}{k_2}^2 \sum_{i=k_1}^{N/2} \binom{\frac{N}{2}}{i} \sum_{j=k_1}^{N/2} \binom{\frac{N}{2}}{j} \left[\begin{array}{l} \int_0^\infty e^{-y\left[\frac{1}{\beta}(N-k_2+1-j)+\frac{1}{1+S}\right]} \left(1-e^{-\frac{y}{\beta}}\right)^{k_2-1+j} dy \\ \int_0^\infty e^{-v\left[\frac{1}{\alpha}(N-k_2+1-i)+\frac{1}{1+S}\right]} \left(1-e^{-\frac{v}{\alpha}}\right)^{k_2-1+i} dv \end{array} \right] \Bigg\}
\end{aligned} \tag{A.18}
$$

Now, applying again the change of variables $y = -\ln(w)$ and $v = -\ln(w)$, the equation (A.18) can then be rewritten as:

$$
\begin{aligned}
P_d = \frac{1}{|\alpha||\beta|}\Bigg\{ & k_1^2 \binom{\frac{N}{2}}{k_1}^2 \sum_{i=k_2}^{N/2}\binom{\frac{N}{2}}{i}\left[\begin{array}{l}\int_0^1 w^{\left[\frac{1}{\beta}\left(\frac{N}{2}-k_1+1\right)+\frac{1}{1+S}-1\right]}\left(1-w^{\frac{1}{\beta}}\right)^{k_1-1} dw \\ \int_0^1 w^{\left[\frac{1}{\alpha}(N-k_1+1-i)+\frac{1}{1+S}-1\right]}\left(1-w^{\frac{1}{\alpha}}\right)^{k_1-1+i} dw\end{array}\right] \\
& + k_1 k_2 \binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2} \sum_{i=k_1}^{N/2}\binom{\frac{N}{2}}{i}\left[\begin{array}{l}\int_0^1 w^{\left[\frac{1}{\beta}\left(\frac{N}{2}-k_1+1\right)+\frac{1}{1+S}-1\right]}\left(1-w^{\frac{1}{\beta}}\right)^{k_1-1} dw \\ \int_0^1 w^{\left[\frac{1}{\alpha}(N-k_2+1-i)+\frac{1}{1+S}-1\right]}\left(1-w^{\frac{1}{\alpha}}\right)^{k_2-1+i} dw\end{array}\right] \\
& + k_1 k_2 \binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2} \sum_{i=k_2}^{N/2}\binom{\frac{N}{2}}{i}\left[\begin{array}{l}\int_0^1 w^{\left[\frac{1}{\beta}\left(\frac{N}{2}-k_2+1\right)+\frac{1}{1+S}-1\right]}\left(1-w^{\frac{1}{\beta}}\right)^{k_2-1} dw \\ \int_0^1 w^{\left[\frac{1}{\alpha}(N-k_1+1-i)+\frac{1}{1+S}-1\right]}\left(1-w^{\frac{1}{\alpha}}\right)^{k_1-1+i} dw\end{array}\right] \\
& + k_2^2 \binom{\frac{N}{2}}{k_2}^2 \sum_{i=k_1}^{N/2}\binom{\frac{N}{2}}{i}\left[\begin{array}{l}\int_0^1 w^{\left[\frac{1}{\beta}\left(\frac{N}{2}-k_2+1\right)+\frac{1}{1+S}-1\right]}\left(1-w^{\frac{1}{\beta}}\right)^{k_2-1} dw \\ \int_0^1 w^{\left[\frac{1}{\alpha}(N-k_2+1-i)+\frac{1}{1+S}-1\right]}\left(1-w^{\frac{1}{\alpha}}\right)^{k_2-1+i} dw\end{array}\right] \\
& - k_1^2 \binom{\frac{N}{2}}{k_1}^2 \sum_{i=k_2}^{N/2}\binom{\frac{N}{2}}{i}\sum_{j=k_2}^{N/2}\binom{\frac{N}{2}}{j}\left[\begin{array}{l}\int_0^1 w^{\left[\frac{1}{\beta}(N-k_1+1-j)+\frac{1}{1+S}-1\right]}\left(1-w^{\frac{1}{\beta}}\right)^{k_1-1+j} dw \\ \int_0^1 w^{\left[\frac{1}{\alpha}(N-k_1+1-i)+\frac{1}{1+S}-1\right]}\left(1-w^{\frac{1}{\alpha}}\right)^{k_1-1+i} dw\end{array}\right] \\
& - k_1 k_2 \binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2} \sum_{i=k_1}^{\frac{N}{2}}\binom{\frac{N}{2}}{i}\sum_{j=k_2}^{\frac{N}{2}}\binom{\frac{N}{2}}{j}\left[\begin{array}{l}\int_0^1 w^{\left[\frac{1}{\beta}(N-k_1+1-j)+\frac{1}{1+S}-1\right]}\left(1-w^{\frac{1}{\beta}}\right)^{k_1-1+j} dw \\ \int_0^1 w^{\left[\frac{1}{\alpha}(N-k_2+1-i)+\frac{1}{1+S}-1\right]}\left(1-w^{\frac{1}{\alpha}}\right)^{k_2-1+i} dw\end{array}\right] \\
& - k_1 k_2 \binom{\frac{N}{2}}{k_1}\binom{\frac{N}{2}}{k_2} \sum_{i=k_2}^{N/2}\binom{\frac{N}{2}}{i}\sum_{j=k_1}^{N/2}\binom{\frac{N}{2}}{j}\left[\begin{array}{l}\int_0^1 w^{\left[\frac{1}{\beta}(N-k_2+1-j)+\frac{1}{1+S}-1\right]}\left(1-w^{\frac{1}{\beta}}\right)^{k_2-1+j} dw \\ \int_0^1 w^{\left[\frac{1}{\alpha}(N-k_1+1-i)+\frac{1}{1+S}-1\right]}\left(1-w^{\frac{1}{\alpha}}\right)^{k_1-1+i} dw\end{array}\right] \\
& - k_2^2 \binom{\frac{N}{2}}{k_2}^2 \sum_{i=k_1}^{N/2}\binom{\frac{N}{2}}{i}\sum_{j=k_1}^{N/2}\binom{\frac{N}{2}}{j}\left[\begin{array}{l}\int_0^1 w^{\left[\frac{1}{\beta}(N-k_2+1-j)+\frac{1}{1+S}-1\right]}\left(1-w^{\frac{1}{\beta}}\right)^{k_2-1+j} dw \\ \int_0^1 w^{\left[\frac{1}{\alpha}(N-k_2+1-i)+\frac{1}{1+S}-1\right]}\left(1-w^{\frac{1}{\alpha}}\right)^{k_2-1+i} dw\end{array}\right]\Bigg\}
\end{aligned}
\tag{A.19}
$$

Now, using the equation (A.11) to evaluate the integrals of the equation (A.19), we find the closed form of the P_d mentioned previously by the equation (28).

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