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Numerical Solution for Solving Two-Points Boundary Value Problems Using Orthogonal Boubaker Polynomials

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NUMERICAL SOLUTION FOR SOLVING TWO-POINTS BOUNDARY VALUE PROBLEMS USING ORTHOGONAL BOUBAKER POLYNOMIALS

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الحل العددي لمسائل القيم الحدودية في نقطتين بإستخدام متعددات حدود بوبكر المتعامدة

الملخص

في هذا البحث, تم تقديم تقنية جديدة لحل مسائل القيمة الحدودية في نقطتين, أُستخدمت الدالة التعامدية لمتعددة حدود بوبكر مع طريقة كالركين لتحويل المسألة الحدودية الى نظام من المعادلات الجبرية الخطية مع العوامل المجهولة والتي يمكن حلها بسهولة للحصول على الحل التقريبي . كذلك تمت إضافة بعض الامثلة العددية مع الرسوم التوضيحية ومقارنة النتائج بالنسبة الى الحل الحقيقي لتوضيح كفاءة وقابلية هذه الطريقة .

Abstract

In this paper, a new technique for solving boundary value problems (BVPs) is introduced. An orthogonal function for Boubaker polynomial was utilizedand by the aid of Galerkin method the BVP was transformed to a system of linear algebraic equations with unknown coefficients, which can be easily solved to find the approximate result. Some numerical examples were added with illustrations, comparing their results with the exact to show the efficiency and the applicability of the method.

Key words-Orthogonal Boubaker polynomials, boundary value problem, Galerkin method.

1. Introduction

Boundary value problems represent a wide fielding the branch of numerical studies related with problems in Applied Physics and Engineering.

Let $I = (a, b) \subseteq P$ be an interval and p, q,r be Continuous functions then Dirichlet First kind Two-point boundary value problem is given by [1]

$$\left.\begin{array}{l} u'' + p(t)u' + q(t)u = r(t) a < t < b \\ \quad \text{and boundaryconditions} \\ \quad u(a) = \gamma_1 \ , \quad u(b) = \gamma_2 \end{array}\right\} \ldots (1)$$

A lot of researches deal with this kind of differential equations, Rahmat Ali Khan proposed generalized quasilinearization technique for solving a second order differential equation with separated boundary conditions [2], Anwar Ja'afar Mohamad-Jawad gave four numerical methods for solving second order on-linear BVPs [3], Naseif J. Al-Jawari et al. studied the controllability of nonlinear boundary value control systems using functional analysis proceeding [4], Getin M. et al, gave a method based on Lucas polynomials for solving high-order linear BVP [5].

Obviously, in recent decades there is a large interesting using different kinds of orthogonal functions like orthogonal polynomials and wavelets for solving linear and nonlinear problems in physics and engineering, many researchers worked in this field. Dr. Suha N. Shihab et al.,used Legendre wavelets method for solving BVPs [6],Olagunju A. and Joseph Folake L., utilized third kind Chebyshev

Polynomials $V_r(x)$ in collocation methods for BVPs For solving BVPs [7].
Siddu Chamabasappa and Kumbinira saiahS. introduced a new generalized operational matrix of integration to solve nonlinear singular BVPs using Hermite wavelets [8], Shirala shettiS. And Srinivasa K. used Hermit wavelets method for solving linear and nonlinear singular initial and boundary value problems [9].
Boubaker polynomial is first utilized for solving heat equation in physical applications then many researches concerning this polynomial have taken place in different proceedings. Since their first appearance it presents a new powerful tool for solving different kinds of differential equations and optimal control problems [10-14].
In this paper, the orthogonal Boubaker function was introduced and used with Galerkin method to solve some boundary value problems. The paper is arranged as follows, the next section gives a fundamental idea about orthogonal Boubaker polynomials and the third section is related with the proposed method for solving BVP's. At last we added some numerical examples presenting different kinds of BVP's.

2. Orthogonal Boubaker polynomials

A set S of Polynomials of an inner product space is called orthogonal if $< f, g >= 0$ for every $f \neq g$ **($f, g \in S$).**
Boubaker polynomial *Bo* presented as in [10] as follows

$$Bo_k(t) = \sum_{r=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{(k-4r)}{(k-r)} \binom{k-r}{r} (-1)^r t^{k-2r} \qquad , k = 0,1,2,...$$

then $Bo_0(t) = 1$, $Bo_1(t) = t$, $Bo_2(t) = t^2+2$, ...

Since Boubaker polynomials are not orthogonal,
the Gram-Schmidt method has been applied to find the orthogonal Boubaker polynomials**,** the first six orthogonal Boubaker polynomials denoted by $B_m(t)$ were found to be[15].

$B_0(t) = 1$,

$B_1(t) = \frac{1}{2}(2t-1)$,

$B_2(t) = \frac{1}{6}(6t^2-6t+1)$,

$B_3(t) = \frac{1}{20}(20t^3-30t^2+12t-1)$,

$B_4(t) = \frac{1}{70}(70\,t^4-140t^3+90t^2-20t+1)$,

$B_5(t) = \frac{1}{252}(252t^5-630\,t^4+560t^3-210t^2+30t-1)$,

$B_6(t) = \frac{1}{924}(924t^6-2772t^5+3150\,t^4-1680t^3+420t^2-42t+1)$.

3.The method

The process in steps is as follows
- Assuming the orthogonal Boubaker polynomial with certain coefficients $a_i's$for the unknown function $u(t)$ defined on [0,1] as follows

$$u(t) \approx \sum_{i=0}^{m} a_i Bo_i(t)\,, i = 1,2,\dots,m \quad ...(2)$$

$B(t) = [Bo_0(t) Bo_1(t) Bo_2(t) Bo_3(t) \dots Bo_m(t)]^T$ represents orthogonal Boubaker polynomials with $a_i's$ as the required coefficients.

- Substituting eq.(2) in (1) and extracting $R(t)$ which represents the residual

$$R(t) = 0 \qquad ...(3)$$

-Now applying Galerkin method to eq.(3). The weight functions were chosen to be the same as the orthogonal Boubaker terms (since it represent linear independence) and integrating their product with $R(t)$ in [0,1] , then we get a system of algebraic equations with unknown coefficients $a_i's$.
-Substituting for some values of $t$to find the values of the unknown coefficients.
Remark:Since the same terms for the weights and trial functions were used, the results would be the same as if Rayleigh-Ritz method was used [16].

4.Numerical Examples:

Example1: Consider the second-order boundary value differential equation:

$$(t^2+1)u^{(2)}(t) + \acute{u}(t) = 1 \qquad ...(4)$$

with boundary conditions:
$u(0) = 0,\ u(1) = 1$
The exact solution for this problem is $u(t) = t$
The steps of the Process would be:
-Assuming the orthogonal Boubaker polynomial with coefficients $a_i's(i=0,\dots,4)$ for the unknown function $u(t)$ defined on [0,1] as follows
$u(t) \approx \sum_{i=0}^{4} a_i Bo_i(t)\,, i = 1,2,\dots,4$ …(5)

- Substituting Eq.(5) in (4) and extracting $R(t)$ which represents the residual

$$R(t) = 0 \qquad ...(6)$$

-Now applying Galerkin method to Eq.(6).

The weight functions were chosen to be the same as the orthogonal Boubaker terms and integrating their product with $R(t)$ in [0,1] , then we get a system of algebraic equations with unknown coefficients $a_i{}'s$.
Using Matlab the required coefficients would be

$a_0 = 0.5$
$a_1=1$
$a_2 = 0$
$a_3 = 0$
$a_4 = 0$

Table (1) shows the numerical results for this example with k=1, M=4 are compared with exact solution, graphically illustrated in Fig.1.

Table (1) Numerical solution of Example1

t	$u_e(t)$	$u_{appr}(t)$	error
0	0	0	0
0.1	0.1	0.1	0
0.2	0.2	0.2	0
0.3	0.3	0.3	0
0.4	0.4	0.4	0
0.5	0.5	0.5	0
0.6	0.6	0.6	0
0.7	0.7	0.7	0
0.8	0.8	0.8	0
0.9	0.9	0.9	0
1.0	1.0	1.0	0

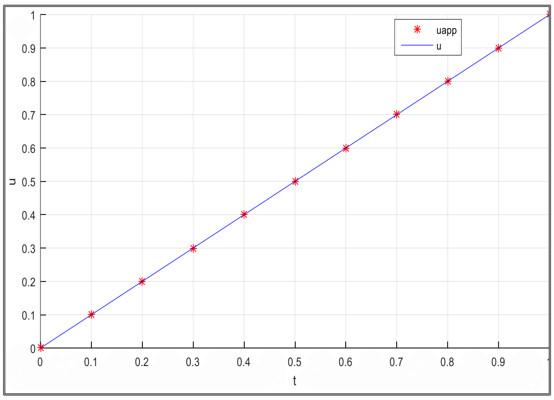


Figure (1)

Example2: Consider the third-order boundary value differential equation:

$$t^2u^{(3)}(t) + u^{(2)}(t) = 2$$

with boundary conditions

$u(0) = 0 , u(1) = 1$

The exact solution for this problem is $u(t) = t^2$

Table (2) shows the numerical results for this example with k =1, M = 4 are compared with exact solution, graphically illustrated in Fig.2

The required coefficients are

$a_0 = 0.333333333333333$

$a_1= 1$

$a_2 =1$

$a_3 = 0$

$a_4= 0$

Table (2) Numerical solution of Example2

t	$u_e(t)$	$u_{appr}(t)$	*error*
0	0	0	0
0.1	0.01	0.01	0
0.2	0.04	0.04	0
0.3	0.09	0.09	0
0.4	0.16	0.16	0
0.5	0.25	0.25	0
0.6	0.36	0.36	0
0.7	0.49	0.49	0
0.8	0.64	0.64	0
0.9	0.81	0.81	0
1.0	1.0	1.0	0

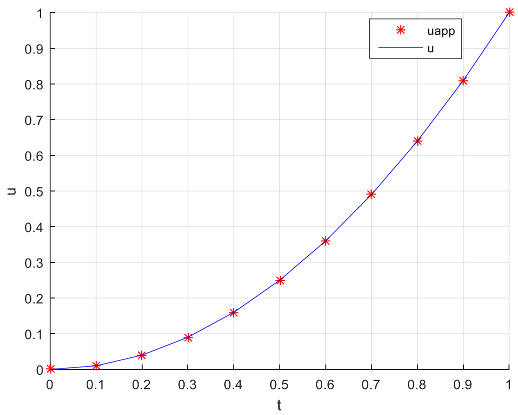


Figure (2)

Example3: Consider the second-order boundary value differential equation:

$$12t^2u^{(2)}(t) + 24t\acute{u}(t) = -30t^4 + 204t^3 - 351t^2 + 110t$$

with boundary conditions $u(0)=1$, $u(1)=2$

The exact solution for this problem is

$$u(t)=\frac{1}{24}(-3t^4+34t^3-117t^2+110t+24)$$

Table (3) shows the numerical results for this example with k=1, M=4 are compared with exact solution, graphically illustrated in Fig.3

The required coefficients are:

a_0 = 1.995833333333333

a_1= 0.8833333333333333

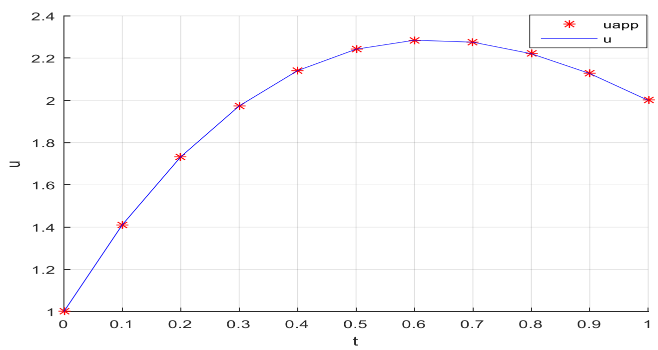
a_2= -2.96428571428571

a_3= 1.166666666666667

a_4 = -0.124999999999999

Table (3) Numerical solution of Example3

t	$u_e(t)$	$u_{appr}(t)$	*error*
0	1.0000000000	1.00000000000	0.00
0.1	1.4109875000	1.41098750000	0.0
0.2	1.7328000000	1.73280000000	0.0
0.3	1.9734875000	1.97348750000	0.0
0.4	2.1408000000	2.14080000000	1x 10^{-14}
0.5	2.2421875000	2.24218750000	0.0
0.6	2.2848000000	2.28480000000	0.0
0.7	2.2754875000	2.27548749999	1x 10^{-14}
0.8	2.2208000000	2.22080000000	0.0
0.9	2.1269874999	2.12698749999	0.0
1.0	2.0000000000	2.00000000000	0.0



Figure(3)

Example4: Consider the second-order boundary value - differential equation:

$$(t+1)u^{(2)}(t)+\acute{u}(t)=0$$

with boundary conditions

$$u(1)=1 \quad ,[(t+1)\acute{u}(t)]_{t=2}=1$$

The exact solution for this problem is

$u(t)$=1+ln $(\frac{t+1}{2})$

Table (4) shows the numerical results for this example with k=1, M=5 are compared with exact solution, graphically illustrated in Fig.4

The required coefficients are

a_0=0.693181818181818,

a_1=0.681818181818182

a_2=-0.233766233766234

a_3=0.113636363636364

a_4=-0.056818181818182

Table (4) Numerical solution of Example4

t	$u_e(t)$	$u_{appr.}(t)$	*Error*
0	0.306852819	0.306818181	0.00003463
0.1	0.402162999	0.402267045	0.000104046
0.2	0.489174376	0.489454545	0.00028016
0.3	0.569217083	0.569539772	0.00032268
0.4	0.643325056	0.643545454	0.00022039
0.5	0.712317927	0.712357954	0.00004002
0.6	0.776856448	0.776727272	0.00005913
0.7	0.837481070	0.837267045	0.00021402504
0.8	0.894639484	0.894454545	0.00018493888
0.9	0.948706705	0.948630681	0.00007602379
1.0	1.0	1.0	0.0

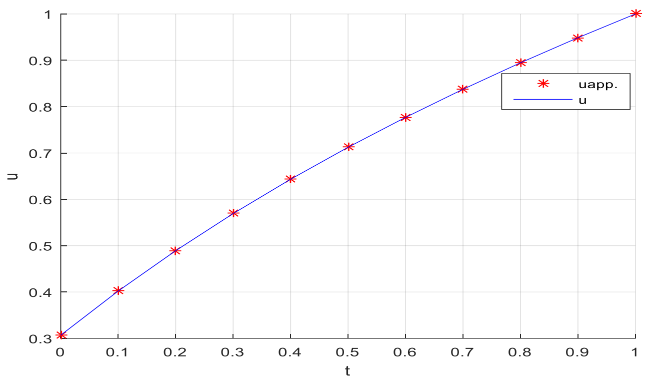


Figure (4)

5. Conclusion

In this paper, the capability of orthogonal Boubaker polynomials with Galerkin method for solving some BVPs was proved, these polynomials have been deduced using Gram-Schmidt method. Also this method can be presented as utilizing Raleigh-Ritz method in addition to Galerkin method according to the condition mentioned in the paper. In all examples the approximate solution is equivalent to the exact solution as shown in all figures .This method can be extended for other special applied kinds of BVPs in Physics and Engineering with other different boundary conditions.

References

[1] Bushra Eesa Al-Robeay, Numerical Solution of Two-Points Boundary Value Problem, thesis, Dept. of Applied Sciences, University of technology (2006).

[2] Rahmat Ali Khan, The Generalized Quasi linearization Technique for A Second Order Differential Equation With Separated Boundary Conditions, Elsevier, Math. and Comp. Modelling,Vol.43,PP.727-742(2006).

[3] Anwar Ja'afar Mohamad-Jawad, Solving Second Order Non-linear Boundary Value Problems by Four Numerical Methods, Eng.&Tech. Journal,Vol.28, No.2, PP.1-11(2009).

[4] Naseif J. Al-Jawari and Ahmed imad N., Controllability of Nonlinear Boundary Value Control Systems in Banach Spaces Using Schauder Fixed Point Theorem, Eng.&Tech. Journal,Vol.33, Part(B), No.4(2015).

[5] Muhammed GETIN, Mehmet SEZER and Huseyin KOCAYIGIT, An Efficient Method Based on Lucas Polynomials for Solving High-order Linear Boundary Value Problems, Gazi university Journal of Science, Vol.28, No.3, PP.483-496(2015).

[6] Dr. Suha N. Shihab, Hayat A. and Bushra E., Legendre Wavelets Method for Solving Boundary Value Problems, Journal of The College of Basic Education,Vol.188, No.2, PP.1437-1444(2007).

[7] Olagunju A. and Joseph Folake L., Third – Kind Chebyshev Polynomials $V_r(x)$ in Collocation Methods of Solving Boundary Value Problems, Journal of Mathematics (IOSR-JM), Vol.7, Issue 2, PP.42-47(2013).

[8] Siddu Chama basappa and kumbin. S., New Generalized Operational Matrix of Integration to Solve Nonlinear Singular Boundary Value Problems Using Hermit Wavelets, Arab Journal of Basic and Applied Sciences, Vol.26, N0. 1, PP. 385-396(2019).

[9] Shirala shetti S. and Srinivasa K., Hermit Wavelets Method for the Numerical Solution of Linear and Nonlinear Singular Initial and Boundary Value Problems, Computational Methods for Differential Equations, Vol.7, No.2, pp.177-198 (2019).

[10] Tinggang Z. and Ben Mahmoud B. K., Some New Properties of the Applied-Physics Related Boubaker Polynomials, Differential Equations and Control Processes, Vol.30(1), pp.8-19 (2009).

[11] Yalcinbas S. and AkkayaT., A Numerical Approach for Solving Linear Integro -differential-difference Equations with Boubaker Polynomial Bases, Ain Shams Engineering Journal , Vol.3, pp.153-161(2012).

[12] Ahmed I. et al., Indirect Method for Optimal Control Problem Using Boubaker Polynomial, Baghdad Science Journal, Vol.13 (1) (2016).

[13] Eman H. Ouda, An Approximate Solution to Calculus of Variational Problems Using Boubaker Polynomials, Baghdad Science Journal, Vol.15 (1), pp.106-109, (2018).

[14] Eman H. Ouda, A New Approach for Solving Optimal Control Problems Using Normalized Boubaker Polynomials, Emirates Journal for Engineering Research, Vol.23(4), pp.33-38 (2018).

[15]Eman H. Ouda , An Efficient Technique for Solving Lane-Emden Equation, Al-Qadisiyah Journal of Pure Science(QJPS), Vol.25, No.1, pp math. 1-10 (2020).

[16] Hoffman Joe D., Numerical Methods for Engineers and Scientists, Marcel DEKKER Inc, Second edition (2001)