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THE GROUPS ACTING ON THE RIEMANN SPHERE

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Under the Supervision of Dr. Jianhua Gong

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Declaration of Original Work

I, Ruba Yousef Wadi, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled "*The Groups Acting on the Riemann Sphere*", hereby, solemnly declare that this thesis is an original research work that has been done and prepared by me under the supervision of Dr. Jianhua Gong, in the College of Science at UAEU. This work has not been previously formed as the basis for the award of any academic degree, diploma or a similar title at this or any other university. The materials borrowed from other sources and included in my thesis have been properly cited and acknowledged.

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Abstract

In this Master thesis we consider the group actions, with emphasis on the group of general Möbius transformations of one complex valuable acting on the Riemann sphere.

We study some invariant subspaces of Riemann sphere under the actions of natural groups of transformations, including the invariant quantities in Hyperbolic Geometry that is a beautiful area of Mathematics.

We use analytic and algebraic points of view to describe some group actions on Riemann sphere; in particular, we present the relationships between isometries of hyperbolic plane, Möbius transformations, and groups of matrices.

Keywords: Group actions, Riemann sphere, general Möbius transformations, transitivity, hyperbolic plane, and hyperbolic isometry.

Title and Abstract (in Arabic)

مجموعة الإجراءات على مجال ريمان

الملخص

في هذه الرسالة سنقوم بدراسة مجموعة الإجراءات (Group Actions) مع التركيز على دراسة نوع تحولات موبيوس العامة (General Möbius transformations) ذات المتغير الواحد والتي تعمل على مجال ريمان

(Riemann Sphere).

سوف ندرس أيضا عملية الإستقرار والثبات تحت تأثير مجموعة الإجراءات الطبيعية للتحولات، وبشكل خاص الكميات الموجودة في الهندسة القطعية (Hyperbolic Geometry), والتي تعتبر من المجالات الممتعة في علوم الرياضيات والتي يقوم الباحثون بالعمل على دراستها والاهتمام بها.

سوف نستخدم أيضا وجهة النظر التحليلية والجبرية لاستكشاف بعض المجموعات الفرعية لمجموعة تحولات موبيوس العامة (General Möbius transformations), وبشكل خاص العلاقات بين: الاقترانات التي تحافظ على المسافة (Isometries) في الفضاء القطعي (Hyperbolic plane) و تحولات موبيوس (Möbius transformations) ومجموعة المصفوفات.

كلمات مفتاحية: مجموعة الإجراءات, مجال ريمان, تحولات موبيوس العامة, خاصية التعدي, الفضاء القطعي, والاقترانات التي تحافظ على المسافة في الفضاء القطعي.

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Special thanks to my family and friends for their help along the way. In particular, I would like to express appreciation to my beloved husband Dr. Ezzeddin Al Wadi for his continuously inspiration, supports and efforts.

Dedication

A special feeling of gratitude to my parents whose words of encouragement and push for tenacity ring in my ears. My siblings who have never left my side and continue to remain very dear and special.

I dedicate this thesis to my friends who have supported me throughout the process. I will always appreciate all they have done, especially Dr. Jianhua Gong for helping me develop my research and analytical skills.

In addition, I dedicate this thesis to my beloved husband Dr. Ezzeddin Al Wadi for his continuously inspiration, supports and efforts, and to my children Mohammed, Bushra, Omar, and Joud.

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Chapter 1: Introduction

Hyperbolic Geometry has an interesting history. One of reasons for the continuing interest in this active and fascinating field of Mathematics is that it touches on a number of different fields such as Complex Analysis, Abstract Algebra, Number Theory, Differential Geometry, and Topology.

Saul [15] described Hyperbolic Geometry from a geometric point of view, and Anderson [1] provided an algebraic approach to geometry of hyperbolic plane.

Each model of planar hyperbolic geometry can be extended to a higher dimensional hyperbolic space. Thurston [16] studied three-dimensional manifolds. Moreover, Ratcliffe [14] gave an exposition of the theoretical foundations of hyperbolic manifolds and developed the theory of hyperbolic orbifolds.

Beardon [3] and Maskit [12] studied discrete subgroups of the group of Möbius transformations. Gong [9] discussed the group of Möbius transformations in higher dimensional space. Ghys [8] gave a well-written and informative survey of group actions on the circle S^1; Kawakubo [10] studied the theory of transformation groups, and Bowditch [4] gave a topological characterization of hyperbolic groups.

In this Master thesis we study some invariant subspaces of Riemann sphere under the actions of natural groups of transformations, including the invariant quantities in Hyperbolic Geometry that is a beautiful area of Mathematics. Hyperbolic Geometry has being actively studying by researchers around the world.

We study some quantities invariant under the action of a natural group of transformations, in particular, those quantities in Hyperbolic Geometry.

We use analytic and algebraic points of view to describe some group actions on Riemann sphere, in particular, we present the relationships between isometries of hyperbolic plane, Möbius transformations, and groups of matrices.

We consider the group actions, with emphasis on the group of general Möbius transformations of one complex valuable acting on the Riemann sphere, that is the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ including the extended real line $\overline{\mathbb{R}}$, upper-half plane $\mathbb{H}^2$, open disc D, circle S^1, and so on.

The group $M\ddot{o}b(\overline{\mathbb{C}})$ of general Möbius transformations of one complex valuable is:

$$M\ddot{o}b(\overline{\mathbb{C}}) = \left\{ \frac{az+b}{cz+d} \text{ or } \frac{a\overline{z}+b}{c\overline{z}+d} : z \in \overline{\mathbb{C}}, a, b, c, d \in \mathbb{C}, \text{ and } ad - bc \neq 0 \right\}.$$

By using analytic and algebraic point of views, we describe some properties of subgroups of $M\ddot{o}b(\overline{\mathbb{C}})$, and present the relationships between isometries of hyperbolic plane, Möbius transformations, and groups of matrices.

In the second chapter, we introduce group actions in the first section. We then define the Riemann sphere and formulate the stereographic projection in the second section. Taking advantages of Complex Analysis, we unify Euclidean lines and Euclidean circles as extended circles in the Riemann sphere; as a result, an equation is formulated in the third section to describe all circles in the Riemann sphere. Finally, in the last section, we extend the continuity to the Riemann sphere so that the group of homeomorphisms can be introduced in the Theorem 2.4.12. One can see that the Riemann sphere plays an important role, including solving the problems of discontinuity in the complex plane for many functions.

We focus in the third chapter on group actions on subspaces of the Riemann Sphere such as the extended real numbers $\overline{\mathbb{R}}$, the unit circle S^1, and the unit open disc D. Three subspaces here are invariant under $M\ddot{o}b(\overline{\mathbb{R}})$, $M\ddot{o}b(S^1)$, and $M\ddot{o}b(D)$, respectively. We present an explicit expression for each element of these three group actions (Theorems 3.4.1, 3.5.2, and 3.5.3) in the fourth and fifth sections.

We explain in the first section that the set $M\ddot{o}b^+(\overline{\mathbb{C}})$ of all Möbius transformations and the set $M\ddot{o}b(\overline{\mathbb{C}})$ of all general Möbius transformations are groups under the operation of composition, and they generated by 2 and 3 generators, respectively (Theorems 3.1.4, 3.1.5, and 3.1.6). They all act on the Riemann sphere by linear fractional transformations. In the second section, we discuss fixed points of Möbius transformations and give the Uniqueness Theorem 3.2.4, and then we can show that

$M\ddot{o}b^+(\overline{\mathbb{C}})$ acts uniquely triply transitively on $\overline{\mathbb{C}}$ (Theorem 3.2.6). As each disc in $\overline{\mathbb{C}}$ is determined by a circle, and each circle in $\overline{\mathbb{C}}$ is determined by three distinct points of $\overline{\mathbb{C}}$. Thus, Möb$^+(\overline{\mathbb{C}})$ acts transitively on the set of the circles in $\overline{\mathbb{C}}$, and on the set of discs in $\overline{\mathbb{C}}$. We take another point of view in the third section to study the group $M\ddot{o}b(\overline{\mathbb{C}})$ by showing that $M\ddot{o}b(\overline{\mathbb{C}})$ coincides the group of homeomorphisms taking circles in $\overline{\mathbb{C}}$ to circles in $\overline{\mathbb{C}}$ (Theorem 3.3.10).

In the last chapter, we turn our attention to the group action of hyperbolic isometries. In the first section we use the upper-half plane as a model of the hyperbolic plane, then we define two types of hyperbolic lines, and explain some differences between Hyperbolic Geometry and usual Euclidean Geometry, especially, the fifth Euclidean Postulate is not held in the hyperbolic plane $\mathbb{H}^2$. In addition, we describe the boundary at infinity of the hyperbolic plane, and then one can see the relationship between $\mathbb{H}^2$ and $\overline{\mathbb{R}}$ at the end of the first section.

In the second section we mainly identify the actions of groups $M\ddot{o}b(\mathbb{H}^2)$ and Möb$^+(\mathbb{H}^2)$, such that the hyperbolic plane $\mathbb{H}^2$ is an invariant subspace of Riemann sphere under these two groups. We will find an explicit expression for an element of either Möb$(\mathbb{H}^2)$ or Möb$^+(\mathbb{H}^2)$ (Theorem 4.2.2 and Corollary 4.2.3).

In the third section we define a generalized length of a piecewise path by using the complex variable, we then define the hyperbolic length and a new distance in hyperbolic plane which is called the hyperbolic distance, which is an invariant quantity under the action of the group Möb$(\mathbb{H}^2)$. At the the end of this section, we present that the hyperbolic distance between two point in $\mathbb{H}^2$ is given by the hyperbolic length of the hyperbolic line segment between these two points.

We present in the last section that the group of hyperbolic isometries acting on the hyperbolic plane is the group of general Möbius transformations preserving the hyperbolic plane: $Isom(\mathbb{H}^2, d_{\mathbb{H}^2}) = M\ddot{o}b(\mathbb{H}^2)$ (Theorem 4.4.4).

Chapter 2: Group Actions and the Riemann Sphere

In this chapter we introduce group actions in the first section. We then define the Riemann sphere and formulate the stereographic projection in the second section. Taking advantages of Complex Analysis, we unify Euclidean lines and Euclidean circles as extended circles in the Riemann sphere; as a result, an equation is formulated in the third section to describe all circles in the Riemann sphere. Finally, in the last section, we extend the continuity to the Riemann sphere so that the notion of homeomorphisms can be introduced there. One can see that the Riemann sphere plays an important role, including solving the problems of discontinuity in the complex plane for many functions.

2.1 Group Actions

Definition 2.1.1 *A function $f : X \to Y$ is called a transformations if it is a bijection, i.e., it is injective (one-to-one) and surjective (onto).*

Definition 2.1.2 *A group G is said to be acting on a set X if every element of G gives a transformation on X.*

Let G be a group, and let X be a set. Suppose ϕ is a function: $\phi : G \times X \to X$ defined by $(g, x) \to \phi(g, x)$, where $\phi(g, x)$ is a point of X, which can be written as $\phi(g)(x)$, or simply as $g(x)$, The function ϕ is called a group action of G on X if it satisfies the following two conditions:

(a) Compatibility: $(gh)(x) = g(h(x))$, for $g, h \in G$ and all $x \in X$, here, gh denotes the result of applying the group operation of G to the elements g and h.

(b) Identity: $e(x) = x$, for $x \in X$, here e is the identity of G.

The set X is called a G-set and the group G is said to act on X.

Now, We can say the group is also called permutation group beside transformation group if the essential elements of the object are described by a set, and the symmetries of the object are described by the symmetric group of this set, which consists of bijective transformations of the set or transformation group. Many objects in mathematics have natural group actions defined on them. In particular, groups can act on other groups, or even on themselves.

As we know, group acting on set is important in mathematics, because when one object acts on another there are more information becomes obtained on both sets and more informations on the structure become available. But here we identify the acting groups on the Riemann sphere $\overline{\mathbb{C}}$. And to identify that, we first define general linear group and special linear group [6].

And because a group action is an extension to the definition of a symmetry group in which every element of the group "acts" of some set, without playing with transformation also this allows for a more including description of the symmetries of an object.

Example 2.1.3 *[22] A polyhedron which is simply a three-dimensional solid consisting of a collection of polygons, usually joined at their edges, by allowing the same group to act on several different sets of features, such as the set of vertices, the set of edges and the set of faces of the polyhedron (see Figure 2.1.1).*

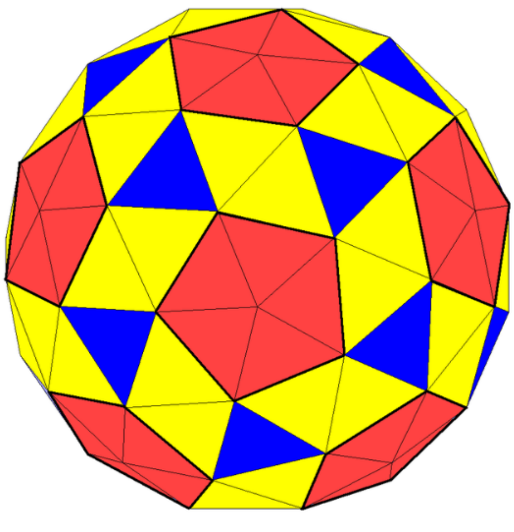

Figure 2.1.1: Polyhedron

Example 2.1.4 *(a) The trivial action of any group G on any set X is defined by:*

$$g(x) = x, \text{ for all } g \in G \text{ and all } x \in X.$$

(b) Given a group G. Let the set $X = G$, then group operation is a group of G :

$$g.x = gx, \forall g, x \in G.$$

Example 2.1.5 *Let G be the set of all the isometrics of the plane $\mathbb{R}^2$ acting on the subset of $\mathbb{R}^2$ by some patterns, such as wallpaper patterns which based on the symmetries in the pattern (see Figure 2.1.2).*

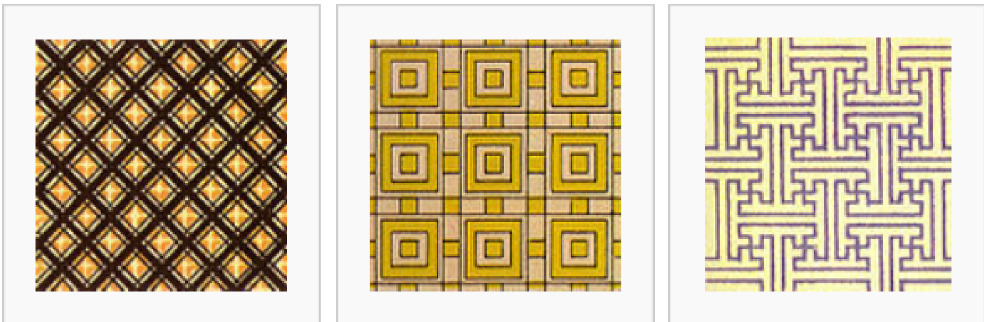

Figure 2.1.2: Wallpaper pattererns

Now, we need to define The general linear group $GL(2, \mathbb{C})$ and the special linear group $SL(2, \mathbb{C})$ to induce the groups acting on Riemann sphere.

Definition 2.1.6 *Let $GL(2, \mathbb{C})$ be the set of all 2×2 complex matrices with non-zero determinant, which is called the general linear group:*

$$GL(2, \mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0 \right\}.$$

Let $SL(2,\mathbb{C}) = \{A \in GL(2,\mathbb{C}) : \det(A) = 1\}$, which is called the special linear group.

2.2 Stereographic Projection of the Riemann Sphere

In this section, we show how the Riemann sphere is visualizing as a sphere by using the stereographic projection. In addition, we find the formula of the stereographic projection.

If any one walk around the two dimensional plane he may walk, walk, and walk $\cdots$, in all directions without stop, so he can say that there is infinity around all the edge of the plane and never see that edge. But we have a chance to imagine what happens if we shrink that infinity edge to a point, and this process looks like tighten the rope on the frame of bag, when we tightening it, the bag is closed and still looks like a distorted sphere [17].

Definition 2.2.1 *Riemann sphere is a model of the extended complex plane, the complex plane plus a point at infinity:*

$$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

One of advantages of the Riemann sphere is that we can merge two different types of the Euclidean lines and circles together by obtaining the space from $\mathbb{C}$ and a single point ∞ which is not contained in $\mathbb{C}$.

In this model, the original point O is near to very small numbers, while the point ∞ is near to vary big numbers (see Figure 2.2.1).

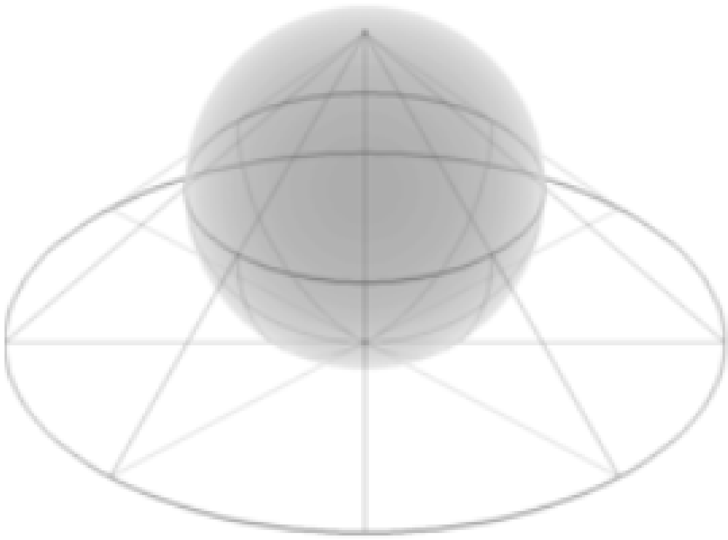

Figure 2.2.1: Riemann sphere

Riemann sphere can be visualized as the complex plane wrapped around a sphere by the stereographic projection.

let us identify the complex $\mathbb{C}$ with the plane $\mathbb{R}^2$, i.e., identify a complex number $z = x_1 + ix_2$ in $\mathbb{C}$ with a point (x_1, x_2) in $\mathbb{R}^2$, and let:

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$$

be the unit sphere in $\mathbb{R}^3$ with north pole $N(0, 0, 1)$.

Define the function $\varphi : S^2 \to \mathbb{C}$: for each point $a \in S^2 - \{N\}$, there is the Euclidean line L_a in $\mathbb{R}^3$ passing through N and a, and let $\varphi(a)$ be the point of intersection $L_a \cap \mathbb{C}$.

If $a = N(0, 0, 1)$, then define $\varphi(N) = \infty$. This function is called a stereographic projection (see Figure 2.2.2).

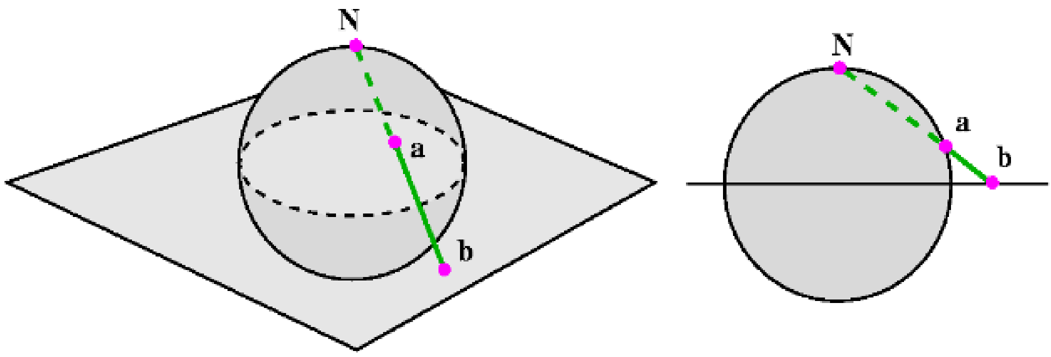

Figure 2.2.2: The Riemann sphere and stereographic projection

Theorem 2.2.2 *Let $\varphi : S^2 - \{N\} \longrightarrow \mathbb{C}$ be the stereographic projection, then it can be given by the formula below:*

$$\varphi(x_1, x_2, x_3) = \begin{cases} \frac{x_1}{1-x_3} + \frac{x_2}{1-x_3}i, & \textit{if } (x_1, x_2, x_3) \neq (0,0,1). \\ \infty & \textit{if } (x_1, x_2, x_3) = (0,0,1). \end{cases},$$

for each $(x_1, x_2, x_3) \in S^2 - \{N\}$.

Proof. Let $a = (x_1, x_2, x_3) \in S^2 - \{N\}$, then $\varphi(a) = (y_1, y_2) = (y_1, y_2, 0) \in \mathbb{R}^3$. Note that $N = (0,0,1)$. Obviously, from the above Figure 2.2 that:

$$\overrightarrow{N\varphi(a)} // \overrightarrow{Na}, \text{ i.e.., } \overrightarrow{N\varphi(a)} = t\overrightarrow{Na}, \text{ where } t \in \mathbb{R}.$$

So, $(y_1, y_2, -1) = t(x_1, x_2, x_3 - 1)$, which gives $y_1 = tx_1$, $y_2 = tx_2$ and $-1 = t(x_3 - 1)$. From the last equation $t = \frac{1}{1-x_3}$. And by substitution, $y_1 = \frac{x_1}{1-x_3}$ and $y_1 = \frac{x_2}{1-x_3}$. Thus:

$$\varphi(x_1, x_2, x_3) = \begin{cases} \frac{x_1}{1-x_3} + \frac{x_2}{1-x_3}i, & \text{if } (x_1, x_2, x_3) \neq (0,0,1). \\ \infty & \text{if } (x_1, x_2, x_3) = (0,0,1). \end{cases}$$

■

The stereographic projection which is a construction of the Riemann sphere

$\overline{\mathbb{C}}$ to the complex plane $\mathbb{C}$ is an example of more general construction known as *the one-point compactification [19].*

And to understand what is the meaning of *one-point compactification,* we need first to state what is the meaning of compact space.

Definition 2.2.3 *A topological space is compact if every open cover of the space contains a finite subcover. Otherwise, it is called non-compact.*

But, we interest here to state that theorem which give us a conditions to a compact subspace in $\mathbb{R}^n$.

Theorem 2.2.4 *A subspace A of $\mathbb{R}^n$ is compact if and only if A is closed and bounded [13].*

Example 2.2.5 $\mathbb{C}$ *is not compact, because it is not bounded.*

Definition 2.2.6 *Compactification is the process of making a topological space into a compact space and the one-point compactification $\overline{X}$ of X for any topological space X is obtained by adding one extra point ∞ and defining the open sets of the new space to be the open sets of X together with the sets of the form $T \cup \{\infty\}$ where T is an open subset of X such that $X - T$ is compact.*

According to the above definition 2.2.6, let $X = \mathbb{C}$ and add ∞ point to it to make $\overline{\mathbb{C}}$. Now, we can say $\overline{\mathbb{C}}$ is compact

2.3 Circles in the Riemann Sphere

Taking advantages of Complex Analysis, we can unify Euclidean lines and Euclidean circle in $\mathbb{R}^2$ as (generalized) circles in $\overline{\mathbb{C}}$. As a result, an equation of z is formulated in this section to describe all circles in the Riemann sphere.

Definition 2.3.1 *A circle in $\overline{\mathbb{C}}$ is either a Euclidean circle in $\mathbb{C}$ or the union of a Euclidean line in $\mathbb{C}$ with $\{\infty\}$.*

It is clear from the Definition 2.3.1 that we use ∞ as a point that added to each Euclidean line and make a circle in $\overline{\mathbb{C}}$. For example, the extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is the circle in $\overline{\mathbb{C}}$ containing real axis $\mathbb{R}$ in $\mathbb{C}$.

Since we can define circles in $\overline{\mathbb{C}}$ as the sets of solutions of an equations in $\overline{\mathbb{C}}$, we have a way in the following theorem 2.3.2 to describe all circles in $\overline{\mathbb{C}}$.

Theorem 2.3.2 *Every circles in $\overline{\mathbb{C}}$ is the set of solutions of an equation in $\overline{\mathbb{C}}$ of the form:*

$$\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0, \text{ where } \alpha, \gamma \in \mathbb{R} \text{ and } \beta \in \mathbb{C}. \tag{2.1}$$

In particular,

(a) If $\alpha \neq 0$, the equation (2.1) is for Euclidean circles.

(b) If $\alpha = 0$, the equation (2.1) is for a Euclidean line with $\{\infty\}$, and it becomes

$$\beta z + \overline{\beta}\overline{z} + \gamma = 0, \text{ where } \gamma \in \mathbb{R} \text{ and } \beta \in \mathbb{C}.$$

Proof. Let the Euclidean circle be:

$$(x-h)^2 + (y-k)^2 = r^2,$$

let $z = x + iy$, and let $z_0 = h + ik$. Then the equation of the circle becomes:

$$|z - z_0|^2 = z\overline{z} - \overline{z_0}z - z_0\overline{z} + |z_0|^2 = r^2.$$

Now, let the Euclidean line be:

$$ax + by + c = 0, \text{ where } a, b, \text{ and } c \in \mathbb{R}. \tag{2.2}$$

Since $z = x + iy$, then $x = \mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$ and $y = \mathrm{Im}(z) = \frac{i}{2}(\overline{z} - z)$, which gives:

$$\begin{aligned} ax + by + c &= a\frac{1}{2}(z + \overline{z}) - b\frac{i}{2}(\overline{z} - z) + c \\ &= \frac{1}{2}(a + ib)z + \frac{1}{2}(a - ib)\overline{z} + c = 0. \end{aligned}$$

Let $\gamma = 2c \in \mathbb{R}$, and let $\beta = a + ib \in \mathbb{C}$, then $\overline{\beta} = a - ib \in \mathbb{C}$. It follows that:

$$\beta z + \overline{\beta}\overline{z} + \gamma = 0, \text{where } \gamma \in \mathbb{R} \text{ and } \beta \in \mathbb{C}.$$

From the above equation (2.2), we can see that the slope of the above line $= -\frac{a}{b}$, which is the quotient of imaginary part and real parts of coefficient of z. Combining these, we can see that every circle in $\overline{\mathbb{C}}$ can be defined as the set of solutions in $\overline{\mathbb{C}}$ to an equation of the form: $\alpha z\overline{z}+\beta z+\overline{\beta}\overline{z} +\gamma = 0$ ($\alpha, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$) and by continuity, we can consider ∞ is a solution of the equation: $\beta z + \overline{\beta}\overline{z} + \gamma = 0$ ($\gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$) because there is a sequence $\{z_n\}$ of points in $\mathbb{C}$ that satisfies this equation and that converges to ∞ in $\overline{\mathbb{C}}$, but for the equation: $\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0$ ($\alpha, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$) we can not view ∞ as a solution by continuity, since z_n can not lie on the circle. ■

Theorem 2.3.3 *Let $X \subseteq \overline{\mathbb{C}}$ and let X is dense in $\overline{\mathbb{C}}$. If $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a continuous function for which $f(x) = x$, for $x \in X$, then $f(z) = z, \forall z \in \overline{\mathbb{C}}$.*

Proof. Let $z \in \overline{\mathbb{C}}$ then $\exists$ a sequence $\{x_n\}$ in X converge to z, because X is dense in $\overline{\mathbb{C}}$. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a continuous function for which $f(x) = x$ for all x in X. So, $\{f_{x_n}\}$ is converge to $f(z)$. ■

Theorem 2.3.4 *Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in $\mathbb{C}$. Let A be a Euclidean circle in $\mathbb{C}$ with centre $re^{i\theta}$ $(r > 1)$, and radius $s > 0$. Then A is perpendicular to S^1 if and only if $s = \sqrt{r^2 - 1}$ [1].*

Proof. Notice that A and S^1 are perpendicular if and only if their tangent lines at the point of intersection are perpendicular. Let x be the point of the intersection of A and S^1, and consider the Euclidean triangle T with three vertices: 0, $re^{i\theta}$, and x. Now the sides of the triangle joining x with 0 and $re^{i\theta}$and these two sides are the radii of A and S^1. So, A and S are perpendicular if and only if the interior angle of the the triangle T at x equals $\frac{\pi}{2}$ and this is happens if and only if $s^2 + 1^2 = r^2$. ■

2.4 Group of Homeomorphisms on the Riemann Sphere

In this last section, we extend the continuity to the Riemann sphere so that the group of homeomorphisms on the Riemann sphere can be introduced. One can see that the extended complex plane $\overline{\mathbb{C}}$ plays an important role here, including solving the problems of discontinuity in $\mathbb{C}$ for many functions.

First of all, we need to define what it means for a subset of $\overline{\mathbb{C}}$ to be open in $\overline{\mathbb{C}}$. let us start to define the (extended) Euclidean open disc with radius ε and center $z = \infty \in \overline{\mathbb{C}}$.

Definition 2.4.1 *The following two subsets of $\overline{\mathbb{C}}$ are called the (extended) Euclidean open discs with radius ε and center z, denoted by $U_\varepsilon(z)$.*

(a) If $z \in \mathbb{C}$,

$$U_\varepsilon(z) = \{w \in \mathbb{C} : |w - z| < \varepsilon\}.$$

(b) If $z = \infty \in \overline{\mathbb{C}}$,

$$U_\varepsilon(z) = U_\varepsilon(\infty) = \{w \in \mathbb{C} : |w| > \varepsilon\} \cup \{\infty\}.$$

Now we can define the open, closed, and bounded subsets in $\overline{\mathbb{C}}$.

Definition 2.4.2 *Suppose that $U, F,$ and X are subsets of $\overline{\mathbb{C}}$.*

(a) The subset U is said to be open in $\overline{\mathbb{C}}$ if for each $z \in U$, there exists an open Euclidean open disc $U_\varepsilon(z)$ such that $U_\varepsilon(z) \subseteq U$, including $z = \infty \in \overline{\mathbb{C}}$.

(b) The subset F is said to be closed in $\overline{\mathbb{C}}$ if its complement $\overline{\mathbb{C}} - F$ is open in $\overline{\mathbb{C}}$.

(c) The subset X is said to be bounded in $\overline{\mathbb{C}}$ if there exists an open Euclidean open disc cent $U_\varepsilon(0)$ such that $X \subseteq U_\varepsilon(0)$.

For examples, every (extended) Euclidean open disc $U_\varepsilon(z)$ is open in $\overline{\mathbb{C}}$, and the following unit disc D is closed in $\overline{\mathbb{C}}$, because it complement $\bar{\mathbb{C}} - D = U_1(0) \cup U_1(\infty)$ is open in $\overline{\mathbb{C}}$.

$$D = \{w \in \mathbb{C} : |w| \leq 1\}.$$

In addition, it is clear that Euclidean open discs $U_\varepsilon(z)(z \neq \infty)$ and D are closed in $\overline{\mathbb{C}}$.

Proposition 2.4.3 *Let U be a subset of $\overline{\mathbb{C}}$. Then U is open in $\overline{\mathbb{C}}$ if and only if U is either open in $\mathbb{C}$ or its complement in $\overline{\mathbb{C}}$ is a compact subset of $\mathbb{C}$.*

Proof. Suppose that U is open in $\overline{\mathbb{C}}$. If $\infty \notin U$ then U is open in $\mathbb{C}$. If $\infty \in U$. Let its complement in $\overline{\mathbb{C}}$ be K then K is closed and bounded, i.e., K is compact in $\mathbb{C}$. ■

For instance, we can see that the upper half plane below, denoted by $\mathbb{H}^2$, is open in $\overline{\mathbb{C}}$ because $\mathbb{H}^2$ is open in $\mathbb{C}$.

$$\mathbb{H}^2 = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

But the unit circle S^1 is not open in $\overline{\mathbb{C}}$, because that any Euclidean open disc $U_\varepsilon(z)$ does not lie in the unit circle.

Given a subset X of $\overline{\mathbb{C}}$, let us define the closure $\overline{X}$ of X in $\overline{\mathbb{C}}$.

Definition 2.4.4 *Let A be a subset of $\overline{\mathbb{C}}$. A subset is said to be the closure of X in $\overline{\mathbb{C}}$, denoted by $\overline{X}$, if*

$$\overline{X} = \{z \in \overline{\mathbb{C}} : U_\varepsilon(z) \cap X \neq \phi, \forall \varepsilon > 0\}.$$

such that X is a subset of $\overline{\mathbb{C}}$, and may be the points in $\overline{X}$ more than the points of X . i.e., $X \subseteq \overline{X}$.

Example 2.4.5 *If $X = \mathbb{R}$, then :*

- *The closure of the set $\mathbb{Z}$ of integers is $\mathbb{Z}$ itself.*
- *But the closure of the set $\mathbb{Q}$ of rational numbers is the whole space $\mathbb{R}$. We say that $\mathbb{Q}$ is dense in $\mathbb{R}$ [13].*

One of advantages of defining open sets in $\overline{\mathbb{C}}$ is to define convergence in the extended complex plane $\overline{\mathbb{C}}$.

Definition 2.4.6 *The sequence $\{z_n\}$ of points in $\overline{\mathbb{C}}$ converges to a point z of $\overline{\mathbb{C}}$ if for each $\varepsilon > 0$, there exists N for all $n > N$, so that $z_n \in U_\varepsilon(z)$.*

Now, we have the following theorem by combining convergence and closure in $\overline{\mathbb{C}}$.

Theorem 2.4.7 *If $X \subseteq \overline{\mathbb{C}}$ and $\{x_n\}$ is a sequence of points of X converges to a point x of $\overline{\mathbb{C}}$, then x is a point of $\overline{X}$.*

Theorem 2.4.8 *Let $X \subseteq \overline{\mathbb{C}}$, then $\overline{X}$ is closed in $\overline{\mathbb{C}}$.*

Proof. We have to show $\overline{\mathbb{C}}-\overline{X}$ is open in $\overline{\mathbb{C}}$ and for that take any element in $\overline{\mathbb{C}} - \overline{X}$ say z.

Suppose $U_\varepsilon(z) \cap \overline{X} \neq \phi$, $\forall\ \varepsilon > 0$. Choose some $z_n \in U_{\frac{1}{n}}(z) \cap \overline{X}$, $\forall\ n \in \mathbb{N}$. Since the intersection between $U_\varepsilon(z)$ and $\overline{X}$ not empty, so $\exists$ some $x_n \in X$ because $z_n \in \overline{X}$ this implies $x_n \in U_\varepsilon(z) \cap X$. Now,

$$\begin{aligned} |x_n - x| &= |x_n + z_n - z_n - x| \\ &\leq |x_n - z_n| + |z_n - x| \\ &\leq \frac{1}{n} + \frac{1}{n} \\ &= \frac{2}{n} \end{aligned}$$

So, we have $x_n \in U_{\frac{2}{n}}(z) \cap X$. This give $z \in \overline{X}$, $\forall n \in \mathbb{N}$. But this is contradiction, because we choose z belong to the complement of $\overline{X}$ ($z \in \overline{\mathbb{C}}-\overline{X}$). In conclusion, $\overline{\mathbb{C}} - \overline{X}$ is open and hence, X is closed. ■

Another advantage of defining open sets in $\overline{\mathbb{C}}$ is to define continuity in the extended complex plane $\overline{\mathbb{C}}$.

Definition 2.4.9 *A function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is continuous at $z \in \overline{\mathbb{C}}$ if every $\varepsilon > 0$, there exists $\delta > 0$ that depends on z and ε, so that if $w \in U_\delta(z)$ then $f(w) \in U_\varepsilon(f\ (z))$.*

A function f is said to be continuous on $\overline{\mathbb{C}}$ if it is continuous at every point $z \in \overline{\mathbb{C}}$.

Continuous functions are useful that they preserve convergent sequences in the sense that, if $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is continuous and if $\{z_n\}$ is a sequence in $\overline{\mathbb{C}}$ converging to z, then $\{f(z_n)\}$ converges to $f(z)$.

Moreover, we can define the notion of homeomorphism in $\overline{\mathbb{C}}$.

Definition 2.4.10 *A function $f : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ is homeomorphism if f is a bijection and both f and f^{-1} are continuous on $\overline{\mathbb{C}}$.*

It is worth mentioning here that there is the point ∞ in $\overline{\mathbb{C}}$, which solves the problem of discontinuity in $\mathbb{C}$ for some functions. let us see the following example.

Example 2.4.11 *The transformation $J : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined by*

$$J(z) = \begin{cases} \frac{1}{z} & \text{if } z \in \mathbb{C} - \{0\} \\ 0 & \text{if } z = \infty \\ \infty & \text{if } z = 0 \end{cases}$$

is continuous on $\overline{\mathbb{C}}$, but it is not continuous on $\mathbb{C}$. In addition, J is a homeomorphism of $\overline{\mathbb{C}}$.

Now we can see from the above example that the Riemann Sphere plays an important role here. J is a homeomorphism in $\overline{\mathbb{C}}$. By the contrast, J is not a homeomorphism in $\mathbb{C}$ because that $J(0)$ is undefined in $\mathbb{C}$.

Moreover, the group of homeomorphisms on the Riemann sphere can be introduced in the following theorem, it is of most interest to us.

Theorem 2.4.12 *The set of all homeomorphisms on $\overline{\mathbb{C}}$ forms a group under the operation of composition, denoted by Homeo($\overline{\mathbb{C}}$), i.e.,*

$$\text{Homeo}(\overline{\mathbb{C}}) = \{f : \overline{\mathbb{C}} \to \overline{\mathbb{C}} : f \;\text{ is a homeomorphism}\}.$$

Proof. Since the composition of two bijections are a bijection, and the composition of two continuous functions are continuous.

It follows that the composition of two homeomorphisms is a homeomorphism as well. Of course, $f(z) = z$ is the identity.

In addition, the inverse of a homeomorphism is also a homeomorphism.

Thus, $\mathrm{Homeo}(\overline{\mathbb{C}})$ is a group under the operation of composition. $\blacksquare$

Chapter 3: Group Actions on Subspaces of the Riemann Sphere

We explain in the first section that the set $M\ddot{o}b^+(\overline{\mathbb{C}})$ of all Möbius transformations and the set $M\ddot{o}b(\overline{\mathbb{C}})$ of all general Möbius transformations are groups under the operation of composition, and they generated by 2 and 3 generators, respectively. They act on the Riemann sphere by linear fractional transformations.

In the second section, we discuss fixed points of Möbius transformations and give the Uniqueness Theorem, and then we can show that $M\ddot{o}b^+(\overline{\mathbb{C}})$ acts uniquely triply transitively on $\overline{\mathbb{C}}$. As each disc in $\overline{\mathbb{C}}$ is determined by a circle, and each circle in $\overline{\mathbb{C}}$ is determined by three distinct points of $\overline{\mathbb{C}}$. Thus, $\text{Möb}^+(\overline{\mathbb{C}})$ acts transitively on the set of the circles in $\overline{\mathbb{C}}$, and on the set of discs in $\overline{\mathbb{C}}$. We take another point of view in the third section to study the group $M\ddot{o}b(\overline{\mathbb{C}})$ by showing that $M\ddot{o}b(\overline{\mathbb{C}})$ coincides the group of homeomorphisms taking circles in $\overline{\mathbb{C}}$ to circles in $\overline{\mathbb{C}}$.

This chapter we focus on group actions on subspaces of the Riemann Sphere such as the extended real numbers $\overline{\overline{\mathbb{R}}}$, the unit circle S^1, and the unit open disc D. Three subspaces here are invariant under $M\ddot{o}b(\overline{\mathbb{R}})$, $M\ddot{o}b(S^1)$, and $M\ddot{o}b(D)$, respectively. We present an explicit expression for each element of these three group actions in the fourth and fifth sections.

3.1 Groups of General Möbius Transformations

The Möbius transformations of the Riemann sphere $\overline{\mathbb{C}}$ are linear fraction transformations, which are defined in the following definition.

Definition 3.1.1 *A transformation $m : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the the following form is called a Mobius transformation:*

$$m(z) = \frac{az+b}{cz+d},\ a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0.$$

Let $M\ddot{o}b^+(\overline{\mathbb{C}})$ *be the set of all Möbius transformations acting on* $\overline{\mathbb{C}}$*, i.e.,:*

$$M\ddot{o}b^+(\overline{\mathbb{C}}) = \left\{ \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0 \right\}.$$

A transformation $m : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *of the following form is called a general Mobius transformation:*

$$\text{either } \frac{az+b}{cz+d} \text{ or } \frac{a\overline{z}+b}{c\overline{z}+d}, \ a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0.$$

Let $M\ddot{o}b(\overline{\mathbb{C}})$ *be the set of all general Möbius transformations acting on* $\overline{\mathbb{C}}$*, i.e.,*

$$M\ddot{o}b(\overline{\mathbb{C}}) = \left\{ \frac{az+b}{cz+d} \text{ or } \frac{a\overline{z}+b}{c\overline{z}+d} : a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0 \right\}.$$

Note that the transformation $C : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined by $C(z) = \overline{z}$ is not an element of $M\ddot{o}b^+(\overline{\mathbb{C}})$. In fact:

If $C(z) = \overline{z} \in M\ddot{o}b^+(\overline{\mathbb{C}})$, then $C(z) = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$. Take some points and substitute in this transformation.

- If $z = 0$, then $C(0) = \overline{0} = 0$ and $C(0) = \dfrac{b}{d}$, so $\dfrac{b}{d} = 0 \Rightarrow b = 0 \Rightarrow C(z) = \dfrac{az}{cz+d}$.
- If $z = 1$, then $C(1) = \overline{1} = 1$ and $C(1) = \dfrac{a}{c+d}$, so $\dfrac{a}{c+d} = 1 \Rightarrow a = c + d \Rightarrow C(z) = \dfrac{(c+d)z}{cz+d}$.
- If $z = -1$, the $C(-1) = -\overline{1} = -1$ and $C(-1) = \dfrac{-c-d}{-c+d}$, so $\dfrac{-c-d}{-c+d} = -1 \Rightarrow 2c = 0 \Rightarrow c = 0 \Rightarrow C(z) = \dfrac{dz}{d} = z$.

So the conclusion is $C(z) = \overline{z} = z, z \in \overline{\mathbb{C}}$. And this contradicts to $C(z) = \overline{z}$. Therefore, $C(z) = \overline{z} \notin \text{Möb}^+(\overline{\mathbb{C}})$.

The next proposition shows that the condition $ad - bc \neq 0$ in $M\ddot{o}b(\overline{\mathbb{C}})$ can be replaced by the condition $ad - bc = 1$.

Proposition 3.1.2 $M\ddot{o}b(\overline{\mathbb{C}}) = \left\{ \dfrac{az+b}{cz+d} \text{ or } \dfrac{a\overline{z}+b}{c\overline{z}+d} : a, b, c, d \in \mathbb{C} \text{ and } ad - bc = 1 \right\}.$

Proof. Let $M = \left\{ \dfrac{az+b}{cz+d} : a,b,c,d \in \mathbb{C} \text{ and } ad-bc=1 \right\}$, then $M \subseteq M\ddot{o}b(\overline{\mathbb{C}})$ as $1 \neq 0$.

We need to show $M\ddot{o}b(\overline{\mathbb{C}}) \subseteq M$.

Let $k = ad - bc \neq 0$. Take $m(z) = \dfrac{az+b}{cz+d} \in M\ddot{o}b(\overline{\mathbb{C}})$, then

$$m(z) = \frac{\frac{a}{\sqrt{k}}z + \frac{b}{\sqrt{k}}}{\frac{c}{\sqrt{k}}z + \frac{d}{\sqrt{k}}}.$$

Since $a,b,c,d \in \mathbb{C}$, then $\frac{a}{\sqrt{k}}, \frac{b}{\sqrt{k}}, \frac{c}{\sqrt{k}}, \frac{d}{\sqrt{k}} \in \mathbb{C}$ and

$$\begin{aligned}
&\left(\frac{a}{\sqrt{k}}\frac{d}{\sqrt{k}}\right) - \left(\frac{b}{\sqrt{k}}\frac{c}{\sqrt{k}}\right) \\
&= \frac{ad-bc}{\sqrt{k}\sqrt{k}} = \frac{ad-bc}{k} \\
&= \frac{ad-bc}{ad-bc} = 1.
\end{aligned}$$

Thus, $m(z) \in M$.

Similarly, if $m(z) = \dfrac{a\overline{z}+b}{c\overline{z}+d} \in M\ddot{o}b(\overline{\mathbb{C}})$, then $m(z) \in M$.

Therefore, $M\ddot{o}b(\overline{\mathbb{C}}) \subseteq M$. ■

We can see in the extended complex plane $\overline{\mathbb{C}}$ that the value $m(\infty)$ is well defined because one of a or c has to be nonzero, as from the definition of Möbius transformations we know that $ad - bc \neq 0$. From the complex analysis observe that if $m(z) = \dfrac{az+b}{cz+d}$ where $a,b,c,d \in \mathbb{C}$ and $d - bc \neq 0$, then $m(\infty) = \lim\limits_{z\to\infty} \dfrac{az+b}{cz+d} = \dfrac{a}{c}$. So, we have that $m(\infty) = \infty$ if and only if $c = 0$. Also, because $m(0) = \dfrac{b}{d}$, we have that $m(0) = 0$ if and only if $b = 0$.

As $\infty \notin \mathbb{C}$, the Möbius transformations of the complex plane $\mathbb{C}$ are linear transformations of the forms: $f(z) = az + b$ or $f(z) = a\overline{z} + b$ such that $a \neq 0$ and $a, b \in \mathbb{C}$.

Furthermore, we can use the traditional natural convention about arithmetic with ∞ [[7],[18]].

Proposition 3.1.3 *There is a one-to-one correspondence between each composition of two Möbius transformations and multiplication of two 2×2 matrices.*

Proof. Let $f(z) = \dfrac{a_1 z + b_1}{c_1 z + d_1}$, where $a_1, b_1, c_1, d_1 \in \mathbb{C}$ and $a_1 d_1 - b_1 c_1 \neq 0$, and let $g(z) = \dfrac{a_2 z + b_2}{c_2 z + d_2}$, where $a_2, b_2, c_2, d_2 \in \mathbb{C}$ and $a_2 d_2 - b_2 c_2 \neq 0$. Then:

$$\begin{aligned}
f \circ g(z) &= f(g(z)) \\
&= \frac{a_1 g(z) + b_1}{c_1 g(z) + d_1} \\
&= \frac{\dfrac{a_1 a_2 z + a_1 b_2}{c_2 z + d_2} + b_1}{\dfrac{c_1 a_2 z + c_1 b_2}{c_2 z + d_2} + d_1} \\
&= \frac{\dfrac{a_1 a_2 z + a_1 b_2 + b_1 c_2 z + d_2 b_1}{c_2 z + d_2}}{\dfrac{c_1 a_2 z + c_1 b_2 + c_2 d_1 z + d_1 d_2}{c_2 z + d_2}} \\
&= \frac{a_1 a_2 z + a_1 b_2 + b_1 c_2 z + d_2 b_1}{c_1 a_2 z + c_1 b_2 + d_1 c_2 z + d_1 d_2} \\
&= \frac{(a_1 a_2 + b_1 c_2)\ z + (a_1 b_2 + d_2 b_1)}{(c_1 a_2 + d_1 c_2)\ z + (c_1 b_2 + d_1 d_2)},
\end{aligned}$$

where $(a_1 a_2 + b_1 c_2)(c_1 b_2 + d_1 d_2) - (a_1 b_2 + d_2 b_1)(c_1 a_2 + d_1 c_2) \neq 0$.

On the other hand, let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, and let $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$. Then,

$$AB = \begin{bmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{bmatrix}.$$

By comparing the composition and multiplication, there is a one-to-one correspondence. ■

Theorem 3.1.4 *$M\ddot{o}b^+(\overline{\mathbb{C}})$ forms a group under the composition.*

Proof. (1) Let f and $g \in M\ddot{o}b^+(\overline{\mathbb{C}})$. By the previous Theorem 3.1.4, there is a one-to-one correspondence between each composition of two Möbius transformations and multiplication of two 2×2 matrices, and hence there are two matrices

$$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \text{ and } B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

such that the composition $f \circ g$ is corresponding to the product AB. Thus,

$$f \circ g = \frac{(a_1a_2 + b_1c_2)\ z + (a_1b_2 + d_2b_1)}{(c_1a_2 + d_1c_2)\ z + (c_1b_2 + d_1d_2)} \in M\ddot{o}b^+(\overline{\mathbb{C}}).$$

In addition, let $f, g,$ and $h \in M\ddot{o}b^+(\overline{\mathbb{C}})$, then there are three matrices $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$, and $C = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}$ such that the compositions $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are corresponding to the products $(AB)C$ and $A(BC)$.

As we know in Linear Algebra:, the set of matrices has the associativity under multiplication. Thus, $(AB)C = A(BC)$, and hence $(f \circ g) \circ h = f \circ (g \circ h)$. Therefore, $M\ddot{o}b^+(\overline{\mathbb{C}})$ has the associativity under the operation of composition.

(2) It's clear that $f(z) = z$ is the identity of $M\ddot{o}b^+(\overline{\mathbb{C}})$ under the operation of composition.

(3) Every $f \in M\ddot{o}b^+(\overline{\mathbb{C}})$, then $f(z) = \dfrac{az + b}{cz + d}$, and the corresponding matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. So A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{k} & \frac{-b}{k} \\ \frac{-c}{k} & \frac{a}{k} \end{bmatrix},$$

where $k = ad - bc$. It follows that $f^{-1} = \dfrac{\frac{d}{k}z + \frac{-b}{k}}{\frac{-c}{k}z + \frac{-a}{k}} \in M\ddot{o}b^+(\overline{\mathbb{C}})$. Therefore, $M\ddot{o}b^+(\overline{\mathbb{C}})$ has the inverse for every element under the operation of composition.

In conclusion, $M\ddot{o}b^+(\overline{\mathbb{C}})$ is a group under the operation of composition. ■

Theorem 3.1.5 *The group $M\ddot{o}b^+(\overline{\mathbb{C}})$ is generated by $f(z) = az + b, a \neq 0, b \in \mathbb{C}$ and $J(z) = \frac{1}{z}$, i.e.,*

$$M\ddot{o}b^+(\overline{\mathbb{C}}) = \left\langle f(z) = az + b, J(z) = \frac{1}{z} \right\rangle.$$

Proof. Let a Möbius transformation be $m(z) = \dfrac{az + b}{cz + d} \in M\ddot{o}b^+(\overline{\mathbb{C}})$ with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Then we have the following two cases.

(a) If $c = 0$, then $ad - bc = ad \neq 0$, and hence $a \neq 0$ and $d \neq 0$.

Now $m(z) = \dfrac{a}{d}z + \dfrac{b}{d} = f(z)$, where $\dfrac{a}{d} \neq 0$ and $\dfrac{b}{d} \in \mathbb{C}$.

(b) If $c \neq 0$, then $m(z) = f(J(g(z)))$, where $g(z) = c^2 z + cd$ with $c^2 \neq 0, cd \in \mathbb{C}$; and $f(z) = -(ad - bc)z + \dfrac{a}{c}$ with $-(ad - bc) \neq 0$, $\dfrac{a}{c} \in \mathbb{C}$. ■

Furthermore, using the similar arguments in the previous proof, we have the following theorem.

Theorem 3.1.6 *The set $M\ddot{o}b(\overline{\mathbb{C}})$ of all general Möbius transformations acting on $\overline{\mathbb{C}}$ is a group, that is generated by $f(z) = az + b$ with $a \neq 0, b \in \mathbb{C}$, $J(z) = \frac{1}{z}$, and $C(z) = \overline{z}$. i.e.,*

$$M\ddot{o}b(\overline{\mathbb{C}}) = \left\langle f(z) = az + b, J(z) = \frac{1}{z}, \text{ and } C(z) = \overline{z} \right\rangle.$$

3.2 Triply Transitiveness

In this section we discuss fixed points of Möbius transformations and give the Uniqueness Theorem. We then can show that $M\ddot{o}b^{+}(\overline{\mathbb{C}})$ acts uniquely triply transitively on $\overline{\mathbb{C}}$. As each disc in $\overline{\mathbb{C}}$ is determined by a circle, and each circle in $\overline{\mathbb{C}}$ is determined by three distinct points of $\overline{\mathbb{C}}$. Thus, Möb$^{+}(\overline{\mathbb{C}})$ acts transitively on the set of the circles in $\overline{\mathbb{C}}$, and on the set of discs in $\overline{\mathbb{C}}$.

Definition 3.2.1 *A fixed point of the Möbius transformation m is a point z of $\overline{\mathbb{C}}$ satisfying $m(z) = z$.*

Example 3.2.2 *(1) Let $m(z) = 7z + 6$, then the fixed points of $m(z)$ satisfies:*

$$7z + 6 = z.$$

Hence, $z = -1$ and $\infty \in \overline{\mathbb{R}}$ are the all fixed points of $7z + 6$.

(2) Let $m(z) = \frac{z}{z+1}$, *then the fixed points of* $m(z)$ *satisfies:*

$$\begin{aligned} \frac{z}{z+1} &= z, \\ i.e., z^2 &= 0. \end{aligned}$$

Hence, $z = 0 \in \mathbb{R}$ *is the unique fixed point of* $\frac{z}{z+1}$.

(3) Let $m(z) = \frac{1}{z}$, *then the fixed points of* $m(z)$ *satisfies:*

$$\begin{aligned} \frac{1}{z} &= z, \\ z^2 &= 1. \end{aligned}$$

Hence $z = \pm 1 \in \mathbb{R}$ *are the all fixed points of* $\frac{1}{z}$.

The previous example shows that we can express the fixed points as the roots of a quadratic equation. it is true in general, see the next theorem.

Lemma 3.2.3 *Let* $m(z) = \dfrac{az+b}{cz+d}$ *is a Möbius transformation. Then we can express the fixed points as the roots of a quadratic equation* $cz^2 + (d-a)z - b = 0$

Proof. Since $m(z) = \dfrac{az+b}{cz+d} = z \Leftrightarrow cz^2 + (d-a)z - b = 0$, so the roots of this expression are:

$$\frac{-(d-a) \pm \sqrt{(d-a)^2 + 4ab}}{2c}.$$

■

Theorem 3.2.4 (Uniqueness Theorem) *Let* $m(z) = \dfrac{az+b}{cz+d}$ *be a Möbius transformation. If it fixes three distinct points, then it is the identity transformation.*

Proof. From the previous Lemma 3.2.3, we can express the fixed points of a Möbius transformation as the roots of a quadratic equation. From that, if there are three roots or more, this is meaning that all coefficients of that equation are zero. In other meaning, $c = b = 0$ and $d = a$. So the Möbius transformation becomes $\frac{az}{d} = z$, and this is the identity transformation. ■

We turn to discuss the uniquely triply transitive action on $\overline{\mathbb{C}}$, it is one of the most important properties of $M\ddot{o}b^+(\overline{\mathbb{C}})$.

Definition 3.2.5 *A group G acts transitively on a set X if for each pair of elements x and y in X, there exists some element $g \in G$ such that $g(x) = y$.*

Theorem 3.2.6 *Let (z_1, z_2, z_3) and (w_1, w_2, w_3) be two triples of distinct points of $\overline{\mathbb{C}}$. Then there exists a unique element m of $M\ddot{o}b^+(\overline{\mathbb{C}})$, such that*

$$m(z_1) = w_1, m(z_2) = w_2, m(z_3) = w_3 \ .$$

Proof. We need to show existence and uniqueness.

First we give the proof of existence. Let (z_1, z_2, z_3) goes to $(0, 1, \infty)$ by m_1, and let (w_1, w_2, w_3) goes to $(0, 1, \infty)$ by m_2, then (z_1, z_2, z_3) goes to (w_1, w_2, w_3) by $m_2^{-1} \circ m_1$.

Second, for the uniqueness: Suppose that there are two elements m and n of $M\ddot{o}b^+(\overline{\mathbb{C}})$ satisfying:

$$n(z_1) = w_1 = m(z_1),\ \ n(z_2) = w_2 = m(z_2), \text{ and } n(z_3) = w_3 = m(z_3).$$

It follows that $m^{-1} \circ n$ fixes three distinct points of $\overline{\mathbb{C}}$:

$$\begin{aligned} m^{-1} \circ n(z_1) &= m^{-1}(w_1) = z_1, \\ m^{-1} \circ n(z_2) &= m^{-1}(w_2) = z_2, \\ m^{-1} \circ n(z_3) &= m^{-1}(w_3) = z_3. \end{aligned}$$

By the Uniqueness Theorem 3.2.4, $m^{-1} \circ n$ is the identity, and hence $n = m$.

■

Notice that each disc in $\overline{\mathbb{C}}$ is determined by a circle, and each circle in $\overline{\mathbb{C}}$ is determined by three distinct points of $\overline{\mathbb{C}}$. Thus, we have the following corollary.

Corollary 3.2.7 *$M\ddot{o}b^+(\overline{\mathbb{C}})$ acts transitively on the set of the circles in $\overline{\mathbb{C}}$, and on the set of discs in $\overline{\mathbb{C}}$.*

3.3 The Group of Homeomorphisms Acting on the Set of Circles

This section we focus on the group of homeomorphisms taking circles in $\overline{\mathbb{C}}$ to circles in $\overline{\mathbb{C}}$. We present that this group coincides with the group $M\ddot{o}b(\overline{\mathbb{C}})$.

Let $Homeo^C(\overline{\mathbb{C}})$ be the set of all homeomorphisms of $\overline{\mathbb{C}}$ taking circles in $\overline{\mathbb{C}}$ to circles in $\overline{\mathbb{C}}$, i.e.,

$$Homeo^C(\overline{\mathbb{C}}) = \left\{ f \in Homeo(\overline{\mathbb{C}}) : f(A) \text{ is a circle in } \overline{\mathbb{C}} \text{ if } A \text{ is a circle in } \overline{\mathbb{C}} \right\}.$$

Then it is clear that $Homeo^C(\overline{\mathbb{C}})$ is a group under the operation of composition.

The simple example is that $C(z) = \overline{z} \in Homeo^C(\overline{\mathbb{C}})$. The following example shows that there is an element of $Homeo(\overline{\mathbb{C}})$ that is not an element of $Homeo^C(\overline{\mathbb{C}})$. That is, $Homeo^C(\overline{\mathbb{C}}) \subsetneqq Homeo(\overline{\mathbb{C}})$.

Example 3.3.1 *Consider the function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ given by:*

$$f(z) = \begin{cases} z, & \text{if } Re(z) \leq 0; \\ z + i\,\mathrm{Re}(z), & \text{if } \mathrm{Re}(z) \geq 0; \\ \infty & \text{if } z = \infty. \end{cases}$$

Obviously, f is continuous, because a sum of continuous functions are also continuous. To see that f is a bijection and f^{-1} is continuous, we give an explicit formula for f^{-1}, namely,

$$f^{-1}(z) = \begin{cases} z, & \text{if } Re(z) \leq 0; \\ z - i\,\mathrm{Re}(z), & \text{if } \mathrm{Re}(z) \geq 0; \\ \infty & \text{if } z = \infty. \end{cases}$$

Hence, we see that $f \in Homeo(\overline{\mathbb{C}})$, but the image of $\overline{\mathbb{R}}$ under f is not a circle in $\overline{\mathbb{C}}$, and so $f \notin Homeo^C(\overline{\mathbb{C}})$.

We are going to identify $Homeo^C(\overline{\mathbb{C}})$. As we know $C(z) = \overline{z} \in Homeo^C(\overline{\mathbb{C}})$. let us consider another two generators of $M\ddot{o}b(\overline{\mathbb{C}})$.

Lemma 3.3.2 *Let transformation $J : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be defined by*

$$J(z) = \begin{cases} \frac{1}{z} & \text{if } z \in \mathbb{C} - \{0\} \\ 0 & \text{if } z = \infty \\ \infty & \text{if } z = 0. \end{cases}$$

Then $J(z) \in Homeo^C(\overline{\mathbb{C}})$.

Proof. Let A be a circle in $\overline{\mathbb{C}}$ defined by the equation:

$$\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0, \text{ where } \alpha, \gamma \in \mathbb{R}, \text{ and } \beta \in \mathbb{C}.$$

Let $w = \frac{1}{z}$, then $z = \frac{1}{w}$. By substituting in the equation of the circle A above, we get:

$$\alpha \frac{1}{w}\frac{\overline{1}}{w} + \beta\frac{1}{w} + \overline{\beta}\frac{\overline{1}}{w} + \gamma = 0.$$

Multiply by $w\overline{w}$,

$$\alpha w\overline{w} + \beta w + \overline{\beta}\overline{w} + \gamma = 0, \text{ where } \alpha, \gamma \in \mathbb{R}, \text{ and } \beta \in \mathbb{C}.$$

Thus, this equation shows that $J(A)$ a circle in $\overline{\mathbb{C}}$. ■

Example 3.3.3 *Let A be a circle in $\overline{\mathbb{C}}$ which given by the equation $2z + 2\overline{z} + 3 = 0$, then $J(A)$ is a circle in $\overline{\mathbb{C}}$.*

In fact, let $w = \frac{1}{z}$, then $z = \frac{1}{w}$.

By the substitution, $f(A)$ is given by the equation:

$$\begin{aligned} 2w + 2\overline{w} + 3w\overline{w} &= 0, \\ w\overline{w} + \frac{2}{3}\overline{w} + \frac{2}{3}w &= 0. \end{aligned}$$

And let:

$$\begin{aligned} r^2 &= |w-s|^2 \\ &= (w-s)\overline{(w-s)} \\ &= (w-s)(\overline{w}-\overline{s}) \\ &= w\overline{w} - s\overline{w} - \overline{s}w + \overline{s}s - r^2 = 0. \end{aligned}$$

So, rewrite the equation as:

$$w\overline{w} - s\overline{w} - \overline{s}w + \overline{s}s - r^2 = 0.$$

Thus, $-s = \frac{2}{3} \Leftrightarrow s = -\frac{2}{3}$ *(Euclidean center).*

Also $\overline{s}s - r^2 = 0 \Longleftrightarrow (-\frac{2}{3})(-\frac{2}{3}) - r^2 = 0 \Leftrightarrow \frac{4}{9} - r^2 = 0 \Longleftrightarrow r = \frac{2}{3}$ *(Euclidean radius).*

Lemma 3.3.4 *Let transformation* $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *be defined by*

$$f(z) = \begin{cases} az+b & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty, \end{cases}$$

where $a, b \in \mathbb{C}$ *and* $a \neq 0$*. Then* $f \in Homeo^C(\overline{\mathbb{C}})$*.*

Proof. We need to show that f takes Euclidean circles in $\mathbb{C}$ to Euclidean circles in $\mathbb{C}$.

Let $w = az+b$, so $z = \frac{1}{a}(w-b)$. By substituting in the equation of the Euclidean circle, $\alpha z\overline{z}+\beta z+\gamma = 0$, we get:

$$\begin{aligned} \alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma &= \alpha\left(\frac{1}{a}(w-b)\right)\overline{\left(\frac{1}{a}(w-b)\right)} + \beta\frac{1}{a}(w-b) + \overline{\beta}\left(\frac{1}{a}(w-b)\right) + \gamma \\ &= \frac{\alpha}{|\ a\ |^2}(w-b)\overline{(w-b)} + \frac{\beta}{a}(w-b) + \frac{\overline{\beta}}{a}\overline{(w-b)} + \gamma \\ &= \frac{\alpha}{|a|^2}\left|w + \frac{\overline{\beta}a}{\alpha} - b\right|^2 + \gamma - \frac{|\beta|^2}{\alpha} = 0, \text{ where } |a|^2 = a\overline{a} \end{aligned}$$

(which is the equation of Euclidean circle in $\mathbb{C}$). ■

Applying for Theorem 3.1.5 and two lemmas above, we can see that $M\ddot{o}b^+(\overline{\mathbb{C}}) \subseteq Homeo^C(\overline{\mathbb{C}})$. Furthermore, by the Theorem 3.1.6, the group $M\ddot{o}b(\overline{\mathbb{C}})$ is generated by $M\ddot{o}b^+(\overline{\mathbb{C}})$ and $C(z) = \overline{z}$. In addition, $C(z) = \overline{z} \in \mathrm{Homeo}^C(\overline{\mathbb{C}})$. Thus, we reach the following theorem.

Theorem 3.3.5 $M\ddot{o}b(\overline{\mathbb{C}}) \subseteq Homeo^C(\overline{\mathbb{C}})$.

Conversely, we can also show that $Homeo^C(\overline{\mathbb{C}}) \subseteq M\ddot{o}b(\overline{\mathbb{C}})$. However, we need some lemmas below.

Lemma 3.3.6 *Let* $f \in Homeo^C(\overline{\mathbb{C}})$*, let* $g \in M\ddot{o}b(\overline{\mathbb{C}})$ *and let* $h = g \circ f$*. If* g *takes* $(f(0), f(1), f(\infty))$ *to* $(0, 1, \infty)$*, then* $h(0) = 0, h(1) = 1$ *and* $h(\infty) = \infty$.

Proof. $h(0) = g \circ f(0) = g(f(0)) = g(0) = 0$, since $0 \xrightarrow{f} f(0) \xrightarrow{g} 0$ so, $0 \to 0$. This is meaning, h fixes the points $0, 1$ and ∞. ■

Lemma 3.3.7 *Let* $f \in Homeo^C(\overline{\mathbb{C}})$*, let* $g \in M\ddot{o}b(\overline{\mathbb{C}})$ *and let* $h = g \circ f$*. If* g *takes* $(f(0), f(1), f(\infty))$ *to* $(0, 1, \infty)$*, then* $h(\mathbb{R}) = \mathbb{R}$.

Proof. As we know $h = g \circ f$ takes circles in $\overline{\mathbb{C}}$ to circles in $\overline{\mathbb{C}}$. Since $\overline{\mathbb{R}}$ is a circle in $\overline{\mathbb{C}}$ and determined by $0, 1$ and ∞. Because h fixes $0, 1$ and ∞ then $h(\overline{\mathbb{R}})$ is circle in $\overline{\mathbb{C}}$, therefore, $h(\overline{\mathbb{R}}) = \overline{\mathbb{R}}$.

In addition $h(\infty) = \infty$ and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Thus, $h(\mathbb{R}) = \mathbb{R}$. ■

Lemma 3.3.8 *Let* $f \in Homeo^C(\overline{\mathbb{C}})$*, let* $g \in M\ddot{o}b(\overline{\mathbb{C}})$ *and let* $h = g \circ f$*. If* $h(\mathbb{R}) = \mathbb{R}$*, then* $h(\mathbb{H}^2) = \mathbb{H}^2$ *or* $h(\mathbb{H}^2) = \mathbb{H}^2_-$ *(lower half-plane).*

Proof. Since $h(\mathbb{R}) = \mathbb{R}$ and fix ∞ in the same time, this is meaning h takes $\mathbb{R}$ to $\mathbb{R}$, so the upper half plane $\mathbb{H}^2$ can be goes to the upper half plane or the lower half plane. And theses are the only two cases. ■

Lemma 3.3.9 *Let* $f \in Homeo^C(\overline{\mathbb{C}})$*, let* $m \in M\ddot{o}b(\overline{\mathbb{C}})$*, such that,* $m = n \circ g$ *and let* $h = g \circ f$*. If* $m \circ f(0) = 0, m \circ f(1) = 1, m \circ f(\infty) = \infty$ *and* $m \circ f(H) = H$*, then* $m \circ f$ *is identity.*

Proof. We want to construct a dense set of points in $\overline{\mathbb{C}}$, each of point fixed by $m \circ f$.

Let $Z = \left\{z \in \overline{\mathbb{C}} : m \circ f(z) = z\right\}$ is the set of points of $\overline{\mathbb{C}}$ fixed by $m \circ f$, with $0, 1, \infty \in Z$.

We can see here that $m \circ f$ takes Euclidean lines in $\mathbb{C}$ to Euclidean lines in $\mathbb{C}$, since $m \circ f \in Homeo^C(\overline{\mathbb{C}})$ and fixes ∞.

Now, let X and Y two Euclidean lines in $\mathbb{C}$ that intersect at z_0, if $m \circ f(X) = X$ and $m \circ f(Y) = Y$, then $m \circ f(z_0) = z_{0,}$ this is meaning $z_0 \in Z$. For that consider two lines:

1) Consider $T(s)$ is the horizontal line in $\mathbb{C}$ that passes through it, if $s \in \mathbb{R}$, then $m \circ f(T) = T$ and $m \circ f(\mathbb{R}) = \mathbb{R}$ are disjoint, since T and $\mathbb{R}$ are disjoint.

2) Consider $V(s)$ is the vertical line in $\mathbb{C}$ that passes through it, if $s \in \mathbb{R}$, then $V(0)$ is the tangent to A at 0, if A is a Euclidean circle with center $\frac{1}{2}$ and radius $\frac{1}{2}$, then $m \circ f(V(0))$ is the tangent line to $m \circ f(A)$ at $m \circ f(0)$ and $m \circ f(V(1))$ is the tangent line to $m \circ f(A)$ at $m \circ f(1)$.

Both: $V(0)$ and $V(1)$ are parallel lines in $\mathbb{C}$, and both $m \circ f(V(0))$ and $m \circ f(V(1))$ are also parallel lines in $\mathbb{C}$. By that, we must get $m \circ f(V(0)) = V(0)$ and $m \circ f(V(1)) = V(1)$.

Now, we want to use same procedure with the two horizontal tangent lines to A :

Consider $T(\frac{1}{2})$ be the tangent line to A at $\frac{1}{2} + \frac{1}{2}i$, then $m \circ f(T(\frac{1}{2})) = T(\frac{1}{2})$, since $m \circ f(T(\frac{1}{2}))$ is a horizontal line in the upper half-plane $\mathbb{H}^2$ tangent to $m \circ f(A) = A$

Until now, we collect some points to be in Z :

- $T(\frac{1}{2}) \cap V(0) = \frac{1}{2}i$.
- $T(\frac{1}{2}) \cap V(1) = 1 + \frac{1}{2}i$.
- $T(-\frac{1}{2}) \cap V(0) = -\frac{1}{2}i$. (the same argument gives: $m \circ f(T(-\frac{1}{2})) = T(-\frac{1}{2})$)
- $T(-\frac{1}{2}) \cap V(1) = 1 - \frac{1}{2}i$. (the same argument gives: $m \circ f(T(-\frac{1}{2})) = T(-\frac{1}{2})$)

- The intersection between : The Euclidean line that take each pair of points in Z to itself by $m \circ f$ and the Euclidean circle that take each triple of non colinear points in Z to itself by $m \circ f$.

By this process, Z contains a dense set of points of $\overline{\mathbb{C}}$, which gives $m \circ f$ is identity. And with the previous theorem that shown, $Homeo^C(\overline{\mathbb{C}}) \subseteq M\ddot{o}b(\overline{\mathbb{C}})$ (see 3.3.5).

The above theorem equivalent to: each element in Möb($\overline{\mathbb{C}}$) takes circles in $\overline{\mathbb{C}}$ to circles in $\overline{\mathbb{C}}$. Conversely, any homeomorphism taking circles in $\overline{\mathbb{C}}$ to circles in $\overline{\mathbb{C}}$ must be a general Möbius transformation in Möb($\overline{\mathbb{C}}$). ■

Theorem 3.3.10 $M\ddot{o}b(\overline{\mathbb{C}}) = Homeo^C(\overline{\mathbb{C}})$.

3.4 The Group Action on the Extended Real Numbers

The set $\mathbb{R} \cup \{\infty\}$ is said to be the extended real numbers, denoted by $\overline{\mathbb{R}}$, it is an extended circle $\overline{\mathbb{R}}$ in $\overline{\mathbb{C}}$.

Let $M\ddot{o}b(\overline{\mathbb{R}})$ be the set of Möbius transformations of $\overline{\mathbb{C}}$ preserve the extended circle $\overline{\mathbb{R}}$ in $\overline{\mathbb{C}}$, then it is a subgroup of $M\ddot{o}b(\overline{\mathbb{C}})$. Thus $\overline{\mathbb{R}}$ is invariant under the group $M\ddot{o}b(\overline{\mathbb{R}})$, equivalently, $M\ddot{o}b(\overline{\mathbb{R}})$ is the stabilizer of $\overline{\mathbb{R}}$:

$$M\ddot{o}b(\overline{\mathbb{R}}) = \left\{m \in M\ddot{o}b(\overline{\mathbb{C}}) : m(\overline{\mathbb{R}}) = \overline{\mathbb{R}}\right\}.$$

In this section we identify the group $M\ddot{o}b(\overline{\mathbb{R}})$ acting on $\overline{\mathbb{R}}$, i.e., the forms of elements of Möb ($\overline{\mathbb{R}}$). An explicit expression for each element of Möb ($\overline{\mathbb{R}}$) are presented in the following theorem.

Theorem 3.4.1 *Every element $m(z)$ of Möb ($\overline{\mathbb{R}}$) has one of the following four forms:*

1. $m(z) = \dfrac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$;
2. $m(z) = \dfrac{a\overline{z}+b}{c\overline{z}+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$;
3. $m(z) = \dfrac{az+b}{cz+d}$ with a, b, c, d purely imaginary and $ad - bc = 1$;

4. $m(z) = \dfrac{a\overline{z}+b}{c\overline{z}+d}$ with a,b,c,d purely imaginary and $ad-bc=1$.

Proof. Notice that if $m(z) \in \left[M\ddot{o}b^{+}(\overline{\mathbb{C}})\right]^{c}$ then $m \circ C(z) \in M\ddot{o}b^{+}(\overline{\mathbb{C}})$, where $C(z) = \overline{z}$, the conjugate function. In fact, let $m(z) = \dfrac{a\overline{z}+b}{c\overline{z}+d} \in \left[M\ddot{o}b^{+}(\overline{\mathbb{C}})\right]^{c}$, where $a,b,c,d \in \mathbb{C}$ and $ad-bc=1$, then the composition of m and $C(z)$ is:

$$m \circ C(z) = m(\overline{z}) = \frac{az+b}{cz+d} \in M\ddot{o}b^{+}(\overline{\mathbb{C}}).$$

Thus, we need to solve $a, b, c,$ and d for each $m(z) = \dfrac{az+b}{cz+d} \in M\ddot{o}b(\overline{\mathbb{R}})$, where $a,b,c,d \in \mathbb{C}$ and $ad-bc=1$. We can solve $a, b,$ and d in terms of c.

let us consider the following three cases.

Case I: Suppose that $a \neq 0$ and $c \neq 0$. Notice that $m(-\frac{d}{c}) = \infty$ and $m(-\frac{b}{a}) = 0$, so $m^{-1}(\infty) = -\frac{d}{c}$ and $m^{-1}(0) = -\frac{b}{a}$. In addition, $m(\infty) = \lim\limits_{z\to\infty} \dfrac{az+b}{cz+d} = \frac{a}{c}$, i.e, $m(\infty) = \frac{a}{c}$.

Since $\overline{\mathbb{R}}$ is invariant under the group $M\ddot{o}b(\overline{\mathbb{R}}) : m(\overline{\mathbb{R}}) = \overline{\mathbb{R}}$, and hence the following three points belong to $\overline{\mathbb{R}}$.

$$m^{-1}(\infty) = -\frac{d}{c}, m(\infty) = \frac{a}{c}, \text{ and } m^{-1}(0) = -\frac{b}{a} \in \overline{\mathbb{R}}.$$

It follows that $a, b,$ and d can be expressed as a multiple of c :

$$\begin{aligned} a &= m(\infty)c, \\ b &= -m^{-1}(0)a = -m^{-1}(0)m(\infty)c, \\ d &= -m^{-1}(\infty)c. \end{aligned}$$

Using the condition $ad-bc=1$, we have

$$ad - bc = c^2 m(\infty)\left[m^{-1}(0) - m^{-1}(\infty)\right] = 1,$$

where $m(\infty), m^{-1}(0)$ and $m^{-1}(\infty) \in \mathbb{R}$. Notice that $i^2 = -1 \in \mathbb{R}$. It follows that c is either real or purely imaginary. Therefore, a, b, c and d are either real or pure imaginary.

Case II: Suppose that $a = 0$ and $c \neq 0$. Since $m(\overline{\mathbb{R}}) = \overline{\mathbb{R}}$, then the two points $m^{-1}(\infty) = -\frac{d}{c}$ and $m(1) = \dfrac{b}{c+d} \in \mathbb{R}$, which give

$$\begin{aligned} d &= -m^{-1}(\infty)c, \\ b &= m(1)(c+d) = cm(1)\left[1 - m^{-1}(\infty)\right]. \end{aligned}$$

Applying for the condition $ad - bc = 1$, we have

$$ad - bc = -bc = c^2 m(1)\left[m^{-1}(\infty) - 1\right] = 1,$$

where $m^{-1}(\infty)$ and $m(1) \in \mathbb{R}$. It follows that c is either real or purely imaginary. Therefore, a, b, c and d are either real or pure imaginary.

Case III: Suppose that $a \neq 0$ and $c = 0$. Since $m(\overline{\mathbb{R}}) = \overline{\mathbb{R}}$, then the two points $m(0) = \dfrac{b}{d}$ and $m(1) = \dfrac{b}{c+d} \in \mathbb{R}$, which give

$$\begin{aligned} b &= m(0)d, \\ a &= m(1)d - b = d\left[m(1) - m(0)\right] \end{aligned}$$

Again, apply for the condition $ad - bc = 1$, we have

$$ad - bc = ad = d^2\left[m(1) - m(0)\right],$$

where $m(1)$ and $m(0) \in \mathbb{R}$. It follows that d is either real or purely imaginary. Again a, b, c and d are either real or pure imaginary.

Note that, when $a = 0$ in the Case II and $c = 0$ in the Case III, we can write a as $0i$, and c as $0i$.

As the summary, every element of Möb $(\overline{\mathbb{R}})$ has one of the following four forms:

(1) $m(z) = \dfrac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$;

(2) $m(z) = \dfrac{a\overline{z}+b}{c\overline{z}+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$;

(3) $m(z) = \dfrac{az+b}{cz+d}$ with a, b, c, d purely imaginary and $ad - bc = 1$;

(4) $m(z) = \dfrac{a\bar{z}+b}{c\bar{z}+d}$ with a, b, c, d purely imaginary and $ad - bc = 1$.

The proof is completed. ■

3.5 The Group Actions on the Unit Circle and the Unit Open Disc

Given a circle A in $\overline{\mathbb{C}}$, the stabilizer of A in $M\ddot{o}b(\overline{\mathbb{C}})$ is the following subgroup of Möb($\overline{\mathbb{C}}$), denoted by $M\ddot{o}b(A)$:

$$M\ddot{o}b(A) = \{m \in M\ddot{o}b(\overline{\mathbb{C}}) : m(A) = A\}.$$

By the Corollary 3.2.7, the group Möb$^+(\overline{\mathbb{C}})$ acts transitively on the set of the circles in $\overline{\mathbb{C}}$ and on the set of discs in $\overline{\mathbb{C}}$, there exists some element $p \in$ Möb$^+(\overline{\mathbb{C}}) \subseteq$ Möb($\overline{\mathbb{C}}$) satisfying $p(\overline{\mathbb{R}}) = A$, i.e., p takes the circle $\overline{\mathbb{R}}$ to the circle A in $\overline{\mathbb{C}}$.

If n is any element of Möb(A), then $n(A) = A$, and hence $p^{-1} \circ n \circ p(\overline{\mathbb{R}}) = \overline{\mathbb{R}}$. Hence, we can write $p^{-1} \circ n \circ p = m$ for some element $m \in M\ddot{o}b(\overline{\mathbb{R}})$, and so $n = p \circ m \circ p^{-1}$. It follows that

$$M\ddot{o}b(A) = \{p \circ m \circ p^{-1} : m \in M\ddot{o}b(\overline{\mathbb{R}})\}.$$

The following proposition shows that the expression above is independent of the choice of p.

Proposition 3.5.1 *The subgroup $M\ddot{o}b(A)$ is independent of the choice of $p \in M\ddot{o}b(\overline{\mathbb{C}})$ in the expression:*

$$M\ddot{o}b(A) = \{p \circ m \circ p^{-1} : m \in M\ddot{o}b(\overline{\mathbb{R}})\}.$$

Proof. Suppose q is another element $\in$ Möb($\overline{\mathbb{C}}$) satisfying $q(\overline{\mathbb{R}}) = A$. If, $p^{-1} \circ q$ takes $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}$, then we can write $q = p \circ s$ for some element $s \in M\ddot{o}b(\overline{\mathbb{R}})$. Hence, for any $m \in M\ddot{o}b(\overline{\mathbb{R}})$, we have that $q \circ m \circ q^{-1} = p \circ (s \circ m \circ s^{-1}) \circ p^{-1}$, and so $\left\{p \circ m \circ p^{-1} \mid m \in M\ddot{o}b(\overline{\mathbb{R}})\right\} = \left\{q \circ m \circ q^{-1} \mid m \in M\ddot{o}b(\overline{\mathbb{R}})\right\}$. ■

Theorem 3.5.2 *Let $\mathbb{S}^1$ be the unit circle, and let $m, n, p \in M\ddot{o}b$. If $h = p \circ m \circ p^{-1}$ and $g = p \circ n \circ p^{-1}$, then an element of $M\ddot{o}b(\mathbb{S}^1)$ has one of the form:*

(1) $h(z) = \dfrac{\alpha z+\beta}{\overline{\beta}z+\overline{\alpha}}$ *with* $a,b,c,d \in \mathbb{R}$ *and* $\alpha\overline{\alpha} - \beta\overline{\beta} = 1$;

(2) $h(z) = \dfrac{\alpha z+\beta}{-\overline{\beta}z-\overline{\alpha}}$ *with* $a,b,c,d \in$ *purely imaginary and* $-\alpha\overline{\alpha} + \beta\overline{\beta} = \beta\overline{\beta} - \alpha\overline{\alpha} = 1$;

(3) $g(z) = \dfrac{\delta\overline{z}+\gamma}{-\overline{\gamma}\overline{z}-\overline{\delta}}$ *with* $a,b,c,d \in \mathbb{R}$ *and* $-\delta\overline{\delta} + \gamma\overline{\gamma} = \gamma\overline{\gamma} - \delta\overline{\delta} = 1$;

(4) $g(z) = \dfrac{\delta\overline{z}+\gamma}{\overline{\gamma}\overline{z}+\overline{\delta}}$ *with* a,b,c,d *purely imaginary and* $\delta\overline{\delta} - \gamma\overline{\gamma} = 1$.

Proof. First of all, let us define the form of the subgroup of $M\ddot{o}b$ for any circle A in $\overline{\mathbb{C}}$ as:

$$M\ddot{o}b(\mathbb{S}^1) = \{m \in M\ddot{o}b : m(A) = A\}.$$

Now, take p which is belong to $M\ddot{o}b$, such that, $p(\overline{\mathbb{R}}) = \mathbb{S}^1$. Since $n(A) = A$, then $g(\overline{\mathbb{R}}) = p \circ n \circ p^{-1}(\overline{\mathbb{R}}) = \overline{\mathbb{R}}$. Hence, we can write $m \in M\ddot{o}b$ as:

$$\begin{aligned} g &= p \circ n \circ p^{-1}. \text{ And so,} \\ h &= p \circ m \circ p^{-1} \end{aligned}$$

Which gives: $M\ddot{o}b(\mathbb{A}) = \left\{p \circ m \circ p^{-1} : m \in M\ddot{o}b(\overline{\mathbb{R}})\right\}$.

In particular, take $p(z) = \frac{z-i}{-iz+1}$ and $m(z) = \dfrac{az+b}{cz+d}$, then:
$h(z) = \frac{[a+d+(b-c)i]z+b+c+(a-d)i}{[b+c-(a-d)i]z+a+d-(b-c)i}$, where $\alpha = [a+d+(b-c)i]$ and $\beta = b+c+(a-d)i$. If $a,b,c,d \in \mathbb{R}$, then we can rewrite h as:

$$\frac{\alpha z+\beta}{\overline{\beta}z+\overline{\alpha}} \text{ with } \alpha\overline{\alpha} - \beta\overline{\beta} = 1.$$

If a,b,c,d are purely imaginary, then we can rewrite h as:

$$\frac{\alpha z+\beta}{-\overline{\beta}z-\overline{\alpha}} \text{ with } \beta\overline{\beta} - \alpha\overline{\alpha} = 1.$$

Now, take $n(z) = \dfrac{a\overline{z}+b}{c\overline{z}+d}$, then: $g(z) = \frac{[a-d-(b+c)i]\overline{z}+b-c-(a+d)i}{[-b+c-(a+d)i]\overline{z}-a+d-(b+c)i}$, where $\delta = [a-d-(b+c)i]$ and $\gamma = b-c-(a+d)i$. If $a,b,c,d \in \mathbb{R}$, then we can rewrite g as:

$$\frac{\delta\overline{z}+\gamma}{-\overline{\gamma}\overline{z}-\overline{\delta}} \text{ with } \gamma\overline{\gamma} - \delta\overline{\delta} = 1.$$

If a, b, c, d are purely imaginary, then we can rewrite g as:

$$\frac{\delta\overline{z}+\gamma}{\overline{\gamma z}+\overline{\delta}} \text{ with } \delta\overline{\delta}-\gamma\overline{\gamma}=1.$$

In conclusion, every element of $M\ddot{o}b(\mathbb{S}^1)$ has one of the following four forms:

(1) $h(z)=\dfrac{\alpha z+\beta}{\overline{\beta}z+\overline{\alpha}}$ with $a, b, c, d \in \mathbb{R}$ and $\alpha\overline{\alpha}-\beta\overline{\beta}=1$;

(2) $h(z)=\dfrac{\alpha z+\beta}{-\overline{\beta}z-\overline{\alpha}}$ with a, b, c, d purely imaginary and $-\alpha\overline{\alpha}+\beta\overline{\beta}=\beta\overline{\beta}-\alpha\overline{\alpha}=1$;

(3) $g(z)=\dfrac{\delta\overline{z}+\gamma}{-\overline{\gamma z}-\overline{\delta}}$ with $a, b, c, d \in \mathbb{R}$ and $-\delta\overline{\delta}+\gamma\overline{\gamma}=\gamma\overline{\gamma}-\delta\overline{\delta}=1$;

(4) $g(z)=\dfrac{\delta\overline{z}+\gamma}{\overline{\gamma z}+\overline{\delta}}$ with a, b, c, d purely imaginary and $\delta\overline{\delta}-\gamma\overline{\gamma}=1$. ■

Theorem 3.5.3 *Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$ which defined as:$\mathbb{D}=\{z \in \mathbb{C} : |z| < 1\}$ and let $m, n, p \in M\ddot{o}b$, If $h = p \circ m \circ p^{-1}$ and $g = p \circ n \circ p^{-1}$, then an element of $M\ddot{o}b(\mathbb{D})$ has either the form:*

(1) $h(z)=\dfrac{\alpha z+\beta}{\overline{\beta}z+\overline{\alpha}}$ with $a, b, c, d \in \mathbb{R}$ and $\alpha\overline{\alpha}-\beta\overline{\beta}=1$; or has the form

(2) $g(z)=\dfrac{\delta\overline{z}+\gamma}{\overline{\gamma z}+\overline{\delta}}$ with a, b, c, d purely imaginary and $\delta\overline{\delta}-\gamma\overline{\gamma}=1$.

Proof. First of all, take $p(z)=\frac{z-i}{-iz+1} \in M\ddot{o}b$ which takes $\overline{\mathbb{R}}$ to $\mathbb{S}^1$ and disc $\mathbb{H}$ to $\mathbb{D}$ because $p(i)=0$. As we know from the previous theorem that every element of $M\ddot{o}b(H)$ either has the form:

(1) $m(z)=\dfrac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}$ and $ad-bc=1$. So,
$h(z)=\frac{[a+d+(b-c)i]z+b+c+(a-d)i}{[b+c-(a-d)i]z+a+d-(b-c)i}$, where $\alpha=[a+d+(b-c)i]$ and $\beta=b+c+(a-d)i$, then we can rewrite h as:

$$\frac{\alpha z+\beta}{\overline{\beta}z+\overline{\alpha}} \text{ with } a, b, c, d \in \mathbb{R} \text{ and } \alpha\overline{\alpha}-\beta\overline{\beta}=1.$$

(2) Or has the form $n(z)=\dfrac{a\overline{z}+b}{c\overline{z}+d}$ where a, b, c, d are purely imaginary and $ad-bc=1$. So, $g(z)=\frac{[a-d-(b+c)i]\overline{z}+b-c-(a+d)i}{[-b+c-(a+d)i]\overline{z}-a+d-(b+c)i}$, where $\delta=[a-d-(b+c)i]$ and $\gamma=b-c-(a+d)i$, then we can rewrite g as:

$$\frac{\delta\overline{z}+\gamma}{\overline{\gamma z}+\overline{\delta}} \text{ with } a, b, c, d \text{ purely imaginary and } \delta\overline{\delta}-\gamma\overline{\gamma}=1.$$

In conclusion, every element of $M\ddot{o}b(\mathbb{D})$ has the form:

(1) $h(z) = \dfrac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}}$ with $a, b, c, d \in \mathbb{R}$ and $\alpha\overline{\alpha} - \beta\overline{\beta} = 1$, or has the form,

(2) $g(z) = \dfrac{\delta\overline{z} + \gamma}{\overline{\gamma}\overline{z} + \overline{\delta}}$ with a, b, c, d purely imaginary and $\delta\overline{\delta} - \gamma\overline{\gamma} = 1$. ■

Chapter 4: The Group of Hyperbolic Isometries Acting on the Hyperbolic Plane

In this last chapter, we turn our attention to the group action of hyperbolic isometries. In the first section we use the upper-half plane as a model of the hyperbolic plane, then we define two types of hyperbolic lines, and explain some differences between Hyperbolic Geometry and usual Euclidean Geometry, especially, the fifth Euclidean Postulate is not held in the hyperbolic plane $\mathbb{H}^2$. In addition, we describe the boundary at infinity of the hyperbolic plane, and then one can see the relationship between $\mathbb{H}^2$ and $\overline{\mathbb{R}}$ at the end of the first section.

In the second section we mainly identify the actions of groups $M\ddot{o}b(\mathbb{H}^2)$ and $\text{Möb}^+(\mathbb{H}^2)$, such that the hyperbolic plane $\mathbb{H}^2$ is an invariant subspace of Riemann sphere under these two groups. We will find an explicit expression for an element of either $\text{Möb}(\mathbb{H}^2)$ or $\text{Möb}^+(\mathbb{H}^2)$.

In the third section we define a generalized length of a piecewise path by using the complex variable, we then define the hyperbolic length and a new distance in hyperbolic plane which is called the hyperbolic distance, which is an invariant quantity under the action of the group $\text{Möb}(\mathbb{H}^2)$. At the the end of this section, we present that the hyperbolic distance between two point in $\mathbb{H}^2$ is given by the hyperbolic length of the hyperbolic line segment between these two points.

We present in the last section that the group of hyperbolic isometries acting on the hyperbolic plane is the group of general Möbius transformations preserving the hyperbolic plane: $Isom(\mathbb{H}^2, d_{\mathbb{H}^2}) = M\ddot{o}b(\mathbb{H}^2)$.

4.1 Upper Half-plane Model and Euclidean Postulates

The upper half-plane is a model of the hyperbolic plane, which is defined as the set of complex numbers with positive imaginary part:

$$\mathbb{H}^2 = \{z \in \mathbb{C} : \text{Im}(z) > 0\}.$$

We explain in this first section some differences between Hyperbolic Geometry and Euclidean Geometry, especially, the fifth Euclidean Postulate is not held in the hyperbolic plane $\mathbb{H}^2$. At the end of this section, we describe the boundary at infinity of the hyperbolic plane, and then we can see the relationship between $\mathbb{H}^2$ and $\overline{\mathbb{R}}$.

Now we have a model to manipulate, the usual notation of points and angles that $\mathbb{H}^2$ inherits from $\mathbb{C}$ are used. For example, the measure of the angle between two curves at the intersection point in $\mathbb{H}^2$ is defined as the angle between two tangent lines to the curves at that point in $\mathbb{C}$. As usual, two hyperbolic lines in $\mathbb{H}^2$ are parallel if they are disjoint.

The hyperbolic lines are defined in the following definition.

Definition 4.1.1 *Hyperbolic lines in $\mathbb{H}^2$ are the following two types:*

(1) The intersection of $\mathbb{H}^2$ with a Euclidean line perpendicular to the real axis $\mathbb{R}$, which is called a vertical hyperbolic line.

(2) The intersection of $\mathbb{H}^2$ with a Euclidean circle centered at the real axis $\mathbb{R}$, which is called a circular hyperbolic line.

We then can define open half-plane in $\mathbb{H}^2$ by using complement of a hyperbolic line in $\mathbb{H}^2$. An open half-plane in $\mathbb{H}^2$ is a component of the complement of a hyperbolic line in $\mathbb{H}^2$.

What are the main differences between Hyperbolic Geometry and Euclidean Geometry among five Euclidean Postulates?

Since there always exists a unique vertical hyperbolic line passing through any two distinct points in $\mathbb{H}^2$ with the same x-coordinate. Also there exists a unique circular hyperbolic line passing through two points in $\mathbb{H}^2$ with distinct x-coordinate. Thus the following first Euclidean Postulate is held in the hyperbolic plane $\mathbb{H}^2$.

Postulate 1**.** To draw straight line from any point to any point.

That is, given two hyperbolic points in $\mathbb{H}^2$, we can draw a hyperbolic line passing through them.

Let γ be a hyperbolic line containing the two points $p = (x_1, y_1)$ and $q = (x_2, y_2)$, if p, while moving along γ, approaches the $x-$axis, then the hyperbolic length $L_H(p, q)$ becomes indefinitely large. Thus the following second Euclidean Postulate is held in the hyperbolic plane $\mathbb{H}^2$.

Postulate 2**.** To produce a finite straight line continuously in a straight line.

Since a curve is a hyperbolic circle if and only if it is a Euclidean circle in the upper half-plane. Thus the following third Euclidean Postulate is held in the hyperbolic plane $\mathbb{H}^2$.

Postulate 3. To describe a circle with any center and radius.

Since the measure of a hyperbolic angle coincides with the measure of a Euclidean angle in the hyperbolic plane $\mathbb{H}^2$. Thus the hyperbolic angle between two hyperbolic lines is the same as Euclidean angle between the corresponding tangent lines. Thus the following fourth Euclidean Postulate is held in the hyperbolic plane $\mathbb{H}^2$.

Postulate 4. That all right angles are equal to one another.

It is well known that the fifth Euclidean Postulate is equivalent to the following Playfair's Postulate (see Figure 4.1).

Postulate 5. Given a straight line L and a point p not on L,

there is a unique straight line that is parallel to L and contains p.

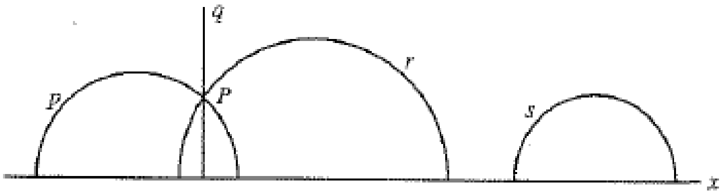

Figure 4.1: Postulate 5

However, the following theorem shows that the fifth Euclidean Postulate is not held in the hyperbolic plane $\mathbb{H}^2$.

Theorem 4.1.2 *Let l be a hyperbolic line in $\mathbb{H}^2$, and let p be a point in $\mathbb{H}^2$ (does not belong to l). Then, there exist infinitely many distinct hyperbolic lines through p that are parallel to l.*

Proof. Here, we are working with two cases.

Case 1: if l is a vertical hyperbolic line, then there exists a Euclidean line L perpendicular to $\mathbb{R}$, such that l is contained in Euclidean line L. On the other hand, there exists a Euclidean line K passing through p that is parallel to L and perpendicular to $\mathbb{R}$. So, one hyperbolic line in $\mathbb{H}^2$ passing through p and parallel to l is the intersection $\mathbb{H}^2 \cap K$.

Using the same procedure we can construct the second hyperbolic line through p and parallel to l. Take a point x on $\mathbb{R}$ between K and L, and let A be the Euclidean circle with center $\mathbb{R}$ and passes through x and p. By construction, A is disjoint from L, and so the hyperbolic line $\mathbb{H}^2 \cap A$ is disjoint from l, and $\mathbb{H}^2 \cap A$ is a second hyperbolic line through p that is parallel to l.

This construction gives infinitely many distinct hyperbolic lines passing through p and parallel to l, because there exist infinitely many points between K and L on $\mathbb{R}$.

Case 2: if l is a circular hyperbolic line, then there exists a Euclidean circle A such that l is contained in A. Let D be a Euclidean circle that is disjoint and with same centre of A and passes through p. Thus, hyperbolic line passing through p and parallel to l. The intersection $\mathbb{H}^2 \cap D$ is another hyperbolic line.

Now, we can construct the second hyperbolic line passing through p and parallel to l by using the similar procedure. Take a point x on $\mathbb{R}$ between A and D, and let E be the Euclidean circle with center on $\mathbb{R}$ and passes through x and p. By construction, E is disjoint from A and $\mathbb{H}^2 \cap E$ is a hyperbolic line passing through p and parallel to l.

This construction gives infinitely many distinct hyperbolic lines though p and parallel to l (because there exist infinitely many points between A and D on $\mathbb{R}$). ∎

At the end of this section, we describe the boundary at infinity of the hyperbolic plane, and then we can see the relationship between $\mathbb{H}^2$ and $\overline{\mathbb{R}}$.

We have defined circles in the Riemann sphere in the chapter one as either Euclidean circle in $\mathbb{C}$, or union of a Euclidean line in $\mathbb{C}$ with $\{\infty\}$.

Now, let us see the complements of circles in $\overline{\mathbb{C}}$, for examples, the unit circles S^1 and the circle $\overline{\mathbb{R}}$ in $\mathbb{C}$.

It is clear that the components $\overline{\mathbb{C}} - S^1$ has two components: One is the unit Euclidean open disc $U_1(0)$ in $\mathbb{C}$, and the other $U_1(\infty)$ can be regarded as open disc in $\overline{\mathbb{C}}$ determined by S^1.

And the component $\overline{\mathbb{C}} - \overline{\mathbb{R}}$ has two components: One is the upper half-plane $\mathbb{H}^2$, and the other is the lower half-plane

$$\mathbb{H}^2_- = \{z \in \mathbb{C}, \mathrm{Im}(z) < 0\},$$

both can be regarded as open discs in $\overline{\mathbb{C}}$ determined by $\overline{\mathbb{R}}$.

Thus, it is not difficult to see that every disc in $\overline{\mathbb{C}}$ determines a unique circle, but every circle in $\overline{\mathbb{C}}$ determines two disjoint discs in $\overline{\mathbb{C}}$.

From now, we will focus on one special disc in $\overline{\mathbb{C}}$: $\mathbb{H}^2$ determined by the circle $\overline{\mathbb{R}}$ in $\overline{\mathbb{C}}$. And we call $\overline{\mathbb{R}}$ as the boundary at infinity of $\mathbb{H}^2$, and we call the points of $\overline{\mathbb{R}}$ as the points at infinity of $\mathbb{H}^2$.

Furthermore, the boundary at infinity of a subset X of $\mathbb{H}^2$ is the intersection $\overline{X} \cap \overline{\mathbb{R}}$. For example, for a circular hyperbolic line A, the boundary at infinity of A is the pair of the two points contained in the intersection $A \cap \overline{\mathbb{R}}$.

Proposition 4.1.3 *Let p a point of $\mathbb{H}^2$ and q be a point of $\overline{\mathbb{R}}$. Then, there is a unique hyperbolic line in $\mathbb{H}^2$ determined by p and q.*

Proof. First, suppose $q = \infty$, so there are infinitely many hyperbolic lines passing through q, but only one line of these lines that contains q in its boundary at infinity, since no hyperbolic line contained in a Euclidean circle contains ∞ in its boundary at infinity, this hyperbolic lines contained in the Euclidean line $\{ z \in \mathbb{C}$, $\mathrm{Re}(z) = \mathrm{Re}(p)\}$.

Now, suppose $q \neq \infty$ and $\mathrm{Re}(q) = \mathrm{Re}(p)$. Then hyperbolic lines contained in the Euclidean line $\{ z \in \mathbb{C}$, $\mathrm{Re}(z) = \mathrm{Re}(p)\}$ is the unique hyperbolic line through p that contains q in its boundary at infinity.

Second, suppose $q \neq \infty$ and $\mathrm{Re}(q) \neq \mathrm{Re}(p)$.By using the perpendicular bisector of the Euclidean line segment that join p with q, we can find the unique Euclidean circle centered on the real axis $\mathbb{R}$ that passes through p and q. Intersecting this circle with $\mathbb{H}^2$ yields the unique hyperbolic line determined by p and q. ■

4.2 The Group Action on the Hyperbolic Plane

In this section, we mainly identify the actions of groups $M\ddot{o}b(\mathbb{H}^2)$ and Möb$^+(\mathbb{H}^2)$ below, such that the hyperbolic plane $\mathbb{H}^2$ is an invariant subspace of Riemann sphere under these two groups. We will find an explicit expression for an element of either Möb$(\mathbb{H}^2)$ or Möb$^+(\mathbb{H}^2)$

$$\begin{aligned} M\ddot{o}b(\mathbb{H}^2) &= \left\{ m \in M\ddot{o}b(\overline{\mathbb{C}}) : m(\mathbb{H}^2) = \mathbb{H}^2 \right\}, \\ M\ddot{o}b^+(\mathbb{H}^2) &= \left\{ m \in M\ddot{o}b^+(\overline{\mathbb{C}}) : m(\mathbb{H}^2) = \mathbb{H}^2 \right\}. \end{aligned}$$

It is clear that $M\ddot{o}b^+(\mathbb{H}^2) = M\ddot{o}b(\mathbb{H}^2) \cap M\ddot{o}b^+(\overline{\mathbb{C}})$.

We will find an explicit expression for an element of either Möb$(\mathbb{H}^2)$ or Möb$^+(\mathbb{H}^2)$.

First, the following theorem states that every element of Möb$(\mathbb{H}^2)$ takes hyperbolic lines in $\mathbb{H}^2$ to hyperbolic lines in $\mathbb{H}^2$. That is, hyperbolic lines are invariant under the group $M\ddot{o}b(\mathbb{H}^2)$.

Theorem 4.2.1 *Every element of $M\ddot{o}b(\mathbb{H}^2)$ takes hyperbolic lines in $\mathbb{H}^2$ to hyperbolic lines in $\mathbb{H}^2$.*

Proof. The elements of Möb($\mathbb{H}^2$) preserve angles between circles in $\overline{\mathbb{C}}$, and every hyperbolic line in $\mathbb{H}^2$ is: the intersection of $\mathbb{H}^2$ with a circle in $\overline{\mathbb{C}}$ perpendicular to $\overline{\mathbb{R}}$. Also, every element of $M\ddot{o}b$ takes circles in $\overline{\mathbb{C}}$ to circles in $\overline{\mathbb{C}}$. ■

Notice that every element of $M\ddot{o}b(\mathbb{H}^2)$ can be written either as $m(z) = \dfrac{az+b}{cz+d}$ or as $n(z) = \dfrac{a\overline{z}+b}{c\overline{z}+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. By using using the composition of $C(z) = \overline{z}$ and $n(z)$ such that $n \circ C(z) = n(\overline{z}) = \dfrac{az+b}{cz+d} \in M\ddot{o}b^+(\mathbb{H}^2)$, so we can consider only the former case $m(z) = \dfrac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. We are determining the conditions imposed on a, b, c, and d by requiring that $m(\mathbb{H}^2) = \mathbb{H}^2$. In fact, we find an explicit expression for an element of either Möb($\mathbb{H}^2$) or Möb$^+$($\mathbb{H}^2$) in the following theorem and corollary.

Theorem 4.2.2 *Every element of $M\ddot{o}b$ ($\mathbb{H}^2$) has the following two forms:*

1. $m(z) = \dfrac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.
2. $n(z) = \dfrac{a\overline{z}+b}{c\overline{z}+d}$ where a, b, c, d are purely imaginary and $ad - bc = 1$.

Proof. Now, we need to use the above formula, to get the form of an element of $M\ddot{o}b(H)$.

By using the result that say: an element of $M\ddot{o}b(\overline{\mathbb{R}})$ is an element of $M\ddot{o}b(H)$ if and only if the imaginary part of $m(i)$ is positive. For that, let us check the value of Im$(m(i))$ for each possible forms of an element of $M\ddot{o}b(\overline{\mathbb{R}})$:

(1) $m(z) = \dfrac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Then:

$$
\begin{aligned}
\mathrm{Im}(m(i)) &= \mathrm{Im}\left(\frac{ai+b}{ci+d}\right) \\
&= \mathrm{Im}\left[\left(\frac{ai+b}{ci+d}\right)\left(\frac{-ci+d}{-ci+d}\right)\right] \\
&= \frac{ad-bc}{c^2+d^2} \\
&= \frac{1}{c^2+d^2} > 0.
\end{aligned}
$$

So, $m(z) \in M\ddot{o}b(H)$.

(2) $n(z) = \dfrac{a\overline{z}+b}{c\overline{z}+d}$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Then:

$$\begin{aligned} \mathrm{Im}(m(i)) &= \mathrm{Im}\left(\frac{-ai+b}{-ci+d}\right) \\ &= \mathrm{Im}\left[\left(\frac{ai+b}{ci+d}\right)\left(\frac{ci+d}{ci+d}\right)\right] \\ &= \frac{-ad+bc}{c^2+d^2} \\ &= \frac{-1}{c^2+d^2} < 0. \end{aligned}$$

So, $m(z) \notin M\ddot{o}b(H)$.

(3) $m(z) = \dfrac{az+b}{cz+d}$, where a, b, c, d pure imaginary and $ad - bc = 1$. First, rewrite the coefficient as:

$$a = \alpha i, b = \beta i, c = \gamma i, d = \delta i \text{ and } \alpha\delta - \beta\gamma = -1$$

Then:

$$\begin{aligned} \mathrm{Im}(m(i)) &= \mathrm{Im}\left(\frac{ai+b}{ci+d}\right) \\ &= \mathrm{Im}\left(\frac{-\alpha+\beta i}{-\gamma+\delta i}\right) \\ &= \mathrm{Im}\left[\left(\frac{-\alpha+\beta i}{-\gamma+\delta i}\right)\left(\frac{-\gamma-\delta i}{-\gamma-\delta i}\right)\right] \\ &= \frac{\alpha\delta-\beta\gamma}{\gamma^2+\delta^2} \\ &= \frac{-1}{\gamma^2+\delta^2} < 0. \end{aligned}$$

So, $m(z) \notin M\ddot{o}b(H)$.

(4) $n(z) = \dfrac{a\overline{z}+b}{c\overline{z}+d}$, where a, b, c, d pure imaginary and $ad - bc = 1$. First, rewrite the coefficient as:

$$a = \alpha i, b = \beta i, c = \gamma i, d = \delta i \text{ and } \alpha\delta - \beta\gamma = -1$$

Then:

$$\begin{aligned}
\mathrm{Im}(m(i)) &= \mathrm{Im}\left(\frac{-ai+b}{-ci+d}\right) \\
&= \mathrm{Im}\left(\frac{\alpha+\beta i}{\gamma+\delta i}\right) \\
&= \mathrm{Im}\left[\left(\frac{\alpha+\beta i}{\gamma+\delta i}\right)\left(\frac{\gamma-\delta i}{\gamma-\delta i}\right)\right] \\
&= \frac{-\alpha\delta+\beta\gamma}{\gamma^2+\delta^2} \\
&= \frac{1}{\gamma^2+\delta^2} < 0.
\end{aligned}$$

So, $m(z) \in M\ddot{o}b(H)$.

In conclusion, every element of $M\ddot{o}b(H)$ either has the form:

(1) $m(z) = \dfrac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$, or has the form;

(2) $n(z) = \dfrac{a\overline{z}+b}{c\overline{z}+d}$ where a, b, c, d are purely imaginary and $ad - bc = 1$. ∎

Since $M\ddot{o}b^+(\mathbb{H}^2) = M\ddot{o}b(\mathbb{H}^2) \cap M\ddot{o}b^+(\overline{\mathbb{C}})$, we have the following corollary 4.2.3

Corollary 4.2.3 *Every element of* $M\ddot{o}b^+(\mathbb{H}^2)$ *has the form* $m(z) = \dfrac{az+b}{cz+d}$ *where* $a, b, c, d \in \mathbb{R}$ *and* $ad - bc = 1$*. i.e.,*

$$M\ddot{o}b^+(\mathbb{H}^2) = \left\{\frac{az+b}{cz+d} : a, b, c, d \in \mathbb{R}, \ \textit{and } ad - bc = 1\right\}.$$

4.3 Hyperbolic Length and Hyperbolic Distance

In this third section we define a generalized length of a piecewise path by using the complex variable, we then define the hyperbolic length and a new distance in hyperbolic plane which is called the hyperbolic distance, which is an invariant quantity under the action of the group Möb($\mathbb{H}^2$).

A path in the plane $\mathbb{C}$ is a function $f : [a, b] \to \mathbb{C}$ that is continuous on $[a, b]$ and analytic (differentiable) on (a, b) with continuous derivative in the same time. Now, we can define the more general path, that is a piecewise.

Definition 4.3.1 *A path* $f : [a, b] \to \mathbb{C}$ *is a piecewise if* f *is continuous and if there is a partition of the interval* $[a, b]$ *into subintervals:*

$$[a, a_1], [a_1, a_2], ..., [a_n, b],$$

so that, f is a path when restricted to each subinterval $[a_k, a_{k+1}]$.

A natural example of a piecewise path that is not a path comes from considering absolute value.

Example 4.3.2 *Let* $f : [-1, 1] \to \mathbb{C}$ *defined by* $f(t) = t + i\,|t|$. *Then this function is not a path because* $|t|$ *is not differentiable at* $t = 0$.

However, if we consider the subinterval $[-1, 0]$, *then* $|t| = -t$, *so* $f(t) = t - it$ *is a path on* $[-1, 0]$. *Similarly,* $f(t) = t + it$ *is a path on the other subinterval* $[0, 1]$.

Thus, f *is a piecewise path on* $[-1, 1]$.

As we know from Calculus, the Euclidean length of a path $f : [a, b] \to \mathbb{C}$ is given by the following path integral:

$$length(f) = \int_f ds = \int_a^b \sqrt{(x'(t))^2 + \ (y'(t))^2}\, dt,$$

where $ds = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$ is the element of arc-length in $\mathbb{C}$.

Furthermore, we can express this formula in terms of the complex variable z in the following lemma. From now on, we write path integrals in complex form, because it is more easier to work with.

Lemma 4.3.3 *Let* f *be a piecewise path* $f : [a, b] \to \mathbb{C}$, *then the Euclidean length of the path* f *can be expressed as the following complex form:*

$$length(f) = \int_f |dz|\,, \ \ where \ \ |dz| = |f\,'(t)|\, dt.$$

Now we can define a generalized element of arc-length and a generalized length of a piecewise path.

Definition 4.3.4 *Let $\rho : \mathbb{C} \to \mathbb{R}^+$ be a non-negative continuous function, and let $f : [a, b] \to \mathbb{C}$ be a path. Then*

$$\rho(z)\,|dz\,| = \rho(f\ (t))\,|f\ '(t)|\,dt$$

is called a generalized element of arc-length.

In addition, we define the generalized length of f with respect to a generalized element of arc-length below, denoted by $length_\rho(f\ )$.

$$length_\rho(f\ ) = \int_f \rho(z)\,|dz| = \int_a^b \rho(f\ (t))\,|f\ '(t)|\,dt.$$

In particular, let $f : [a, b] \to \mathbb{H}^2$ be a piecewise path in the hyperbolic plane $\mathbb{H}^2$. Let $\rho(z) = \frac{1}{\mathrm{Im}(z)}$, then ρ is anon-negative continuous nonzero function on $\mathbb{H}^2 \subseteq \mathbb{C}$. Thus we can define a hyperbolic length in the following definition. Later we will show that this hyperbolic length is an invariant quantity under the action of the group Möb($\mathbb{H}^2$) in Proposition 4.3.10.

Definition 4.3.5 *For a piecewise path $f : [a, b] \to \mathbb{H}^2$, define the hyperbolic length of f by:*

$$length_{\mathbb{H}^2}(f) = \int_f \frac{1}{\mathrm{Im}(z)}\,|dz| = \int_a^b \frac{1}{\mathrm{Im}(f(t))}\,|f\ '(t)|\,dt.$$

Example 4.3.6 *Let $f : [0, 1] \to \mathbb{H}^2$ be a piecewise path in the hyperbolic plane $\mathbb{H}^2$ given by $f(t) = t + i(t + 1)$. Then the hyperbolic length of the piecewise path f is:*

$$\begin{aligned} length_{\mathbb{H}^2}(f) &= \int_f \frac{1}{\mathrm{Im}(z)}\,|dz| = \int_a^b \frac{1}{\mathrm{Im}(f(t))}\,|f\ '(t)|\,dt \\ &= \int_0^1 \frac{\sqrt{2}}{1+t}dt = \left[\sqrt{2}\ln(1+t)\right]_0^1 = \sqrt{2}\ln 2. \end{aligned}$$

It is time to define the hyperbolic distance in the hyperbolic plane $\mathbb{H}^2$. Let's recall the Euclidean distance between two points $x(x_1, y_1)$ and $y(x_2, y_2)$:

$$d(x, y) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Notice that the Euclidean distance between two points x and y is the shortest Euclidean length of all piecewise paths between two points x and y in Euclidean plane $\mathbb{R}^2$.

Thus, the hyperbolic distance between two points x and y should be the shortest hyperbolic length of all piecewise paths between x and y in the hyperbolic plane $\mathbb{H}^2$.

Let $\Gamma[x, y]$ be the set of all piecewise paths from x to y in the hyperbolic plane $\mathbb{H}^2$, and each piecewise path $f : [a, b] \to \mathbb{H}^2$ with $f(a) = x$ and $f(b) = y$. One see that $\Gamma[x, y]$ is not empty, because we can parametrize the hyperbolic line segment joining x to y by a piecewise path, and every piecewise path f in $\Gamma[x, y]$ has finite hyperbolic length $length_{\mathbb{H}^2}(f)$.

Definition 4.3.7 *Define the hyperbolic distance between two points x and y in $\mathbb{H}^2$ is the shortest hyperbolic length of all paths between x and y in the hyperbolic plane $\mathbb{H}^2$, denoted by $d_{\mathbb{H}^2}(x, y)$.*

$$d_{\mathbb{H}^2}(x, y) = inf\{length_{\mathbb{H}^2}(f) \mid f \in \Gamma[x, y]\}.$$

It is clear that the hyperbolic distance is a function defined as $d_{\mathbb{H}^2} : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R}$, and it is a metric on $\mathbb{H}^2$. Let us recall the definition of a metric on a set.

Definition 4.3.8 *A metric on a set X is a function $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ such that $\forall$ $x, y, z \in X$:*

1. *$d(x, y) \geqslant 0$, $\forall x, y \in X$ (nonnegative) and $d_{\mathbb{H}^2}(x, y) = 0$ if and only if $x = y$ (non degeneracy).*

2. *$d(x, y) = d(y, x), \forall x, y \in X$ (symmetry).*

3. *$d(x, z) \leqslant d(x, y) + d(y, z) \forall x, y, z \in X$ (triangle inequality)[14].*

Theorem 4.3.9 *Let* $d_{\mathbb{H}^2} : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R}$ *is a metric on* $\mathbb{H}^2$ *if it satisfies the following three conditions:*

1. $d_{\mathbb{H}^2}(x, y) \geqslant 0, \forall x, y \in \mathbb{H}^2$, *and* $d_{\mathbb{H}^2}(x, y) = 0$ *if and only if* $x = y$.

2. $d_{\mathbb{H}^2}(x, y) = d_{\mathbb{H}^2}(y, x), \forall x, y \in \mathbb{H}^2$.

3. $d_{\mathbb{H}^2}(x, z) \leqslant d_{\mathbb{H}^2}(x, y) + d_{\mathbb{H}^2}(y, z) \forall x, y, z \in \mathbb{H}^2$ *(the triangle inequality).*

At the end of this section, we present that the hyperbolic distance between two point in $\mathbb{H}^2$ is given by the hyperbolic length of the hyperbolic line segment between these two points.

Theorem 4.3.10 *For every element* $\gamma \in$ *Möb*$(\mathbb{H}^2)$ *and* $\forall (x, y) \in$ *points of* $\mathbb{H}^2$, *we have* $d_{\mathbb{H}^2}(x, y) = d_{\mathbb{H}^2}(\gamma(x), \gamma(y))$ *[1].*

Proof. We need to proof that $d_{\mathbb{H}^2}(x, y) = d_{\mathbb{H}^2}(\gamma(x), \gamma(y))$.

Now, Take a path $f : [a, b] \to \mathbb{H}^2$ in $\Gamma[x, y]$, so that $f(a) = x$ and $f(b) = y$. So, $\gamma \circ f(a) = \gamma(f(a)) = \gamma(x)$ and $\gamma \circ f(b) = \gamma(f(b)) = \gamma(y)$, in conclusion $\gamma \circ f$ lies in $\Gamma[\gamma(x), \gamma(y)]$. we have $length_{\mathbb{H}^2}(\gamma \circ f) = length_{\mathbb{H}^2}(f)$. And it is important here to note that $\{\gamma \circ f \mid f \in \Gamma[x, y]\} \subseteq \Gamma(\gamma(x), \gamma(y))$.

As $length_{\mathbb{H}^2}(f)$ is stable (invariant) under the action of Möb$(\mathbb{H}^2)$, $\forall f \in \Gamma[x, y]$, and so:

$$\begin{aligned} d_{\mathbb{H}^2}(\gamma(x), \gamma(y)) &= inf\{length_{\mathbb{H}^2}(g) \mid g \in \Gamma[\gamma(x), \gamma(y)] \\ &\leq inf\{length_{\mathbb{H}^2}(\gamma \circ f) \mid f \in \Gamma[x, y]\} \\ &\leq inf\{length_{\mathbb{H}^2}(f) \mid f \in \Gamma[x, y]\} \\ &= d_{\mathbb{H}^2}(x, y). \end{aligned}$$

Again note that $\{\gamma^{-1} \circ g \mid g \in \Gamma[\gamma(x), \gamma(y)] \subseteq \Gamma[x, y]$. As γ is invertible and

$\gamma^{-1}$is an element of Möb($\mathbb{H}^2$) so:

$$\begin{aligned} d_{\mathbb{H}^2}(x,y) &= inf\{length_{\mathbb{H}^2}(f) \mid f \in \Gamma[x,y]\} \\ &\leq inf\{length_{\mathbb{H}^2}(\gamma^{-1} \circ g) \mid g \in \Gamma[\gamma(x),\gamma(y)]\} \\ &\leq inf\{length_{\mathbb{H}^2}(g) \mid g \in \Gamma[\gamma(x),\gamma(y)]\} \\ &= d_{\mathbb{H}^2}(\gamma(x),\gamma(y)). \end{aligned}$$

Finally, our result is $d_{\mathbb{H}^2}(x,y) = d_{\mathbb{H}^2}(\gamma(x),\gamma(y))$.This completes the proof of the proposition. ■

Theorem 4.3.11 *$d_{\mathbb{H}^2}(x,y) = length(f)$, where f is the hyperbolic segment between x and y.*

Proof. As the hyperbolic length of a path is invariant under the action of Möb($\mathbb{H}^2$), the hyperbolic distance is also invariant under the action of Möb($\mathbb{H}^2$) as we saw in the previous proposition 4.3.10. ■

4.4 Group Action of Hyperbolic Isometries

In the last section, we define isometries in the hyperbolic plane, and explain some important properties of isometries in hyperbolic plane.

Given two metric spaces X and Y. A bijection $\varphi : X \to Y$ is called an isometry if it preserves the distances. And the inverse of an isometry and the composite of two isometries are also an isometry [14].

Moreover, it's not difficult to see that if f is an isomey then f is injective and continuous. However, an isometry is, in general, not a homeomorphism.

Example 4.4.1 *Let* $g(n,m) = \begin{cases} 0 & if\ n = m, \\ 1 & if\ n \neq m. \end{cases}$ *This function g gives a metric on Z that is different from the usual metric on Z. The function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(m) = 2m$ is an isometry but is not surjective, for that it is not a homeomorphism.*

On the other hand, we can say that an isometry is a homeomorphism onto it is image. If $f : (X, d_X) \to (Y, d_Y)$ is an isometry then f is a homeomorphism between X and $f(X) \subseteq Y$.

In fact, we have:

$$d(z,w) = d(f(f^{-1}(z)), f(f^{-1}(w))) = d(f^{-1}(z), f^{-1}(w)), \text{ for all } z \text{ and } w \in Y.$$

Hence, f^{-1} is also an isometry, and so it is continuous.

In the Euclidean plane $\mathbb{R}^2$, an isometry is a homomorphism preserving Euclidean distance, and we know that a Euclidean isometry must be one of the actions: a translation, a rotation, a reflection, and a glide reflection.

Definition 4.4.2 *For hyperbolic plane $\mathbb{H}^2$ and the metric $d_{\mathbb{H}^2} = \frac{|dz|}{\operatorname{Im} z}$, a homeomorphism $f : \mathbb{H}^2 \to \mathbb{H}^2$ is called a hyperbolic isometry if it preserves the distance:*

$$d_{\mathbb{H}^2}(x,y) = d_{\mathbb{H}^2}(f(x), f(y)), \text{ for all } x \text{ and } y \in \mathbb{H}^2.$$

It is clear that we have the following proposition.

Proposition 4.4.3 *The set of all isometrics of $(\mathbb{H}^2, d_{\mathbb{H}^2})$ is a group under the operation of composition, denoted by $Isom(\mathbb{H}^2, d_{\mathbb{H}^2})$.*

The Theorem 4.3.10 states that

$$M\ddot{o}b(\mathbb{H}^2) \subseteq Isom(\mathbb{H}^2, d_{\mathbb{H}^2}).$$

Furthermore, we have the following theorem.

Theorem 4.4.4 $Isom(\mathbb{H}^2, d_{\mathbb{H}^2}) = M\ddot{o}b(\mathbb{H}^2)$.

Proof. Let $M\ddot{o}b_-(\mathbb{H}^2) = \left\{ \dfrac{a\overline{z}+b}{c\overline{z}+d} : a,b,c,d \text{ are purely imaginary, and } ad-bc=1 \right\}$. Equivalently, $M\ddot{o}b_-(\mathbb{H}^2) = \left\{ \dfrac{a\overline{z}+b}{c\overline{z}+d} : a,b,c,d \in \mathbb{R}, \text{ and } ad-bc=-1 \right\}$.

$$M\ddot{o}b(\mathbb{H}^2) = M\ddot{o}b^+(\mathbb{H}^2) \cup M\ddot{o}b_-(\mathbb{H}^2).$$

Here $M\ddot{o}b^{+}(\mathbb{H}^2) = \left\{ \dfrac{az+b}{cz+d} : a, b, c, d \in \mathbb{R}, \text{ and } ad - bc = 1 \right\}$, and Chang [5] showed that the set of orientating-preserving isometries is $M\ddot{o}b^{+}(\mathbb{H}^2)$. If an isometry f is not orientating-preserving, then f is an anti-orientating-preserving.

Let $C(z) = \overline{z}$, then $f \circ C$ is orientating-preserving, and $f \circ C \in M\ddot{o}b^{+}(\mathbb{H}^2)$. It follows that $f = C \circ f \circ C \in M\ddot{o}b_{-}(\mathbb{H}^2)$.

Thus, $Isom(\mathbb{H}^2, d_H) \subseteq M\ddot{o}b(\mathbb{H}^2)$. From Theorem 4.3.10 $M\ddot{o}b(\mathbb{H}^2) \subseteq Isom(\mathbb{H}^2, d_{\mathbb{H}^2})$.

That is $Isom(\mathbb{H}^2, d_H) = M\ddot{o}b(\mathbb{H}^2)$. ■

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