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NUMERICAL SOLUTION FOR SOLVING TWO-POINTS BOUNDARY VALUE PROBLEMS USING ORTHOGONAL BOUBAKER POLYNOMIALS

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Abstract

In this paper, a new technique for solving boundary value problems (BVPs) is introduced. An orthogonal function for Boubaker polynomial was utilized and by the aid of Galerkin method the BVP was transformed to a system of linear algebraic equations with unknown coefficients, which can be easily solved to find the approximate result. Some numerical examples were added with illustrations, comparing their results with the exact to show the efficiency and the applicability of the method.

Key words-Orthogonal Boubaker polynomials, boundary value problem, Galerkin method.

1. Introduction

Boundary value problems represent a wide fielding the branch of numerical studies related with problems in Applied Physics and Engineering. Let \( I = (a, b) \subseteq \mathbb{R} \) be an interval and \( p, q, r \) be Continuous functions then Dirichlet First kind Two-point boundary value problem is given by [1]

\[
\begin{align*}
&u'' + p(t)u' + q(t)u = r(t) \quad a < t < b \\
&u(a) = \gamma_1, \quad u(b) = \gamma_2
\end{align*}
\]

A lot of researches deal with this kind of differential equations, Rahmat Ali Khan proposed generalized quasilinearization technique for solving a second order differential equation with separated boundary conditions [2], Anwar Ja'afar Mohamad-Jawad gave four numerical methods for solving second order on-linear BVPs [3], Naseif J. Al-Jawari et al. studied the controllability of nonlinear boundary value control systems using functional analysis proceeding [4], Getin M. et al, gave a method based on Lucas polynomials for solving high-order linear BVP [5].

Obviously, in recent decades there is a large interesting using different kinds of orthogonal functions like orthogonal polynomials and wavelets for solving linear and nonlinear problems in physics and engineering, many researchers worked in this field. Dr. Suha N. Shihab et al., used Legendre wavelets method for solving BVPs [6], Olagunju A. and Joseph Folake L., utilized third kind Chebyshev
Polynomials $V_f(x)$ in collocation methods for BVPs For solving BVPs [7]. Siddu Chamahasappa and Kumbinira saiahS. introduced a new generalized operational matrix of integration to solve nonlinear singular BVPs using Hermite wavelets [8], Shirala shettiS. And Srinivasa K. used Hermite wavelets method for solving linear and nonlinear singular initial and boundary value problems [9].

Boubaker polynomial is first utilized for solving heat equation in physical applications then many researches concerning this polynomial have taken place in different proceedings. Since their first appearance it presents a new powerful tool for solving different kinds of differential equations and optimal control problems [10-14]. In this paper, the orthogonal Boubaker function was introduced and used with Galerkin method to solve some boundary value problems. The paper is arranged as follows, the next section gives a fundamental idea about orthogonal Boubaker polynomials and the third section is related with the proposed method for solving BVP's. At last we added some numerical examples presenting different kinds of BVP's.

2. Orthogonal Boubaker polynomials
A set $S$ of Polynomials of an inner product space is called orthogonal if $<f, g> = 0$ for every $f \neq g \in S$.

Boubaker polynomial $B_0$ presented as in [10] as follows

$$B_0(t) = \frac{k^2}{2} \sum_{r=0}^{k-3} \frac{(k-4r)}{(k-r)} (-1)^r, k = 0,1,2,...$$

then $B_0(0) = 1, B_0(t) = t, B_0(t) = t^2 + 2, ...$

Since Boubaker polynomials are not orthogonal, the Gram-Schmidt method has been applied to find the orthogonal Boubaker polynomials, the first six orthogonal Boubaker polynomials denoted by $B_{m}(t)$ were found to be [15].

$$B_0(t) = 1,$$
$$B_1(t) = \frac{1}{2} (2t - 1),$$
$$B_2(t) = \frac{1}{6} (6t^2 - 6t + 1),$$
$$B_3(t) = \frac{1}{20} (20t^3 - 30t^2 + 12t - 1) ,$$
$$B_4(t) = \frac{1}{70} (70t^4 - 140t^3 + 90t^2 - 20t + 1),$$
$$B_5(t) = \frac{1}{252} (252t^5 - 630t^4 + 560t^3 - 210t^2 + 30t - 1),$$
$$B_6(t) = \frac{1}{924} (924t^6 - 2772t^5 + 3150t^4 - 1680t^3 + 420t^2 - 42t + 1).$$

3. The method
The process in steps is as follows

- Assuming the orthogonal Boubaker polynomial with certain coefficients $a_i$ for the unknown function $u(t)$ defined on $[0,1]$ as follows

$$u(t) \approx \sum_{i=0}^{m} a_i B_0(t), i = 1,2,...,m ... (2)$$

$B(t) = [B_0(t)B_1(t)B_2(t)...B_m(t)]^T$ represents orthogonal Boubaker polynomials with $a_i$'s as the required coefficients.

- Substituting eq.(2) in (1) and extracting $R(t)$ which represents the residual $R(t) = 0$ ... (3)

- Now applying Galerkin method to eq.(3). The weight functions were chosen to be the same as the orthogonal Boubaker terms (since it represent linear independence) and integrating their product with $R(t)$ in $[0,1]$ , then we get a system of algebraic equations with unknown coefficients $a_i$'s.

- Substituting for some values of t to find the values of the unknown coefficients.

Remark: Since the same terms for the weights and trial functions were used, the results would be the same as if Rayleigh-Ritz method was used [16].

4. Numerical Examples:
Example 1: Consider the second-order boundary value differential equation:

$$(t^2 + 1)u^{(2)}(t) + u(t) = 1$$

with boundary conditions:

$$u(0) = 0, \quad u(1) = 1$$

The exact solution for this problem is

$$u(t) = t$$

The steps of the Process would be:

- Assuming the orthogonal Boubaker polynomial with coefficients $a_i's(i=0,...,4)$ for the unknown function $u(t)$ defined on $[0,1]$ as follows

$$u(t) \approx \sum_{i=0}^{4} a_i B_0(t), i = 1,2,...,4 \quad ... (5)$$

- Substituting Eq.(5) in (4) and extracting $R(t)$ which represents the residual $R(t) = 0$ ... (6)

- Now applying Galerkin method to Eq.(6).
The weight functions were chosen to be the same as the orthogonal Boubaker terms and integrating their product with $R(t)$ in $[0,1]$, then we get a system of algebraic equations with unknown coefficients $a_i$’s.

Using Matlab the required coefficients would be

- $a_0 = 0.5$
- $a_1 = 1$
- $a_2 = 0$
- $a_3 = 0$
- $a_4 = 0$

Table (1) shows the numerical results for this example with $k=1, M=4$ are compared with exact solution, graphically illustrated in Fig.1.

Table (1) Numerical solution of Example1

<table>
<thead>
<tr>
<th>$t$</th>
<th>$u(t)$</th>
<th>$u_{app}(t)$</th>
<th>error</th>
</tr>
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<td>0</td>
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<tr>
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</table>

Table (2) shows the numerical results for this example with $k=1, M=4$ are compared with exact solution, graphically illustrated in Fig.2.

The required coefficients are

- $a_0 = 0.333333333333333$
- $a_1 = 1$
- $a_2 = 1$
- $a_3 = 0$
- $a_4 = 0$

Table (2) Numerical solution of Example2

<table>
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<tr>
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<th>$u(t)$</th>
<th>$u_{app}(t)$</th>
<th>error</th>
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<td>1.0</td>
<td>0</td>
</tr>
</tbody>
</table>

Example2: Consider the third-order boundary value differential equation:

$$t^2u^{(3)}(t) + u^{(2)}(t) = 2$$

with boundary conditions

$u(0) = 0$, $u(1) = 1$

The exact solution for this problem is $u(t) = t^2$

Example3: Consider the second-order boundary value differential equation:

$$12t^2u^{(2)}(t) + 24tu(t) = -30t^4 + 204t^3 - 351t^2 + 110t$$

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with boundary conditions $u(0) = 1, u(1) = 2$

The exact solution for this problem is

$$u(t) = \frac{1}{24}(-3t^4 + 34t^3 - 117t^2 + 110t + 24)$$

Table (3) shows the numerical results for this example with $k=1, M=4$ are compared with exact solution, graphically illustrated in Fig.3

The exact solution for this problem is

$$u(t) = 1 + \ln \left( \frac{t+1}{t} \right)$$

Table (4) shows the numerical results for this example with $k=1, M=5$ are compared with exact solution, graphically illustrated in Fig.4

The required coefficients are:

$$a_0 = 1.995833333333333$$
$$a_1 = 0.883333333333333$$
$$a_2 = -2.96428571428571$$
$$a_3 = 1.166666666666667$$
$$a_4 = -0.124999999999999$$

**Example 4:** Consider the second-order boundary value - differential equation:

$$(t + 1)u''(t) + u(t) = 0$$

with boundary conditions $u(1) = 1, [(t + 1)u(t)]_{t=2} = 1$.

The required coefficients are:

$$a_0 = 0.693181818181818$$
$$a_1 = 0.681818181818182$$
$$a_2 = -0.233766233766234$$
$$a_3 = 0.113636363636364$$
$$a_4 = 0.056818181818182$$

**Table (4) Numerical solution of Example 4**

<table>
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<th>$u(t)$</th>
<th>$u_{app}(t)$</th>
<th>Error</th>
</tr>
</thead>
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<td>1.0</td>
<td>1.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Figure (3)

Figure (4)

**5. Conclusion**

In this paper, the capability of orthogonal Boubaker polynomials with Galerkin method for solving some BVPs was proved, these polynomials have been deduced using Gram-Schmidt method. Also this method can be presented as utilizing Raleigh-Ritz method in addition to Galerkin method according to the
condition mentioned in the paper. In all examples the approximate solution is equivalent to the exact solution as shown in all figures. This method can be extended for other applied kinds of BVPs in Physics and Engineering with other different boundary conditions.

References
