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BLOCK AND WEDDLE METHODS FOR SOLVING NTH ORDER LINEAR RETARDED VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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طريقتي بلوك و ويدل لحل معادلات فولتيرا التراجعية التكاملية- التفاضلية الخطية من الرتبة n

ملخص

يقدم البحث طريقة مقترحة لحل معادلات فولتيرا التكاملية التفاضـلية التراجعية الخطية من الرتبة n عدديا باسـتعمال طريقة بلوك من الرتبة الرابعة و طريقة ويدل. حيث تمت مقارنة النتائج العددية و الحقيقية من خلال بعض الأمثلة والرسوم لتوثيق دقة النتائج للطريقة المقترحة.

Abstract

A proposed method is presented to solve nth order linear retarded Volterra integro-differential equations (RVIDE's) numerically by using fourth-order block and Weddle methods. Comparison between numerical and exact results has been given in numerical examples for conciliated the accuracy of the results of the proposed scheme.

1. INTRODUCTION

One of the most important and applicable subjects of developing modern applied mathematics is the integral equations. The names of many modern mathematicians notably, Volterra, Cauchy and others are associated with this topic [1].

Integral equation was introduced by Bois-Reymond in 1888. However, Volterra equation, was introduced by Volterra in 1884 and in 1959 Volterra's book "Theory of Functional and of Integro-Differential Equations" appeared [2].

The integral and integro-differential equations formulation of physical problems are more elegant and compact than the differential equation formulation, since the boundary conditions can be satisfied and embedded in the integral or integro-differential equation. Also the form of the solution to an integro-differential equation is often more stable for today's extremely fast machine computation. Delay integro-differential equation has been developed over twenty years ago where one of its types widely is used in control systems and digital communication systems as, lag-lead compensation and spread spectrum designs [3,4].

2. Retarded Integro-Differential Equation (RIDE) [5,6]:

The delay integro-differential equation (DIDE) is a delay differential equation in which the unknown function $u(x)$ can appear under an integral sign.

Consider the nth order DIDE:

$$\sum_{i=0}^{n} p_i(x)\frac{d^i u(x)}{dx^i}+\sum_{i=1}^{n} q_i(x)\frac{d^i u(x-\tau_i)}{dx^i}+\sum_{i=0}^{n} r_i(x)u(x-\tau_i)=g(x)+ \lambda\int_a^{b(x)} k(x,t)\,u(t-\tau)dt, \qquad x\in[a,b(x)] \quad \dots (1)$$

with initial functions:

$$\left.\begin{aligned} u(x) &= \phi(x) \\ u'(x) &= \phi'(x) \\ &\vdots \\ u^{(n-1)}(x) &= \phi^{(n-1)}(x) \end{aligned}\right\} \quad for \quad x_0 - \max(\tau, \tau_i) \le x \le x_0 \,, \quad i = 0,1,\ldots,n$$

.

where $g(x), p_i(x), q_i(x), k(x,t)$ are known functions of x, $k(x,t)$ is called the kernel of the integral equation, $u(x)$ is the unknown function, λ is a parameter equal 1, $\tau, \tau_0, \tau_1, \ldots, \tau_n$ are fixed positive numbers. eq.(1) is called retarded type if the delay comes in u only and the delay appears in the integrand unknown function (i.e. $\tau \neq 0$) [7,8] which is :-

$$\sum_{i=0}^{n} p_i(x)\frac{d^i u(x)}{dx^i} + \sum_{i=0}^{n} r_i(x) u(x-\tau_i) = g(x) + \lambda \int_a^{b(x)} k(x,t) u(t-\tau) dt$$
$$x \in [a, b(x)] \qquad \ldots (2)$$

with initial functions:

$$\left.\begin{aligned} u(x) &= \phi(x) \\ u'(x) &= \phi'(x) \\ &\vdots \\ u^{(n-1)}(x) &= \phi^{(n-1)}(x) \end{aligned}\right\} \quad for \quad x_0 - \max(\tau, \tau_i) \le x \le x_0 \,, i = 0,1,\ldots,n$$

3. Block and Weddle Methods

Block method was employed together with Weddle method to treat numerically RVIDE.

Weddle Method [1,5]

Weddle method is one of basic formula of quadrature approximation methods for integration. The composite Weddle rule is obtained as:

$$\int_a^b f(t)dx = \frac{3H}{10}\left[\begin{aligned} & f_0 + 5f_1 + f_2 + 6f_3 + f_4 + 5f_5 + 2f_6 + 5f_7 + \\ & f_8 + 6f_9 + f_{10} + 5f_{11} + \cdots + 2f_{N-6} + 5f_{N-5} + \\ & f_{N-4} + 6f_{N-3} + f_{N-2} + 5f_{N-1} + f_N \end{aligned}\right] \qquad \ldots (3)$$

where $H = \frac{(b-a)}{N}$, N is the number of intervals ($[t_0,t_1],[t_1,t_2],\ldots,[t_{N-1},t_N]$) which is the multiple of (6), $f_i = f(t_i)$, $t_0 = a$, $t_N = b$ and $t_i = a + iH$ are called the integration nodes which are lying in the interval $[a,b]$ where $i = 0,1,\ldots,N$.

Block Method [9,10]

The concept of block method is essentially an extrapolation procedure and has the advantage of being self-starting.

Consider the differential equation:

$$y' = f(t, y(t)), \qquad y(t_0) = y_0 \qquad \ldots (4)$$

Let

$$\left.\begin{aligned} B_1 &= f(t_n, y(t_n)) \\ B_2 &= f(t_n + h, y(t_n) + hB_1) \\ B_3 &= f\left(t_n + h, y(t_n) + \frac{h}{2}B_1 + \frac{h}{2}B_2\right) \\ B_4 &= f(t_n + 2h, y(t_n) + 2hB_3) \\ B_5 &= f\left(t_n + h, y(t_n) + \frac{h}{12}(5B_1 + 8B_3 - B_4)\right) \\ B_6 &= f\left(t_n + 2h, y(t_n) + \frac{h}{3}(B_1 + B_4 + 4B_5)\right) \end{aligned}\right\} \qquad \ldots (5)$$

Then, the fourth order block method may be written in the form:

$$y_{n+1} = y_n + \frac{h}{12}(5B_1 + 8B_3 - B_4) \qquad \ldots (6)$$

$$y_{n+2} = y_n + \frac{h}{3}(B_1 + 4B_5 + B_6) \qquad \ldots (7)$$

A fourth order block method is said to be stable, since when applied the test equation $y' = \lambda y$ for it yields $y_{n+i} = 6(h\lambda) y_{n+i-1}, i = 1,2$ which is divergent for $|6(h\lambda)| > 1$ and convergent otherwise, where λ is a complex constant with $\operatorname{Re}\lambda < 0$. Hence, the absolute stability region is the set: $\{h\lambda \in ¢ : |6(h\lambda)| \le 1\}$.

4. The Solutions of n[th] Order Linear RVIDE Using Block and Weddle Methods

The general formula of n[th] order linear RVIDE in eq.(2) can be written as:

$$\frac{d^n u(x)}{dx^n} = f\begin{pmatrix} x, p_0(x)u(x), p_1(x)u'(x), \ldots, p_{n-1}(x)u^{(n-1)}(x), \\ q_1(x)u'(x-\tau_1), \ldots, q_n(x)u^{(n)}(x-\tau_n), \\ r_0(x)u(x-\tau_0), \ldots, r_n(x)u(x-\tau_n), g(x), I[Q(x,t)] \end{pmatrix} \qquad \ldots (8)$$

with

$$\left.\begin{aligned} u(x) &= \phi(x) \\ u'(x) &= \phi'(x) \\ &\vdots \\ u^{(n-1)}(x) &= \phi^{(n-1)}(x) \end{aligned}\right\} \quad x_0 - \max(\tau, \tau_i) \le x \le x_0 \,, \quad i = 0,1,\ldots,n$$

where $I[Q(x,t)]$ is the finite integral on $[a,x]$, $x \ge a$ and $Q(x,t) = k(x,t)\, u(t-\tau)$.

Clearly, eq.(8) can be replaced by a system of first order RVIDE's as follows:

Let

$$\begin{aligned} v_1(x) &= u(x) \\ v_2(x) &= u'(x) \\ &\vdots \\ v_{n-1}(x) &= u^{(n-2)}(x) \\ v_n(x) &= u^{(n-1)}(x) \end{aligned}$$

Hence, we can write the initial functions as:

$$\left.\begin{aligned} v_1(x) &= \phi(x) \\ v_2(x) &= \phi'(x) \\ &\vdots \\ v_n(x) &= \phi^{(n-1)}(x) \end{aligned}\right\} \quad for \quad x_0 - \max(\tau, \tau_i) \le x \le x_0 \ , \quad i = 0,1,\ldots,n$$

At this point the system of first order equations can be gotten next one:

$$\begin{aligned} v_1'(x) &= v_2(x) \\ v_2'(x) &= v_3(x) \\ &\vdots \\ v_{n-1}'(x) &= v_n(x) \\ v_n'(x) &= f\begin{pmatrix} x, p_0(x)v_1(x), p_1(x)v_2(x), \ldots, p_{n-1}(x)v_n(x), \\ r_0(x)\phi(x-\tau_0), \ldots, r_n(x)\phi(x-\tau_n), g(x), I[Q(x,t)] \end{pmatrix} \end{aligned} \quad \ldots (9)$$

The above system of RVIDE's can be treated numerically by using 4th block with Weddle methods as follows:

$$v_i(x_{j+1}) = v_i(x_j) + \frac{h}{12}(5B_{1i} + 8B_{3i} - B_{4i}) \quad \ldots (10)$$

$$v_i(x_{j+2}) = v_i(x_j) + \frac{h}{3}(B_{1i} + 4B_{5i} + B_{6i}) \quad \ldots (11)$$

where

$$B_{1i} = f_i\begin{pmatrix} x_j, p_0(x_j)v_1(x_j), \ldots, p_{n-1}(x_j)v_n(x_j), r_0(x_j)\phi(x_j - \tau_0), \\ \ldots, r_n(x_j)\phi(x_j - \tau_n), g(x_j), Weddle(Q(x_j,t), a, x_j, N) \end{pmatrix}$$

$$B_{2i} = f_i\begin{pmatrix} x_j + h, p_0(x_j+h)v_1(x_j) + hB_{11}, \ldots, p_{n-1}(x_j+h)v_n(x_j) + \\ hB_{1n}, r_0(x_j+h)\phi(x_j + h - \tau_0), \ldots, r_n(x_j+h)\phi(x_j + h - \tau_n) \\ , g(x_j+h), Weddle(Q(x_j+h,t), a, x_j+h, N) \end{pmatrix}$$

$$B_{3i} = f_i\begin{pmatrix} x_j + h, p_0(x_j+h)v_1(x_j) + \frac{h}{2}B_{11} + \frac{h}{2}B_{21}, \ldots, \\ p_{n-1}(x_j+h)v_n(x_j) + \frac{h}{2}B_{1n} + \frac{h}{2}B_{2n}, r_0(x_j+h)\phi(x_j + h - \tau_0), \\ \ldots, r_n(x_j+h)\phi(x_j + h - \tau_n), g(x_j+h), \\ Weddle(Q(x_j+h,t), a, x_j+h, N) \end{pmatrix}$$

$$B_{4i} = f_i\begin{pmatrix} x_j + 2h, p_0(x_j+2h)v_1(x_j) + 2hB_{31}, \ldots, \\ p_{n-1}(x_j+2h)v_n(x_j) + 2hB_{3n}, r_0(x_j+2h)\phi(x_j + 2h - \tau_0) \\ , \ldots, r_n(x_j+2h)\phi(x_j + 2h - \tau_n), \\ g(x_j+2h), Weddle(Q(x_j+2h,t), a, x_j+2h, N) \end{pmatrix}$$

$$B_{5i} = f_i\begin{pmatrix} x_j + h, p_0(x_j+h)v_1(x_j) + \frac{h}{12}(5B_{11} + 8B_{31} - B_{41}), \\ \ldots, p_{n-1}(x_j+h)v_n(x_j) + \frac{h}{12}(5B_{1n} + 8B_{3n} - B_{4n}), \\ r_0(x_j+h)\phi(x_j + h - \tau_0), \ldots, r_n(x_j+h)\phi(x_j + h - \tau_n), \\ g(x_j+h), Weddle(Q(x_j+h,t), a, x_j+h, N) \end{pmatrix}$$

$$B_{6i} = f_i\begin{pmatrix} x_j + 2h, p_0(x_j+2h)v_1(x_j) + \frac{h}{3}(B_{11} + B_{41} + 4B_{51}), \\ \ldots, p_{n-1}(x_j+2h)v_n(x_j) + \frac{h}{3}(B_{1n} + B_{4n} + 4B_{5n}), \\ r_0(x_j+2h)\phi(x_j + 2h - \tau_0), \ldots, r_n(x_j+2h)\phi(x_j + 2h - \tau_n), \\ g(x_j+2h), Weddle(Q(x_j+2h,t), a, x_j+2h, N) \end{pmatrix} \quad \ldots (12)$$

for each i=1,2,…,n. and j=0,1,…,m where (m + 1) is the number of points ($x_0, x_1, \ldots, x_m$) and *Weddle*(Q(x,t),a,x,N) is:

$$Weddle(Q(x_j,t), a, x_j, N) = \frac{3H}{10}\begin{bmatrix} Q(x_j,t_0) + 5Q(x_j,t_1) + Q(x_j,t_2) + \\ 6Q(x_j,t_3) + Q(x_j,t_4) + 5Q(x_j,t_5) + \\ \cdots + 2Q(x_j,t_{N-6}) + 5Q(x_j,t_{N-5}) + \\ Q(x_j,t_{N-4}) + 6Q(x_j,t_{N-3}) + Q(x_j,t_{N-2}) + \\ 5Q(x_j,t_{N-1}) + Q(x_j,t_N) \end{bmatrix}$$

where $t_k = a + kH$, $H = \dfrac{(b-a)}{N}$ and $k = 0,1,\ldots,N$.

The numerical solution can be summarized by the following algorithm:

<u>BWM-RVIDE Algorithm:</u>

<u>1:</u> Input a, x_0 , m, N, n where x_0 is the initial value and n is the order of RVIDE.

<u>2:</u> Define $Q(x,t)$ as in eq.(8) and $g(x)$ in RVIDE.

<u>3:</u> put $h = \dfrac{(x_m - x_0)}{m}$ and j = 0.

<u>4:</u> $\forall$ i=1,2,…,n compute B_{1i} in eq.(12).

<u>5:</u> $\forall$ i=1,2,…,n compute B_{2i} in eq.(12).

<u>6:</u> $\forall$ i=1,2,…,n compute B_{3i} in eq.(12)

<u>7:</u> $\forall$ i=1,2,…,n compute B_{4i} in eq.(12)

<u>8:</u> $\forall$ i=1,2,…,n compute B_{5i} in eq.(12)

<u>9:</u> $\forall$ i=1,2,…,n compute B_{6i} in eq.(12)

<u>10:</u> $\forall$ i=1,2,…,n compute:

$$v_i(x_{j+1}) = v_i(x_j) + \frac{h}{12}(5B_{1i} + 8B_{3i} - B_{4i})$$

$$v_i(x_{j+2}) = v_i(x_j) + \frac{h}{3}(B_{1i} + 4B_{5i} + B_{6i})$$

$$x_{j+1} = x_j + h$$

<u>11:</u> Put $j = j$+1

<u>12:</u> If $j = m$ then stop.
Else go to step (4)

5. NUMERICAL EXAMPLES

Example (1):

Consider the following 2nd order RVITE:

$$\frac{d^2u(x)}{dx^2} - xu(x-1) = e^{x+\frac{1}{2}} + \cos x - x\left(xe^{x-\frac{1}{2}} + e^{-\frac{1}{2}} - \tfrac{1}{2}x^2 + 1\right) + \int_0^x xt\,u(t-1)dt \qquad x \ge 0 \quad \ldots (13)$$

with initial functions :

$$\left.\begin{aligned} u(x) &= e^{x+\frac{1}{2}} + 1 \\ u'(x) &= e^{x+\frac{1}{2}} \end{aligned}\right\} \qquad -1 \le x \le 0$$

and exact solution:

$$u(x) = 2 - \cos x + e^{x+\frac{1}{2}} \qquad 0 \le x \le 1 \ .$$

The above RVIDE can be replaced by the system:

$$\left.\begin{aligned} &v_1'(x) = v_2(x), && x \ge 0 \\ &v_2'(x) = xv_1(x-1) + e^{x+\frac{1}{2}} + \cos x - \\ &\quad x\left(xe^{x-\frac{1}{2}} + e^{-\frac{1}{2}} - \tfrac{1}{2}x^2 + 1\right) + \int_0^x xtv_1(t-1)dt && x \ge 0 \end{aligned}\right\} \quad \ldots(14)$$

with initial functions:

$$\left.\begin{aligned} v_1(x) &= e^{x+\frac{1}{2}} + 1 \\ v_2(x) &= e^{x+\frac{1}{2}} \end{aligned}\right\} \qquad -1 \le x \le 0$$

and exact solutions:

$$exact_1 = v_1(x) = 2 - \cos x + e^{x+\frac{1}{2}} \qquad 0 \le x \le 1$$

$$exact_2 = v_2(x) = \sin x + e^{x+\frac{1}{2}} \qquad 0 \le x \le 1$$

When the algorithm (BWM-RVIDE) is applied, table (1) gives the comparison between the exact and numerical results of eq.(14) for m=10, h=0.1, $x_j = jh$, $j = 0,1,\ldots,m$ with least square error (L.S.E.).

Figure (1) shows the solution of linear RVIDE, which was given in table (1).

Table (1) The solution of RVIDE for Ex.(1).

x	*Exact₁*	*(BWM-RVIDE)* $v_1(x)$	*Exact₂*	*(BWM-RVIDE)* $v_2(x)$
0	2.6487	2.6487	1.6487	1.6487
0.1	2.8271	2.8271	1.9220	1.9220
0.2	3.0337	3.0337	2.2124	2.2124
0.3	3.2702	3.2702	2.5211	2.5211
0.4	3.5385	3.5385	2.8490	2.8490
0.5	3.8407	3.8407	3.1977	3.1977
0.6	4.1788	4.1788	3.5688	3.5688
0.7	4.5553	4.5553	3.9643	3.9643
0.8	4.9726	4.9726	4.3867	4.3867
0.9	5.4336	5.4336	4.8385	4.8385
1	5.9414	5.9414	5.3232	5.3232
L.S.E		0.2707e-9	L.S.E	0.5163e-8

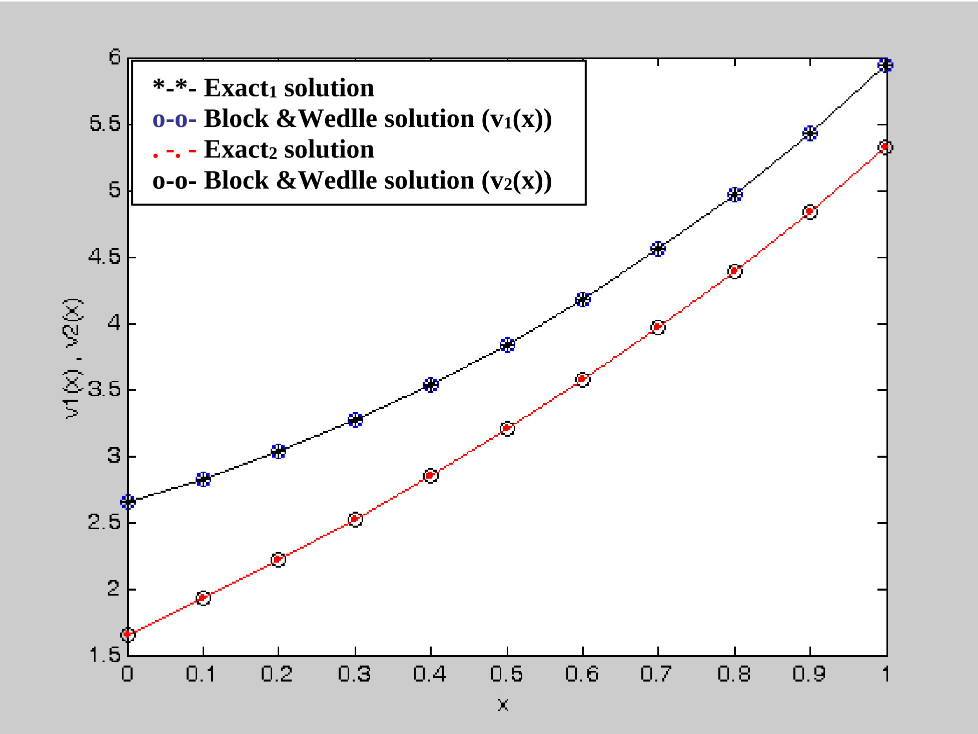

Figure.(1) The comparison between exact and BWM-RVIDE algorithm in Ex.(1)

Example (2):

Consider the RVIDE of third order:

$$\frac{d^3u(x)}{dx^3}+2\frac{du(x)}{dx}-u(x)+u(x-\tfrac{1}{2})=$$
$$\left(-\frac{5}{6}x^3-\frac{9}{4}x^2+\frac{3}{2}\right)+\int_0^x (x+t)u(t-\tfrac{1}{2})dt \quad x\ge 0 \qquad \dots(15)$$

with initial functions :

$$\left.\begin{aligned} u(x)&=x+2\\ u'(x)&=1\\ u''(x)&=0\end{aligned}\right\} \quad -2\le x\le 0$$

The exact solution of the above linear DVIDE is: $u(x)=x+2 \qquad 0\le x\le 0.5$

Table (2) presents the comparison between the exact and numerical results of eq.(15) using (BWM-RVIDE) algorithm for m=10, h=0.05, $x_j=jh,\ j=0,1,\dots,m$ and m=100, h=0.005 , depending on (L.S.E.).

Figure (2) shows the solution of example (2) which was given in table (2).

Table (2) The solution of RVIDE for Ex.(2).

x	*Exact₁*	*(BWM-RVIDE) v₁(x)*		*Exact₂*	*(BWM-RVIDE) v₂(x)*		*Exact₃*	*(BWM-RVIDE) v₃(x)*	
		h=0.05	*h=0.005*		*h=0.05*	*h=0.005*		*h=0.05*	*h=0.005*
0	2.0000	2.0000	2.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.05	2.0500	2.0500	2.0500	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.10	2.1000	2.1000	2.1000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.15	2.1500	2.1500	2.1500	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.20	2.2000	2.2000	2.2000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.25	2.2500	2.2500	2.2500	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.30	2.3000	2.3000	2.3000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.35	2.3500	2.3500	2.3500	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.40	2.4000	2.4000	2.4000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.45	2.4500	2.4500	2.4500	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
0.50	2.5000	2.5000	2.5000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
L.S.E.		0.12e-31	0.0000	L.S.E.	0.0000	0.0000	L.S.E.	0.15e-45	0.0000

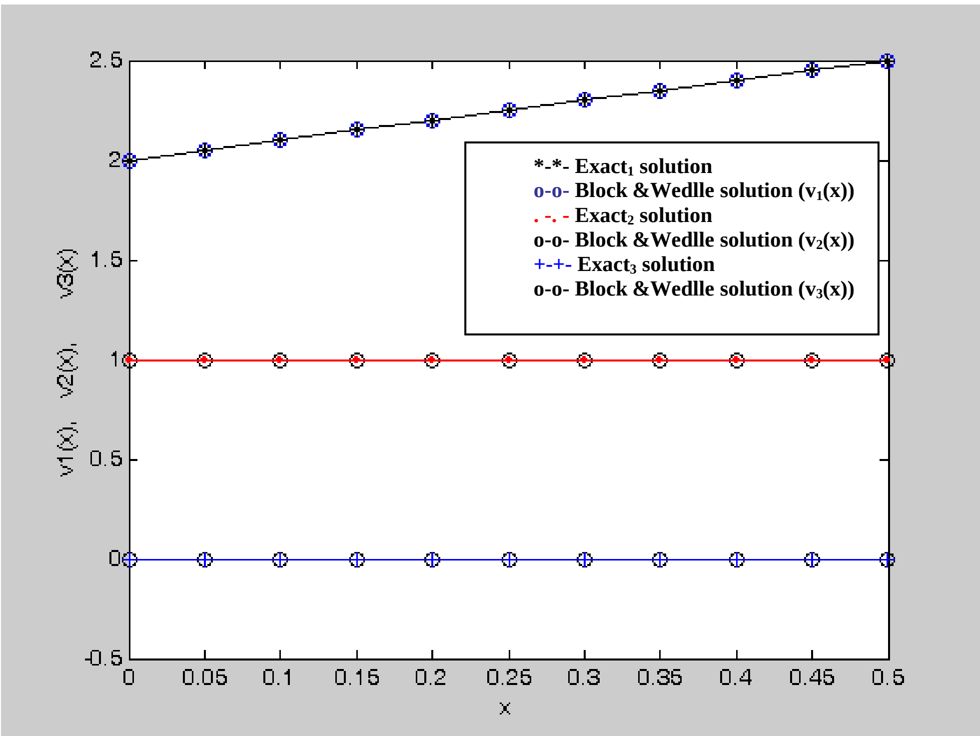

Figure (2) The comparison for RVIDE in Ex.(2).

Conclusion

Block with Weddle methods have been presented to treat numerically n^{th}-order linear RVIDE's. The results show a marked improvement in L.S.E. Examples are concludes the following points:

1. Block and Weddle methods give qualified way for solving n^{th}-order linear RVIDE's.
2. The accurate results depend upon the value of h, if h is decreased then the nodes increases and L.S.E. approaches to zero.
3. Block and Weddle methods solved linear RVIDE of any order by reducing the equation to a system of first order equations.

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