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SAID-BALL POLYNOMIALS FOR SOLVING LINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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كثيرات حدود سعيد-بول لحل انظمة خطية من المعادلات التفاضلية العادية

ملخص

استخدمت طريقة التجميع بالاعتماد على متعددات حدود سعيد- بول لإيجاد الحل العددي لنظام من المعادلات التفاضلية الخطية العادية. حيث حصلنا على صيغ المصفوفة التشغيلية للحل و المشتقات و الشروط المعطاة. هذه الطريقة حولت المسألة المعطاة الى نظام من المعادلات الجبرية الخطية في معاملات متعددات حدود سعيد- بول. تم حل هذا النظام الخطي و وجدنا هذه المعاملات و من ثم حصلنا على الحلول المضبوطة و التقريبية للمسألة المعطاة. النتائج العددية التي حصلنا عليها أثبتت دقة و فعالية الطريقة المقترحة عند مقارنتها مع اعمال أخرى و الحلول المضبوطة.

Abstract

Said-Ball polynomials with collocation method are used to numerically solve a system of linear ordinary differential equations. The matrix forms of Said-Ball polynomials of the solution, derivatives, and conditions are done. The linear system of ordinary differential equations with appropriate conditions is reduced to the linear algebraic equations system with unknown Said-Ball coefficients. Solving the resulting system determines the coefficients of Said-Ball polynomials. By Substituting these values in the polynomial, we get the problem's exact and approximate solutions. The obtaining numerical results show the proposed method's accuracy and reliability when compared with the other works and exact solutions.

Keywords: Said-Ball polynomials, collocation method, linear ordinary differential equations, matrix form.

1. Introduction

Differential equations play an important role in engineering and physics sciences. One of the challenging problems in applied mathematics is finding the solution to differential equations or their systems. Also, the most difficult has lain in the case of higher-order systems of differential equations. There are various methods to solve some systems of differential equations numerically. For instance, Akyüz-Daşcıoğlu and Sezer [1] used Chebyshev polynomial approach to solve the systems of high-

order linear differential equations with variable coefficients. Biazar et al. [2] employed the Adomian decomposition method to solve ordinary differential equations. Rationalized Haar functions method is used to solve the linear integro-differential equations system [3]. In [4], Sezer et al. implemented the Taylor polynomial approach to obtain the solutions of systems of linear differential equations with variable coefficients. Jafari and Daftardar-Gejji [5] solved systems of ordinary and fractional differential equations using the revised Adomian decomposition method. [6] Abdel and Hassan used the differential transformation method to solve systems of differential equations. Tatari and Dehghan [7] employed the Improvement of He's Variational iteration method to solve systems of differential equations. Thongmoon and Pusjuso [8] obtained the numerical solutions of the system of differential equations using the differential transform method and the Laplace transform method. In [9], Javidi presented a Modified homotopy perturbation method to solve the linear Fredholm integral equations system. The homotopy analysis method is used to handle systems of fractional differential equations [10]. Yüzbaşi et al. solved linear differential equation systems using the Bessel collocation method [11]. Yüzbaşi [12] employed an efficient algorithm to solve the multi-pantograph equation systems. Yüksel et al. [13] solved high-order linear Fredholm-Volterra integro-dfferential equations via Chebyshev polynomial approach. Ramadan and Abd El Salam [14] applied the exponential Chebyshev collocation method to solve systems of ordinary differential equations in unbounded domains. In [15], Yuzbasi_ and Yldirim solved the systems of first-order linear differential equations using the Laguerre collocation method. For more methods, see [16-25]. Recently, Yüzbaşi and Yildirim [26] used Laguerre polynomial approach to solve the systems of linear differential equations. Sezer and Kürkçü [27] obtained the Charlier series solutions of the systems of the delay differential equations of the first order based on the Charlier polynomials and the collaboration of the matrices.
In this study, we will consider the following linear system of ordinary differential equations [26]:

$$\sum_{k=o}^{n}\sum_{j=1}^{r} P_{i,j}^{k}(x)\ y_j^{(k)}(x) = f_i(x),\ \ i = 1,2,\cdots,r, \tag{1}$$

Subject to mixed conditions

$$\sum_{j=o}^{n-1} \alpha_{i,j}^{k}\, y_k^{(j)}(\alpha) + \beta_{i,j}^{k}\, y_k^{(j)}(\beta) = \eta_{n,i} \tag{2}$$

Where $i = 0,1,\cdots,n-1,\ k = 1,2,\cdots,r$. The functions $P_{i,j}^{k}(x)$ and $f_i(x)$ are given and defined on the nonnegative interval $[\alpha,\beta]$, $y_j^{(0)}(x) = y_j(x)$ is an unknown function to be determined and $\alpha_{i,j}^{k}, \beta_{i,j}^{k}$ and $\eta_{n,i}$ are suitable real constants.
The purpose of this study is to apply a matrix method using Said-Ball Polynomials [28, 29, 30, 31, 32] to solve the linear system of ordinary differential equations (1) with some appropriate conditions (2).

This paper is organized as follows: Section 2 presents some concepts of Ball polynomials, Said-Ball polynomials, and Said-Ball monomial formulas. Relations of the fundamental matrix are given in section 3. In section 4, residual error estimation and solutions accuracy is presented. In section 5, numerical examples are presented. The conclusion is presented in section 6.

2. Said-Ball Polynomial and Said-Ball monomial formulas:

Said-Ball Polynomial basis function $S_i^m(x)$ of degree m is defined as [33, 34, 35,38]:

$$S_i^m(x) = \begin{cases} \binom{\frac{m-1}{2}+i}{i} x^i (1-x)^{\frac{m}{2}+\frac{1}{2}}, & 0 \le i \le \frac{m}{2}-\frac{1}{2}, \\ \binom{\frac{3m-2i}{2}-\frac{1}{2}}{m-i} x^{\frac{m}{2}+\frac{1}{2}} (1-x)^{m-i}, & \frac{m}{2}+\frac{1}{2} \le i \le m \end{cases} \quad (3)$$

when m is an odd and

$$S_i^m(x) = \begin{cases} \binom{\frac{m}{2}+i}{i} x^i (1-x)^{\frac{m}{2}+1}, & 0 \le i \le \frac{m}{2}-1, \\ \binom{m}{m/2} x^{\frac{m}{2}} (1-x)^{\frac{m}{2}}, & i = \frac{m}{2}, \\ \binom{\frac{3m}{2}-i}{m-i} x^{\frac{m}{2}+1} (1-x)^{m-1}, & \frac{m}{2} \le i \le m \end{cases} \quad (4)$$

When m is an even.

Said-Ball curve $S^m(x)$ of degree m with $m+1$ control points, denoted by $\{\zeta_i\}_{i=0}^m$, can be expressed as the following form in power basis:

$$S^m(x) = \sum_{i=0}^{m} \sum_{j=0}^{m} \zeta_i \, s_{i,j} \, x^j, \quad 0 \le x \le 1, \quad (5)$$

Where,

$$s_{i,j} = \begin{cases} (-1)^{j-i} \binom{i+\lfloor \frac{m}{2} \rfloor}{i} \binom{\lfloor \frac{m}{2} \rfloor + 1}{j-i}, & 0 \le i \le \lceil \frac{m}{2} \rceil - 1, \\ (-1)^{j-i} \binom{m}{i} \binom{i}{j-i}, & i = \frac{m}{2}, \\ (-1)^{j-\lfloor \frac{m}{2} \rfloor - 1} \binom{\lfloor \frac{m}{2} \rfloor + m - i}{m-i} \binom{m-i}{j-\lfloor \frac{m}{2} \rfloor - 1}, & \lfloor \frac{m}{2} \rfloor + 1 \le i \le m. \end{cases} \quad (6)$$

The Said-Ball monomial matrix is given by:

$$S_{(m+1)\times(m+1)} = \begin{bmatrix} s_{0,0} & s_{0,1} & \cdots & \cdots & s_{0,m} \\ s_{1,0} & s_{1,1} & \cdots & \cdots & s_{1,m} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ s_{m,0} & s_{m,1} & \cdots & \cdots & s_{m,m} \end{bmatrix} \quad (7)$$

Where $s_{i,j}$ is defined in Eq. (6).

To obtain approximate solutions of Eq. (1) with conditions (2). We will use truncated Said-Ball Series in the form:

$$y_{i,M}(x) = \sum_{m=0}^{M} c_{i,m} \, S^m(x), \qquad i = 1, 2, \cdots, r, \quad (8)$$

Where $c_{i,m}$, $m = 0,1,\cdots,M$ are unknown Said-Ball coefficients to be determined.

3.1 Relations of the fundamental matrix

We can be written the approximate solutions of Eq.(8) in matrix form as:

$$y_{j,M}(x) = \boldsymbol{S}(x)\boldsymbol{C}_j, \ \ j = 1,2,\cdots,r, \tag{9}$$

Where, $\boldsymbol{S}(x) = [S^0(x)\ \ S^1(x)\ \ \cdots\ \ S^M(x)]$ and $\boldsymbol{C}_j = \left[c_{j,0}\ \ c_{j,1}\ \ \cdots\ \ c_{j,M}\right]^T$, $j = 1,2,\cdots,r$.

Now, the first derivative of Eq.(9) is

$$y'_{j,M}(x) = \boldsymbol{S}'(x)\boldsymbol{C}_j\,, \qquad j = 1,2,\cdots,r, \tag{10}$$

Where,

$$\boldsymbol{S}'(x) = \left[S^{0'}(x)\ \ S^{1'}(x)\ \ \cdots\ \ S^{M'}(x)\right]$$
$$= \boldsymbol{S}(x)\boldsymbol{Q}, \tag{11}$$

Where, $\boldsymbol{Q} = \mathcal{S}^{-1}V\mathcal{S}$, and

$$\boldsymbol{V} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & M \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(M+1)\times(M+1)}$$

Therefore, Eq.(10) becomes

$$y'_{j,M}(x) = \boldsymbol{S}(x)\boldsymbol{Q}\boldsymbol{C}_j\,, \ j = 1,2,\cdots,r, \tag{12}$$

Similarly, the k-th order derivative of $y_j^{(k)}(x)$ in Eq.(1) can be derived as the following steps:

$$y_{j,M}^{(k)}(x) = \boldsymbol{S}^{(k)}(x)\boldsymbol{C}_j\,, \qquad j = 1,2,\cdots,r, \tag{13}$$
$$= \left[S^{0^{(k)}}(x)\ \ S^{1^{(k)}}(x)\ \ \cdots\ \ S^{M^{(k)}}(x)\right]\boldsymbol{C}_j$$
$$= [S^0(x)\ \ S^1(x)\ \ \cdots\ \ S^M(x)]\left(\boldsymbol{Q}^{\boldsymbol{k}}\right)\boldsymbol{C}_j$$
$$= \boldsymbol{S}(x)(\boldsymbol{Q}^{\boldsymbol{k}})\ \boldsymbol{C}_j\ \ j = 1,2,\cdots,r \tag{14}$$

Hence, $y^{(k)}(x)$ can be written as the following matrix form

$$\boldsymbol{y}^{(k)}(x) = \widehat{\boldsymbol{S}}(\boldsymbol{x})(\widehat{\boldsymbol{Q}^{\boldsymbol{k}}})\ \boldsymbol{C}, \quad k = 0,1,2,\cdots,r, \tag{15}$$

where,

$$\boldsymbol{y}^{(k)}(x) = \begin{bmatrix} y_1^{(k)}(x) \\ y_2^{(k)}(x) \\ \vdots \\ y_l^{(k)}(x) \end{bmatrix}_{r\times 1}, \widehat{\boldsymbol{S}}(\boldsymbol{x}) = \begin{bmatrix} S(x) & 0 & \cdots & 0 \\ 0 & S(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S(x) \end{bmatrix}_{r\times r}$$

$$\widehat{\boldsymbol{Q}^{\boldsymbol{k}}}(x) = \begin{bmatrix} \boldsymbol{Q}^k & 0 & \cdots & 0 \\ 0 & \boldsymbol{Q}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{Q}^k \end{bmatrix}_{r\times r}, \qquad \boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_1 \\ \boldsymbol{C}_2 \\ \vdots \\ \boldsymbol{C}_r \end{bmatrix}_{r\times 1}$$

3.2 Relation of the fundamental matrix via collocations points

The matrix form of the linear system (1) can be written as

$$\sum_{k=o}^{n} \boldsymbol{P}_k(x)\ \boldsymbol{y}^{(k)}(x) = \boldsymbol{f}(x), \tag{16}$$

where,

$$\boldsymbol{P}_k(x) = \begin{bmatrix} P_{1,1}^k(x) & P_{1,2}^k\,(x) & \cdots & P_{1,r}^k(x) \\ P_{2,1}^k(x) & P_{2,2}^k(x) & \cdots & P_{2,r}^k(x) \\ \vdots & \vdots & \ddots & \vdots \\ P_{n,1}^k(x) & P_{n,2}^k(x) & \cdots & P_{n,n}^k(x) \end{bmatrix}_{n\times n},$$

$$\boldsymbol{y}^{(m)}(x) = \begin{bmatrix} y_1^{(m)}(x) \\ y_2^{(m)}(x) \\ \vdots \\ y_n^{(m)}(x) \end{bmatrix}_{n\times 1}, \boldsymbol{f}(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}_{n\times 1}$$

The Said-Ball polynomials solution of system (16) in the form (8) can be obtained with the following collocations points

$$x_\tau = \frac{\beta+\alpha}{2} - \frac{\beta-\alpha}{2}\cos\left(\frac{\tau\,\pi}{M}\right) \quad ,\tau = 0,1,\cdots,M \tag{17}$$

Where, x belongs to the nonnegative interval $[\alpha,\beta]$ and $\alpha = x_0 < x_1 < \cdots < x_n = \beta$. Substituting the collocations points (17) into the system (16), we get the system

$$\sum_{k=o}^{n} \boldsymbol{P}_k(x_\tau)\ \boldsymbol{y}^{(k)}(x_\tau) = \boldsymbol{f}(x_\tau), \qquad \tau = 0,1,\cdots,M. \tag{18}$$

System (18) can be written in the following new matrix form

$$\sum_{k=o}^{n} \boldsymbol{P}_k\ \boldsymbol{Y}^{(k)} = \boldsymbol{F}, \tag{19}$$

where,

$$\boldsymbol{P}_k = \begin{bmatrix} \boldsymbol{P}_k(x_0) & 0 & \cdots & 0 \\ 0 & \boldsymbol{P}_k(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{P}_k(x_M) \end{bmatrix}, \boldsymbol{Y}^{(k)} = \begin{bmatrix} \boldsymbol{y}^{(k)}(x_0) \\ \boldsymbol{y}^{(k)}(x_1) \\ \vdots \\ \boldsymbol{y}^{(k)}(x_M) \end{bmatrix}, \boldsymbol{F} = \begin{bmatrix} \boldsymbol{f}(x_0) \\ \boldsymbol{f}(x_1) \\ \vdots \\ \boldsymbol{f}(x_M) \end{bmatrix}$$

Substitute (17) in (15). We get the following matrix from system

$$\boldsymbol{y}^{(k)}(x_\tau) = \widehat{\boldsymbol{S}}(x_\tau)\left(\widehat{\boldsymbol{Q}^{\boldsymbol{k}}}\right)\boldsymbol{C}, \quad \tau = 0,1,2,\cdots,M \tag{20}$$

System (20) can be written briefly as

$$\boldsymbol{Y}^{(k)} = \boldsymbol{S}\left(\widehat{\boldsymbol{Q}^{\boldsymbol{k}}}\right)\boldsymbol{C}, \tag{21}$$

Where,

$$\boldsymbol{S} = [\widehat{\boldsymbol{S}}(x_0) \quad \widehat{\boldsymbol{S}}(x_1) \quad \cdots \quad \widehat{\boldsymbol{S}}(x_M)]^T,$$

$$\widehat{\boldsymbol{S}}(x_\tau) = \begin{bmatrix} \boldsymbol{S}(x_\tau) & 0 & \cdots & 0 \\ 0 & \boldsymbol{S}(x_\tau) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{S}(x_\tau) \end{bmatrix}, \tau = 0,1,2,\cdots,M$$

Substituting (21) into the system (19), we obtain the system in matrix form as

$$\sum_{k=o}^{n} \boldsymbol{P}_k\ \boldsymbol{S}\left(\widehat{\boldsymbol{Q}^{\boldsymbol{k}}}\right)\boldsymbol{C} = \boldsymbol{F}, \tag{22}$$

which can be written in the following form

$$\boldsymbol{\Psi C} = \boldsymbol{F} \qquad or \qquad [\boldsymbol{\Psi}; \boldsymbol{F}], \tag{23}$$

where,

$$\boldsymbol{\Psi} = \sum_{k=o}^{n} \boldsymbol{P}_k \ \boldsymbol{S}(\widehat{\boldsymbol{Q}^k}) = [\Psi_{\mu,\rho}], \qquad \mu, \rho = 1, 2, \cdots, r(M+1) \tag{24}$$

System (23) is an algebraic linear of $r(M+1)$ equations in $r(M+1)$ unknowns Said-Ball coefficients.

Now, we formulate the matrix form of the mixed conditions (2). Using Eq. (15), then we have the matrix form of the mixed conditions (2) as

$$\sum_{j=o}^{n-1} \left[\alpha_j \widehat{\boldsymbol{S}}(\alpha) + \beta_j \ \widehat{\boldsymbol{S}}(\beta)\right] (\widehat{\boldsymbol{Q}})^j \ \boldsymbol{C} \ = \eta, \tag{25}$$

Where,

$$\alpha_j = \begin{bmatrix} \alpha_j^1 & 0 & \cdots & 0 \\ 0 & \alpha_j^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_j^r \end{bmatrix}, \qquad \beta_j = \begin{bmatrix} b_j^1 & 0 & \cdots & 0 \\ 0 & b_j^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_j^r \end{bmatrix}, \qquad \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_r \end{bmatrix},$$

$$\alpha_j^i = \begin{bmatrix} \alpha_{0,j}^i \\ \alpha_{1,j}^i \\ \vdots \\ \alpha_{n-1,j}^i \end{bmatrix}, \qquad \beta_j^i = \begin{bmatrix} \beta_{0,j}^i \\ \beta_{1,j}^i \\ \vdots \\ \beta_{n-1,j}^i \end{bmatrix}, \qquad \eta_i = \begin{bmatrix} \eta_{i,0} \\ \eta_{i,1} \\ \vdots \\ \eta_{i,n-1} \end{bmatrix}, i = 1, 2, \cdots, r.$$

Eq. (25) can be written as

$$\boldsymbol{ZC} = \eta \quad or \qquad [\boldsymbol{Z}; \eta], \tag{26}$$

Where,

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}_0 \\ \mathbf{Z}_1 \\ \vdots \\ \mathbf{Z}_{n-1} \end{bmatrix}, \qquad \mathbf{Z}_j = \sum_{j=o}^{n-1} \left[\alpha_j \ \widehat{\boldsymbol{S}}(\alpha) + \beta_j \ \widehat{\boldsymbol{S}}(\beta)\right] (\widehat{\boldsymbol{Q}})^j$$

By replacing all the rows of the matrix $[\mathbf{Z}; \eta]$ with the last rows of the matrix $[\boldsymbol{\Psi}; \boldsymbol{F}]$, we get a new system

$$\breve{\boldsymbol{\Psi}}\boldsymbol{A} = \breve{\boldsymbol{F}} \quad or \quad [\breve{\boldsymbol{\Psi}}; \breve{\boldsymbol{F}}] \tag{27}$$

Where,

$$[\tilde{\boldsymbol{\Psi}}; \tilde{\boldsymbol{F}}] = \begin{bmatrix} \Psi_{1,1} & \Psi_{1,2} & \cdots & \Psi_{1,r(M+1)} & ; & f_1(x_0) \\ \Psi_{2,1} & \Psi_{2,2} & \cdots & \Psi_{2,r(M+1)} & ; & f_2(x_0) \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \Psi_{r,1} & \Psi_{r,2} & \cdots & \Psi_{r,r(M+1)} & ; & f_r(x_0) \\ \Psi_{r+1,1} & \Psi_{r+1,2} & \cdots & \Psi_{r+1,r(M+1)} & ; & f_1(x_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{r(M-(n-1)),1} & \Psi_{r(M-(n-1)),2} & \cdots & \Psi_{r(M-(n-1)),r(M+1)} & ; & f_r(x_{M-n}) \\ v_{1,1} & v_{1,2} & \cdots & v_{1,r(M+1)} & ; & \eta_{1,0} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,r(M+1)} & ; & \eta_{1,1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{n,1} & v_{n,2} & \cdots & v_{n,r(M+1)} & ; & \eta_{1,n-1} \\ v_{n+1,1} & v_{n+1,2} & \cdots & v_{n+1,r(M+1)} & ; & \eta_{2,0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{nr,1} & v_{nr,2} & \cdots & v_{nr,r(M+1)} & ; & \eta_{r,n-1} \end{bmatrix}$$

Solve the linear system of linear algebraic equations (27) to calculate the coefficients $c_{i,m}$, $m = 0, 1, \cdots, M, i = 1, 2, \cdots, r$ in Eq. (8); hence we obtained the approximate solution of the original problem (1) and (2).

4. Accuracy of solutions and residual errors analysis

We used the upper bound of the mean error to check the accuracy of the solutions, which were obtained using the presented method. Then if we substituted the approximate solution (8) and its derivatives in the system (1), we obtain

$$\Re_{\lambda,M}(x_\tau) = \sum_{k=o}^{n} \sum_{j=1}^{r} P_{i,j}^{k}(x_\tau)\ y_j^{(k)}(x_\tau) - f_i(x_\tau) \cong 0,\ i = 1,2,\cdots,r,$$
$$\lambda = 1,2,3, \forall\ x = x_\tau \in [0,1], \tau = 0,1,2,\cdots \quad (28)$$

or

$$\Re_{\lambda,M}(x_\tau) \le 10^{-\theta_\tau}, \quad \forall\ \theta_\tau \in \mathbb{Z}^+. \quad (29)$$

The truncation limits M increases until the difference $\Re_{\lambda,M}(x_\tau) < 10^{-\theta}, \forall\ x_\tau \in [0,1]$, $\forall\ \theta \in \mathbb{Z}^+$. If $\max 10^{-\theta_\tau} = 10^{-\theta}$ and M is sufficiently large enough, then the error decreases, that is, $\Re_{\lambda,M}(x_\tau) \to 0$. On the other hand, we can estimate the solution's accuracy and error by using the residual function $\Re_{\lambda,M}(x)$ and its mean value $|\Re_{\lambda,M}(x)|$ on the interval $[0,1]$.

Hence, the upper bound of the mean error $\bar{\Re}_{k,M}$, is derived as follows [36, 37]:

$$\left|\int_0^1 \Re_{\lambda,M}(x)\right|\ dx \le \int_0^1 \left|\Re_{\lambda,M}(x)\right|\ dx$$
$$\because \left|\int_0^1 \Re_{\lambda,M}(x)\right| = \left|\Re_{\lambda,M}(\gamma)\right|, \qquad \gamma \in [0,1]$$
$$\therefore \left|\int_0^1 \Re_{\lambda,M}(x)\right| = \left|\Re_{\lambda,M}(\gamma)\right| \le \int_0^1 \left|\Re_{\lambda,M}(x)\right|\ dx$$

Therefore,

$$\left|\Re_{\lambda,M}(\gamma)\right| \le \int_0^1 \left|\Re_{\lambda,M}(x)\right|\ dx = \bar{\Re}_{\lambda,M}, \quad \gamma \in [0,1]. \quad (30)$$

5. Numerical examples

In this section, we presented three examples and compared our numerical results with the exact solutions and other works mentioned in the literature [8, 15] (see tables and figures)

Example 1: Consider the system**[26]**

$$\left.\begin{array}{r} y_1''(x) + x\,y_1(x) + x\,y_2(x) = 2 \\ y_2''(x) + 2x\,y_2(x) + 2x\,y_1(x) = -2 \end{array}\right\}, 0 \le x \le 1 \quad (31)$$

With the conditions:

$$y_1(0) = y_1(1) = 0,\ y_2(0) = y_2(1) = 0. \quad (32)$$

The exact solutions of this system are $y_1(x) = x^2 - x$ and $y_2(x) = -x^2 + x$.
Here $r = 2$ and $n = 2$. $f_1(x) = 2$, $f_2(x) = -2$,

$$P_{1,1}^0(x) = P_{1,2}^0(x) = x, P_{2,1}^0(x) = P_{2,2}^0(x) = 2x,$$
$$P_{1,1}^1(x) = P_{1,2}^1(x) = P_{2,1}^1(x) = P_{2,2}^1(x) = 0,$$
$$P_{1,2}^2(x) = P_{2,1}^2(x) = 0, P_{1,1}^2(x) = P_{2,2}^2(x) = 1.$$

To apply the solution method with $M = 2$. Then the collocation points are $x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1$. The matrix equations of (1) and (2) are

$$\boldsymbol{\Psi}\ \boldsymbol{C} = \left[\boldsymbol{P}_0\boldsymbol{S} + \boldsymbol{P}_1\boldsymbol{S}\widehat{\boldsymbol{Q}} + \boldsymbol{P}_2\boldsymbol{S}(\widehat{\boldsymbol{Q}})^2\right]\boldsymbol{C} = \boldsymbol{F} \quad (33)$$

and

$$\mathbf{Z}\boldsymbol{C} = \boldsymbol{\eta}, \tag{34}$$

where

$$\boldsymbol{P}_0 = \begin{bmatrix} \boldsymbol{P}_0(0) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{P}_0\left(\frac{1}{2}\right) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{P}_0(1) \end{bmatrix}, \quad \boldsymbol{P}_1 = \begin{bmatrix} \boldsymbol{P}_1(0) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{P}_1\left(\frac{1}{2}\right) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{P}_1(1) \end{bmatrix},$$

$$\boldsymbol{P}_2 = \begin{bmatrix} \boldsymbol{P}_2(0) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{P}_2\left(\frac{1}{2}\right) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{P}_2(1) \end{bmatrix}, \boldsymbol{S} = \begin{bmatrix} \widehat{\boldsymbol{S}}(0) \\ \widehat{\boldsymbol{S}}\left(\frac{1}{2}\right) \\ \widehat{\boldsymbol{S}}(1) \end{bmatrix}, \widehat{\boldsymbol{S}}(0) = \begin{bmatrix} \boldsymbol{S}(0) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{S}(0) \end{bmatrix},$$

$$\widehat{\boldsymbol{S}}\left(\frac{1}{2}\right) = \begin{bmatrix} \boldsymbol{S}\left(\frac{1}{2}\right) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{S}\left(\frac{1}{2}\right) \end{bmatrix}, \widehat{\boldsymbol{S}}(1) = \begin{bmatrix} \boldsymbol{S}(1) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{S}(1) \end{bmatrix}, \widehat{\boldsymbol{Q}} = \begin{bmatrix} \boldsymbol{Q} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{Q} \end{bmatrix},$$

Where $\boldsymbol{Q} = \mathcal{S}^{-1} V \mathcal{S},\ V = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix},\ \mathcal{S} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix}$, and

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_1 \\ \boldsymbol{C}_2 \end{bmatrix}, \boldsymbol{C}_1 = \begin{bmatrix} c_{1,0} \\ c_{1,1} \\ c_{1,2} \end{bmatrix}, \boldsymbol{C}_2 = \begin{bmatrix} c_{2,0} \\ c_{2,1} \\ c_{2,2} \end{bmatrix}, \boldsymbol{F} = \begin{bmatrix} f(0) \\ f\left(\frac{1}{2}\right) \\ f(1) \end{bmatrix}, \boldsymbol{f}(x) = \begin{bmatrix} 2 \\ -2 \end{bmatrix},$$

$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}_0 \\ \mathbf{Z}_1 \end{bmatrix};\ \mathbf{Z}_j = \sum_{j=o}^{1} \left[\alpha_j\, \widehat{\boldsymbol{S}}(0) + \beta_j\, \widehat{\boldsymbol{S}}(1)\right] (\widehat{\boldsymbol{Q}})^j\ \ \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}; \eta_i = \begin{bmatrix} \eta_{i,0} \\ \eta_{i,1} \end{bmatrix}, i = 1,2.$

Therefore, we have

$$[\boldsymbol{\Psi}; \boldsymbol{F}] = \begin{bmatrix} 2 & -4 & 2 & 0 & 0 & 0 & ; & 2 \\ 0 & 0 & 0 & 2 & -4 & 2 & ; & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 & ; & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & ; & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & ; & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & ; & -2 \end{bmatrix}$$

and

$$[\boldsymbol{Z}; \eta] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & ; & 0 \end{bmatrix}$$

Replacing $[\boldsymbol{Z}; \eta]$ by the last four rows in $[\boldsymbol{\Psi}; \boldsymbol{F}]$, we obtain a new augmented matrix

$$[\breve{\boldsymbol{\Psi}}; \breve{\boldsymbol{F}}] = \begin{bmatrix} 2 & -4 & 2 & 0 & 0 & 0 & ; & 2 \\ 0 & 0 & 0 & 2 & -4 & 2 & ; & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & ; & 0 \end{bmatrix}$$

Solving this system gives Said-ball coefficients as the following matrix form

$\boldsymbol{C} = [\boldsymbol{C}_1 \quad \boldsymbol{C}_2]^T;\ \boldsymbol{C}_1 = [0 \quad -1/2 \quad 0]^T, \boldsymbol{C}_2 = [0 \quad 1/2 \quad 0]^T$

By substituting these values into Eq. (8) (or equivalent Eq. (9)), we obtain the approximate solutions:

$$y_1(x) = x^2 - x \text{ and } y_2(x) = -x^2 + x,$$

Which are identical to the exact solutions.

Example 2: Consider the system **[15]**

$$\left.\begin{aligned} y_1'(x) - y_3(x) = -\cos(x) \\ y_2'(x) - y_3(x) = -e^x \\ y_3'(x) - y_1(x) + y_2(x) = 0 \end{aligned}\right\}, \qquad 0 \le x \le 1 \qquad (35)$$

With the conditions:

$$y_1(0) = 1,\ y_2(0) = 0,\ y_3(0) = 2. \qquad (36)$$

The exact solutions of this system are $y_1(x) = e^x$, $y_2(x) = \sin(x)$, and $y_3(x) = e^x + \cos(x)$.

Here $r = 3,\ n = 1$. Assume $M = 6$. The relation of the fundamental matrix is

$$\left[\boldsymbol{P}_0\boldsymbol{S} + \boldsymbol{P}_1\boldsymbol{S}\widehat{\boldsymbol{Q}}\right]\boldsymbol{C} = \boldsymbol{F}, \qquad (37)$$

where

$$\boldsymbol{P}_0(x) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \boldsymbol{P}_1(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{f}(x) = \begin{bmatrix} -\cos(x) \\ -e^x \\ 0 \end{bmatrix}$$

After substitutions in the above fundamental matrix relation, we get the augmented matrix as (27). Solve this system gives the Said-ball coefficients matrix form $\boldsymbol{C}$, by substituting these values into Eq. (9), we obtain the approximate solutions

$$\begin{aligned} y_{1,6}(x) &= 0.00213413784932x^6 + 0.00745756931915x^5 \\ &+0.042113234152x^4 + 0.166570358068x^3 \\ &+0.500006352039x^2 + x + 1 \\ y_{2,6}(x) &= -0.000547923902642x^6 + 0.00890196529882x^5 \\ &-0.000266276054369\ x^4 - 0.166613370962\ x^3 \\ &-0.00000329205964518x^2 + x \\ y_{3,6}(x) &= 0.000876793066706x^6 + 0.0072689742529x^5 \\ &+ 0.0838826332659\ x^4 + 0.16654765199\ x^3 \\ &+0.00000785760278896x^2 + x + 2. \end{aligned}$$

Table 1 compares the absolute errors of the presented method (PM)with other works [8, 15].

Table 2 indicates the upper bound of the mean error $\overline{\Re}_{\lambda,M}$.

Figure 1. indicates the absolute errors between the exact solutions $y_i(x)$ and approximate solutions $y_{i,M}$, $i = 1,2,3$ at $M = 3, 5, 7$ of this example.

Table1: Comparison of the absolute errors $(e_{i,M})$ between the exact solution (y_i) and approximate solution $(y_{i,M})$, $i = 1, 2, 3$ when $M = 6$ for Example 2

	DTM [8]	**Laguerre [15]**	**PM**
x_i	$e_{1,6}$	$e_{1,6}$	$e_{1,6}$
0	0	1.7319e-14	0
0. 2	2.6046e-09	8.9691e-08	3.7032e-08
0. 4	3.4209e-07	9.4300e-08	2.7334e-08
0. 6	6.0004e-06	1.1665e-07	2.9758e-08
0. 8	4.6173e-05	1.9907e-07	1.1140e-07
1. 0	2.2627e-04	1.5209e-06	1.7703e-07
x_i	$e_{2,6}$	$e_{2,6}$	$e_{2,6}$
0	0	5.8398e-14	0
0. 2	2.5383e-09	2.6784e-08	1.8075e-08
0. 4	3.2436e-07	1.9940e-09	2.9618e-08
0. 6	5.5266e-06	2.9723e-08	3.1570e-09

0. 8	4.1242e-05	2.3020e-08	5.0603e-08
1. 0	1.9568e-04	1.1272e-06	1.1751e-07
x_i	$e_{3,6}$	$e_{3,6}$	$e_{3,6}$
0	0	2.7756e-17	0
0. 2	2.6681e-09	1.1003e-07	4.6082e-08
0. 4	3.5831e-07	1.0917e-07	3.6352e-08
0. 6	6.4153e-06	1.2560e-07	3.8633e-08
0. 8	5.0305e-05	2.1407e-07	1.4104e-07
1. 0	2.5080e-04	2.0229e-06	2.2415e-07

Table 2: The upper bound of the mean error $\overline{\mathfrak{R}}_{\lambda,M}$ for Example 2.

M / $\overline{\mathfrak{R}}_{\lambda,M}$	2	3	4	5	6	7
$\overline{\mathfrak{R}}_{1,M}$	8.0269E-01	6.9349E-03	4.9005E-04	2.5523E-05	1.0295E-06	3.6657E-08
$\overline{\mathfrak{R}}_{2,M}$	7.2381E-01	9.3232E-04	2.6357E-04	7.5972E-06	6.5977E-07	7.3179E-08
$\overline{\mathfrak{R}}_{3,M}$	9.6392E-02	1.2221E-02	9.6268E-05	9.6680E-06	1.3096E-06	5.9326E-08

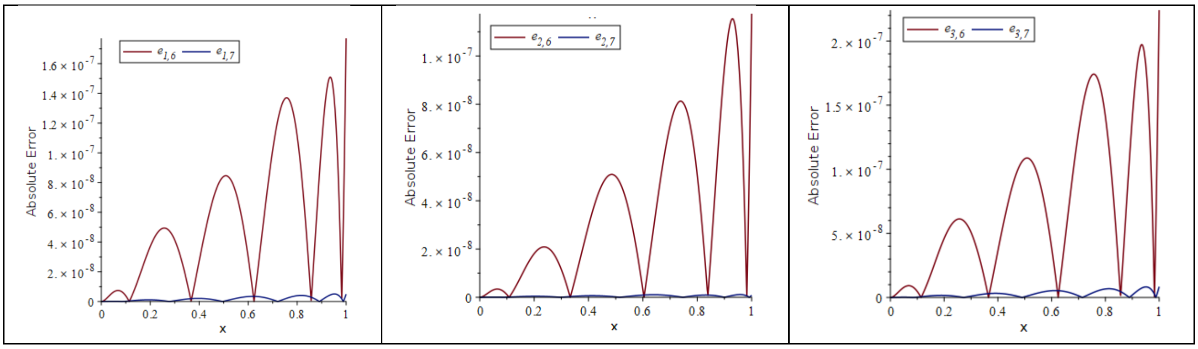

Fig.1: The absolute error $(e_{i,M})$ between the exact solution (y_i) and the approximate solution $(y_{i,M})$, where $i = 1, 2, 3$ and $M = 6, 7$ for Example 2.

Table 1 shows that the presented method gives better results than the differential transform method, Laplace transform method [8], and Laguerre collocation method [15].
Figure 1 indicates that the presented method obtained highly accurate solutions when M increased.

Example 3: Consider the system **[26]**

$$\left.\begin{array}{l} y_1^{(4)}(x) - \cos(x)\, y_2^{(2)}(x) + x y_3^{(1)}(x) - y_1(x) = xe^x + \cos^2(x) \\ y_2^{(4)}(x) + \sin(x)\, y_1^{(3)}(x) + \cos(x)\; y_1(x) - \cos(x)\; y_3(x) = \cos(x)(1 - e^x) \\ e^{-x} y_3^{(4)}(x) + y_2^{(2)}(x) - \cos(x)\, y_1^{(1)}(x) + y_2(x) = \sin^2(x) \end{array}\right\}, \quad (38)$$

with the conditions:

$$y_1(0) = 0, \quad y_2(0) = 1, \quad y_3(0) = 1, \quad y_1^{(1)}(0) = 1, \quad y_2^{(1)}(0) = 0,$$
$$y_3^{(1)}(0) = 1, \quad y_1^{(2)}(0) = 0, \quad y_2^{(2)}(0) = -1, \quad y_3^{(2)}(0) = 1,$$
$$y_1^{(3)}(0) = -1, \quad y_2^{(3)}(0) = 0, \quad y_3^{(3)}(0) = 1. \quad (39)$$

The exact solutions are $y_1(x) = \sin(x)$, $y_2(x) = \cos(x)$, and $y_3(x) = e^x$.

Here $r = 3, n = 4$. Let $M = 5$. We have the relation of the fundamental matrix is

$$\left[\boldsymbol{P}_0\boldsymbol{S} + \boldsymbol{P}_1\boldsymbol{S}\widehat{\boldsymbol{Q}} + \boldsymbol{P}_2\boldsymbol{S}(\widehat{\boldsymbol{Q}})^2 + \boldsymbol{P}_3\boldsymbol{S}(\widehat{\boldsymbol{Q}})^3 + \boldsymbol{P}_4\boldsymbol{S}(\widehat{\boldsymbol{Q}})^4\right]\boldsymbol{C} = \boldsymbol{F} \qquad (40)$$

Here,

$$\boldsymbol{P}_0(x) = \begin{bmatrix} -1 & 0 & 0 \\ \cos(x) & 0 & -\cos(x) \\ 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{P}_1(x) = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ -\cos(x) & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{P}_2(x) = \begin{bmatrix} 0 & -\cos(x) & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{P}_3(x) = \begin{bmatrix} 0 & 0 & 0 \\ \sin(x) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{P}_4(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-x} \end{bmatrix}, \quad \boldsymbol{f}(x) = \begin{bmatrix} xe^x + \cos^2(x) \\ \cos(x)(1 - e^x) \\ \sin^2(x) \end{bmatrix}$$

Applying the proposed method above, we finally obtained the approximate solutions as the following formulas

$$y_{1,5} = 0.00832037291941x^5 - 0.166666666667x^3 + x$$

$$y_{2,5} = -0.000397548628658x^5 + 0.0416666666667x^4 - 0.5x^2 + 1.0$$

$$y_{3,5} = 0.0087445195381x^5 + 0.0416666666667x^4 + 0.166666666667x^3 + 0.5x^2 + x + 1.0\,.$$

In Figure 2, we plotted the exact solution $y_i(x)$ and approximate solution $y_{i,M}$, $i = 1,2,3$ at $M = 3, 4, 5$.

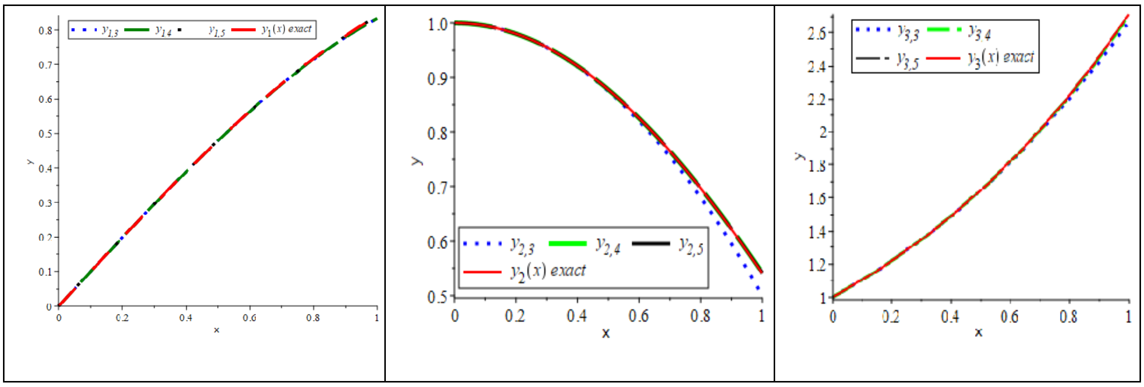

Fig.2: Comparison of the exact solution (y_i) and the approximate solution $(y_{i,M})$, where $i = 1, 2, 3$ and $M = 3, 4, 5$ for Example 3.

Table3: Comparison of the absolute errors $(e_{i,M})$ between the exact solution (y_i) and approximate solution $(y_{i,M})$, $i = 1, 2, 3$ when $M = 3, 4, 5$ for Example 3

x_i	$e_{1,3}$	$e_{2,3}$	$e_{3,3}$
0	0	0	0
0. 2	2.6641e-06	6.6578e-05	6.9425e-05
0. 4	8.5009e-05	1.0610e-03	1.1580e-03
0. 6	6.4247e-04	5.3356e-03	6.1188e-03
0. 8	2.6894e-03	1.6707e-02	2.0208e-02
1. 0	8.1377e-03	4.0302e-02	5.1615e-02
x_i	$e_{1,4}$	$e_{2,4}$	$e_{3,4}$
0	0	0	0
0. 2	2.6641e-06	8.8825e-08	2.7582e-06

0. 4	8.5009e-05	5.6727e-06	9.1364e-05
0. 6	6.4247e-04	6.4385e-05	7.1880e-04
0. 8	2.6894e-03	3.5996e-04	3.1409e-03
1. 0	8.1377e-03	1.3644e-03	9.9485e-03
x_i	$e_{1,5}$	$e_{2,5}$	$e_{3,5}$
0	0	0	0
0. 2	1.6091e-09	3.8390e-08	4.0086e-08
0. 4	1.9164e-07	1.6018e-06	1.8204e-06
0. 6	4.5188e-06	3.3472e-05	3.8827e-05
0. 8	3.6996e-05	2.2969e-04	2.7552e-04
1. 0	1.8272e-04	9.6681e-04	1.2040e-03

It is clear from Table 1 that the more increase of M, the fewer errors and the more accuracy of the presented method.
Figure 2 illustrates that the presented method gave highly accurate solutions when M increases.

6. Conclusion

In this article, Said-Ball polynomials with the collocation method are used to solve the systems of high-order linear ordinary differential equations. This method reduces the given problem of the linear ordinary differential equations system with suitable conditions to the linear algebraic equations system. Solutions of the resulting system gave the exact solution and sometimes good approximate solutions compared with the other works, as shown in tables and figures. This means the presented method is effective and accurate.

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