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PRICING EUROPEAN OPTION UNDER A MODIFIED CEV MODEL

Wafaa Ibrahim AbuZarqa

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United Arab Emirates University

College of Science

Department of Mathematics

PRICING EUROPEAN OPTION UNDER A MODIFIED CEV MODEL

Wafaa Ibrahim AbuZarqa

This thesis is submitted in partial fulfilment of the requirements for the degree of
Master of Science in Mathematics

Under the Supervision of Dr. Youssef El-Khatib

May 2015

Declaration of Original Work

I, Wafaa Ibrahim AbuZarqa, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled “*Pricing European Options Under a Modified CEV Model*”, hereby, solemnly declare that this thesis is an original research work that has been done and prepared by me under the supervision of Dr. Youssef El-Khatib, in the College of Sciences at UAEU. This work has not been previously formed as the basis for the award of any academic degree, diploma or a similar title at this or any other university. The materials borrowed from other sources and included in my thesis have been properly cited and acknowledged.

Student's Signature _____

Date _____

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Abstract

A financial derivative is an instrument whose payoff is derived from the behavior of another underlying asset. One of the most commonly used derivatives is the option which gives the right to buy or to sell an underlying asset at a pre-specified price at (European) or at and before (American) an expiration date. Finding a fair price of the option is called the option pricing problem and it depends on the underlying asset prices during the period from the initial time to expiration date. Thus, a “good” model for the underlying asset price trajectory is needed.

In this work, we are interested in European call options. We propose a new Constant Elasticity of Variance (CEV) model that covers the post-crash situations. First, we set up the modified CEV model for markets with high volatility. Then we find a numerical solution for the stochastic differential equation of the underlying price. The risk-neutral valuation method shows that the option price can be written as an expected value of the discounted underlying asset price at maturity. Then we use Monte Carlo methods for finance this to find a numerical solution for the price of a European option under a CEV model with high volatility.

Keywords: European option, stochastic calculus, stochastic volatility models, CEV model, Post-Crash markets, numerical analysis, Monte Carlo methods.

Title and Abstract (in Arabic)

تسعير عقود الخيارات المالية الأوروبية لنموذج " فرق بياني ذات مرونة ثابتة " معدل

الملخص

المشتقات المالية هي أداة مردودها ناتج عن سلوك أصل آخر. ومن أكثر المشتقات المالية استعمالاً هو الخيار الذي يعطي الحق ببيع أصل مدرج بسعر محدد مسبقاً عند تاريخ الانتهاء (في حالة الخيار الأوروبي) وأما عند أو قبل تاريخ الانتهاء (في حالة الخيار الأمريكي). إيجاد سعر الخيار العادل يطلق عليه "تسعير الخيار" ويعتمد على أسعار الأصول المندرجة خلال الفترة الممتدة بين وقت البدء وحتى تاريخ الانتهاء. لذا فإن نموذجاً مناسباً لمسار سعر الأصل المدرج يعد ضرورياً. اهتمامنا في هذا البحث على خيار الطلب الأوروبي ونقترح نموذجاً جديداً لمرونة التباين الثابتة ومن خلال هذا النموذج يغطي مواقف ما بعد الانهيار.

في البداية نضع النموذج المعدل للأسواق كثيرة التقلب . ثم نجد حلاً عددياً لمعادلة السعر المندرج التفاضلية العشوائية. تبين طريقة التثمين متعادل المخاطر أن سعر الخيار يمكن التعبير عنه كقيمة متوقعة لسعر الأصل المندرج للخصوم عند حلول وقت السداد. وأخيراً نستخدم طريقة مونت كارلو المالية لإيجاد حل عددي لسعر الخيار الأوروبي ضمن نموذج مرونة تباين ثابتة يؤخذ بعين الاعتبار كثرة التقلب السوقي.

كلمات مفتاحية: الخيار الأوروبي، التفاضل والتكامل العشوائي، نماذج التقلب العشوائي، نموذج مرونة التباين الثابتة، أسواق ما بعد الانهيار، التحليلات العددية، طرق مونت كارلو.

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Last but not the least I would like to thank my family: my parents my brothers and sister for their constant and meaningful presence in my life, and for their support and guidance in every aspect of my life.

Dedication

A special feeling of gratitude to my parents whose words of encouragement and push for tenacity ring in my ears. My siblings who have never left my side and continue to remain very dear and special.

I also dedicate this dissertation to my mentors and many friends who have supported me throughout the process. I will always appreciate all they have done, especially Dr. Youssef El-Khatib for helping me develop my research and analytical skills.

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List of Abbreviations

CEV	Constant Elasticity Variance
SDE	Stochastic Differential Equation
BSM	Black and Scholes Method
MC	Monte Carlo

Chapter 1: Introduction

Options can be seen as a powerful tool in financial risk management. The holder, of a European call option for example, gets the right (and not the obligation) to buy the underlying asset at the strike price on the maturity date. On the other hand, the holder has to pay the premium (the option price at the initial time). The maximum loss of the owner of the option is simply the premium but no limit to his profit. This shows how much it is important to find a fair price for the options.

The famous Black-Scholes formula (F.Black, 1973) is usually used for this purpose. However, the Black Scholes model has many shortcomings, among others: the constant volatility.

Empirical studies show clearly that the volatility is stochastic. To surmount this deficit, many researchers recommended models where the volatility is stochastic (stochastic volatilities models). We point out for example: Constant Elasticity of Variance (CEV) model (Cox, J.C., and S.A. Ross, 1976), Heston model (S.Heston, 1993) or more recently hybrid models, see for example (Y. El-Khatib, and A. Hatemi-J, 2015). However, these stochastic models do not treat markets suffering from crisis. On the other hand many researchers worked on models that account for the financial crisis see for instance (G.Chahda, 2005), (Y.El-Khatib, A.Hatemi-J, 2013/1), (F.Mantenga, 2003), and (D.Sornette, 2003). In the reference (Y.El-Khatib, A.Hatemi-J, 2013/2) a closed form solution for a model with crisis is obtained.

This work, we deal with the pricing problem for European options under a modified Constant Elasticity of Variance (CEV) model (Cox, J.C., and S.A. Ross, 1976). We

consider a hybrid model that combines the CEV model and a particular case of the model with crisis considered in (Y.El-Khatib, A.Hatemi-J, 2013/2).

The main advantage of the CEV model is that it accounts for the leverage effect: the volatility of a stock increases as its price goes down.

We start by an introduction to financial derivatives and to stochastic calculus, then we provide a presentation on the Black and Scholes and the CEV models. Our results start by suggesting the modified CEV model where a new parameter α is added to the stochastic differential equation of the underlying asset price:

$$dS_t = rS_t dt + \left(\sigma S_t^{\frac{\beta}{2}} + \alpha \right) dW_t, \quad S_0 > 0.$$

We solve the above equation numerically since to the best knowledge it does not have a closed form solution. Then, we find a numerical solution for the pricing of European option problem using the Monte Carlo methods for finance.

The rest of the thesis is structured as follows, in chapter 2, we provide an introduction to the financial derivatives. Chapter 3 is devoted to the stochastic calculus and the different tools needed to solve our research problem. In chapter 4, we discuss the CEV model under the Black-Scholes model and we give the different numerical methods for solving stochastic differential equations. We use Euler scheme to solve numerically the stochastic differential equation of the underlying asset price. Chapter 4 is dedicated to our proper research work; we first suggest the modified CEV model. Then, we solve numerically the asset price stochastic differential equation using Euler scheme. Moreover, we find a numerical solution for the European call option price of the high volatile model.

Chapter 2: Prior Knowledge of Financial derivatives

Financial derivatives are very important tools for risk management. Currently, derivatives markets exist in many countries over the world. Studying financial derivatives lead to two major problems: pricing and hedging. In the present work, we are interested in the pricing problem. This chapter is dedicated to an introduction to financial derivatives.

2.1 Definition of Derivative

A derivative security is a financial agreement whose value “derives” from cash market tools like stocks, bonds, currencies and commodities. (N.Neftci, 16 Dec 2013)

The cash market tool is also denoted to as the underlying asset. Therefore, an underlying asset can be

- Stocks: claims to real returns generated in the production sector of goods and services.
- Currencies: banks.
- Soft: such as cocoa, sugar, coffee
- Grains and oilseeds: barley, corn, cotton, oat, palm oil
- Metals: copper, nickel, tin
- Precious metals: gold, platinum, silver
- Energy: basic oil, gasoline.

Definition 2.1.1 A financial contract is a derivative security if its value at a stated date T , called the expiration time, is determined by the market price of the fundamental asset at that time T . (C.Hull, 2005)

2.2 Major Categories of Derivatives

There are many types of financial derivatives. In the coming subsections we list some of them.

2.2.1 Forwards and Futures

Forwards and futures contract are very similar the two parties are in the obligation.

Definition 2.2.1 A forward contract is a contract (or agreement) to buy or to sell a positive asset at a certain future period for a certain price. (N.Neftci, 16 Dec 2013)

Definition 2.2.2. A futures contract is a contract between two parties to buy or sell an asset at a sure time in the future for a sure price. (N.Neftci, 16 Dec 2013)

2.2.2 Option

Options differ from the futures and forward by giving the right to the holder (and not the obligation) to exercise. We have many types of options, below is a general definition of an option. (MörTERS & Peres, 2010)

Definition2.2.3 An option offers the holder with the right to buy or to sell a specified amount of an underlying asset at a fixed price (called a strike price) at or before the expiration date of the option. (Cohen, 2013)

Since it is a right and not an obligation, the holder can choose not to exercise the right and consent the option to expire. There are two kinds of options - call options (right to buy) and put options (right to sell).

Call Options

A call option gives the buyer of the choice the right to buy the underlying asset at a fixed price (strike price or K) at any time proceeding to the expiration date of the option. The buyer pays a price for this right.

From expiration:

- If the price of the underlying asset (S) $>$ Strike Price (K)
 - Buyer makes the difference: $S - K$

- If the value of the underlying asset (S) $<$ Strike Price (K)
 - Buyer does not exercise

More generally:

- The value of a call grows as the value of the underlying asset grows
- The value of a call decreases as the value of the underlying asset decreases

Put Options

A put option provides the buyer of the option the right to sell the underlying asset at a fixed price at any time previous to the expiration date of the option. The buyer pays a price for this correct.

From expiration:

- If the value of the underlying asset (S) $<$ Strike Price (K)
 - Buyer makes the difference: $K - S$

- If the value of the underlying asset (S) $>$ Strike Price (K)

– Buyer does not exercise

More generally:

- The value of a put decreases as the value of the underlying asset increases
- The value of a put rises as the value of the underlying asset decline

2.3 American versus European Options

An American option can be practical at any time prior to its expiration, while a European option can be exercised only at expiration.

The opportunity of early exercise makes American options more appreciated than otherwise similar European options.

However, in most cases, the time premium associated with the remaining life of an option makes early exercise sub-optimal.

2.4 Example

If we take an example, say a prepackaging food company who may be subject to volatile price actions and changes. In order for them to sustain and maintain steady and consistent prices for the customers, they are required to purchase supplies for the company at a market friendly rate, which is relatively consistent to maintain their prices. (Bjork, 2009)

In order to be able to do this they must enter an options contract with their agricultural suppliers and buy a certain amount of crops at a certain time at a certain rate and upon a certain timeframe. If a product such as wheat went up suddenly then the company would

still be able to purchase the goods at the best value rate rather than purchase at the supplier's rate for that particular time (Wheat prices raised)

The company is then getting value for their money by signing up for the options contract and also are assured the best price at all times. Both parties involved are in a win situations as the company is guaranteed a competitive prices and the supplier is assured fair value for their goods, plus consistency in dealing with the company.

In this instance, the value of the option is "derived" from an underlying asset; in this case, a certain number of bushels of wheat. (Cohen, 2013)

A European style call (put) option is a right, but not an obligation, to purchase (sell) an asset at a strike price on option maturity date, T. An American style option is a European option that can be exercised prior to T.

Chapter 3: Introduction to Stochastic Calculus

This chapter provide to some probability notions and elementary specific tools form stochastic calculus that are needed in our study.

3.1 Sample Space

Definition 3.1.1 A random experiment in the theory of probability is an experiment whose outcomes cannot be determined in advance. (Neftci, 2000)

These experiments are done mentally most of the time. When an attempt is practiced, the output is called the sample space, and we will define it as Ω . In the stock markets it can be accepted as a world case, known by all acceptable cases that are available. The number of cases that affect the financial market is big. Those will include all the possible costs for the vector parameter which describe the world and is always wide. (Mörters & Peres, 2010).

For some simple experiments the sample space is much smaller. For instance, flipping a coin will produce the sample space with two states $\{H, T\}$, while rolling a die yields a sample space with six states. Choosing randomly a number from 0 to 1 corresponds to a sample space which is the entire segment $[0, 1]$.

3.2 Events and Probability

The chance of incidence of an experience is measured by a probability function (N.Neftci, 16 Dec 2013)

$P: F \rightarrow [0, 1]$ which satisfies the following two properties:

- $P(\Omega) = 1$;

- For any mutually disjoint events $A_1, A_2, \dots \in F$, $P(A_1 \cup A_2 \cup \dots) = P(A_1) + P(A_2) + \dots$
- The triplet (Ω, F, P) is called a probability space. This is the main setup in which the probability theory works.

Example 1.3.1 In the case of flipping a coin, the probability space has the following elements:

- $\Omega = \{H, T\}, F = \{\phi, \{H\}, \{T\}, \{H, T\}\}$
- P defined by $P(\phi) = 0, P(\{H\}) = \frac{1}{2}, P(\{T\}) = \frac{1}{2}, P(\{H, T\}) = 1$

3.3 Probability Space

Definition 3.3.1 A probability space is a measure space with total measure one.

The standard notation is (Ω, F, P) where:

- Ω is a set (occasionally called a sample space in simple probability). Elements of Ω are signified ω and are sometimes called outcomes.
- F is an σ -algebra of subsets of Ω . Sets in F are called events.
- P is a function from F to $[0, 1]$ with $P(\Omega) = 1$ and such that if $E_1, E_2 \in F$ are disjoint, we say “probability of E” for P (E).

$$\mathbb{P}\left[\bigcup E_j\right] = \sum_{j=1}^{\infty} \mathbb{P}[E_j] \quad (3.1)$$

3.4 Distribution Functions

3.4.1 Introduction

Let X be a random variable on the probability space (Ω, F, P) . The distribution function of X is the function $f(x): R \rightarrow [0, 1]$ defined by $F(x) = P(\omega: X(\omega) \leq x)$ (Privault, 2013)

It is worth observing that since X is a random variable, then the set $\{ \omega: X(\omega) \leq x \}$ belongs to the information set F .

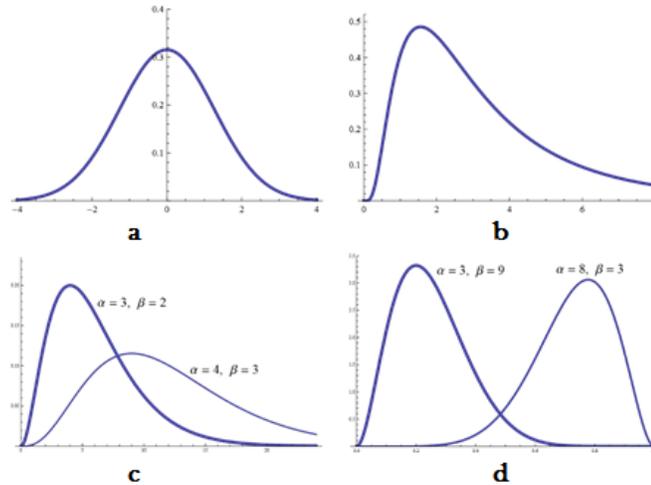


Figure 1: a) Normal distribution; b) Log-normal distribution; c) Gamma distribution; d) Beta distributions

$$\lim_{x \rightarrow -\infty} F_X(x) = 0 \quad (3.2)$$

$$\lim_{x \rightarrow \infty} F_X(x) = 1 \quad (3.3)$$

If we have $\frac{d}{dx} F_X(x) = p(x)$, then we say that $p(x)$ is the probability density function of

X . a useful property which follows from the Fundamental Theorem of Calculus is

$$P(a < X < b) = P(\omega; a < X(\omega) < b) = \int_a^b p(x) dx \quad (3.4)$$

3.4.2 Basic Distributions

We shall recall a few basic distributions, which are most often seen in applications.

Normal distribution: a random variable X is said to have a normal distribution if its probability density function is given by:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (3.5)$$

, with μ and $\sigma > 0$ constant parameters, see Fig.1.1a. The mean and variance are given by $E(X) = \mu, Var[X] = \sigma^2$. If X has a normal distribution with mean μ and variance σ^2 , we shall write $X \sim N(\mu, \sigma^2)$. (C.Hull, 2005)

3.4.3 Independent Random Variables

Roughly speaking, two random variables X and Y are independent if the occurrence of one of them does not change the probability density of the other. More precisely, if for any sets $A, B \subset R$, the events $\{\omega; X(\omega) \in A\}$, $\{\omega; Y(\omega) \in B\}$ are independent, then X and Y are called independent random variables. (C.Hull, 2005)

Proposition 3.4.3.1 Let X and Y be independent random variables with probability density functions $P_X(X)$ and $P_Y(Y)$. Then the joint probability density function of (X, Y) is given by $P_{X,Y}(X, Y) = P_X(X)P_Y(Y)$ (C.Hull, 2005)

3.5 Brownian Motion

The Brownian motion is one of the most used stochastic processes. It has another name: the Wiener process.

3.5.1 Definition of the Brownian Motion

Below, we give its definition.

Definition 3.5.1 A Brownian motion process is a stochastic process $W_t, t \geq 0$, which satisfies

- $W_0 = 0$, starting at 0,
- W_t has stationary, independent increments

- The process W_t is continuous in t ; the increments $W_t - W_s$ are normally distributed with mean zero and variance $|t - s|$, $W_t - W_s \sim N(0, |t - s|)$.
- The process $x_t = x + W_t$ has all the properties of a Brownian motion that starts at x since $W_t - W_s$ is stationary, its distribution function depends only on the time interval $t - s$, i.e.

$$P(W_{t+s} - W_s \leq a) = P(W_t - W_0 \leq a) = P(W_t \leq a) \quad (3.6)$$

It is worth noting that even if W_t is continuous, it is nowhere differentiable. From condition 4 we get that W_t is normally distributed with mean $E[W_t] = 0$ and $Var[W_t] = t$, $W_t \sim N(0, t)$.

This suggests also that the second moment is $E[W_t^2] = t$. Let $0 < s < t$. Since the increments are independent, we can write:

$$\begin{aligned} E[W_s W_t] &= E[(W_s - W_0)(W_s - W_t) + W_s^2] = E[W_s - W_0]E[W_s - W_0] + E[W_s^2] \\ &= s \end{aligned} \quad (3.7)$$

Consequently, W_s and W_t are not independent.

3.5.2 Geometric Brownian Motion

The geometric Brownian motion used usually to describe the underlying asset price trajectory. Its SDE is given by

$$dX_t = \alpha X_t dt + \sigma X_t dW_t \text{ or } \frac{dX_t}{X_t} = \alpha dt + \sigma dW_t$$

A process with a constant expected return over time and a constant variance of return.

This is a simplified but more natural model of stock prices:

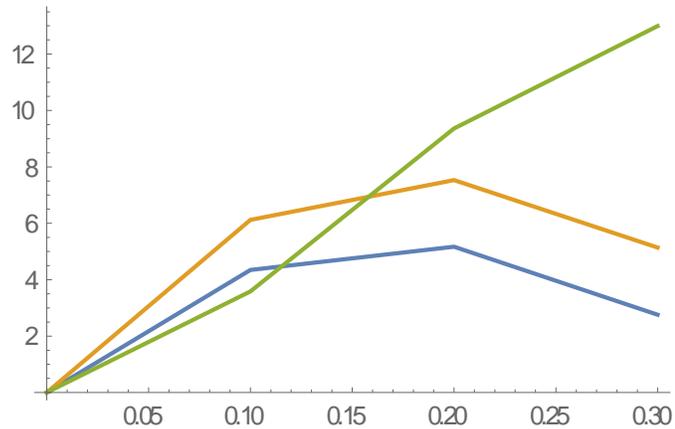


Figure 2: Geometric Brownian motion process with drift μ , volatility σ , and initial value S_0 in difference values

3.6 Stochastic Differential Equation of the Stock Price

We are going to assume that the stochastic process that generates stock prices is as follows:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (3.8)$$

where

- μ is the expected return per time unit
- σ is the standard deviation of return per time unit.
- W_t represents a Wiener process

One way of looking at Equation (3.8) is that the first term is the deterministic share, while the second term is the stochastic or random component. We could assume a number of other stochastic processes that would imply different price processes for stocks. For example, a mean reverting process could be written as

$$dS_t = \mu(100 - S_t)dt + \sigma S_t dW_t \quad (3.9)$$

Here, the assumption is that, as the price moves away from 100, the drift component makes the change in price move back toward 100.

The assumptions for the equation reflect different assumptions about how stock prices move. As you will see, defining how values change over time is an important part of option pricing.

3.7 Ito's Lemma

If $x(t)$ is described by Equation (3.8) and there is another function that depends on $x, y(x, t)$, then what is the distribution or the functional form of $y(x, t)$ In option pricing, we have the option price, $y(x, t)$, which depends on the stock price, $x(t)$. In order to examine this question, we must know the following theorem:

Ito's Lemma: If $y(t) = y(x, t)$ is a continuous differentiable function of t and twice differential in x , and x satisfies Equation 1, then $y(t)$ satisfies the following stochastic differential equation:

$$dy = \left(y_t + y_x f(x, t) + \frac{1}{2} y_{xx} \sigma^2(x, t) \right) dt + y_x \sigma dW_t, \quad (3,10)$$

where:

- $y_t \equiv$ the first derivative of y with respect to time t
- $y_x \equiv$ the first derivative of y with respect to x , and
- $y_{xx} \equiv$ the second derivative of y with respect to x .

The evidence of this suggestion is very difficult and far beyond anything we really need for this note. However, the following examples demonstrate how Ito's Lemma is applied and how stochastic differential equations are solved.

Examples of Ito's Lemma

1. Consider an Ito process $y_t = W_t^2$ for $t \geq 0$, where W_t is a standard Brownian motion. Find dY_t . We first identify X_t then $f(\cdot)$ and finally compute f_x, f_{xx} , and f_t in order to use the lemma.

$$\alpha_t = 0, \sigma_t = 1 \text{ and } X_t = W_t \text{ so } dX_t = dW_t$$

$$Y_t = f(X_t, t) = X_t^2 \text{ and } f: \mathfrak{R} \times [0, t] \rightarrow \mathfrak{R} \quad \forall X_t \in \mathfrak{R},$$

then

$$f_x(X_t, t) = 2X_t \quad f_{xx}(X_t, t) = 2 \text{ and } f_t(X_t, t) = 0,$$

and we arrive at

$$\begin{aligned} dY_t &= \left[f_x(X_t, t)\alpha_t + f_t(X_t, t) + \frac{1}{2}f_{xx}(X_t, t)\sigma^2 \right] dt + f_x(X_t, t)\sigma_t dW_t \\ &= \left[2X_t \cdot 0 + 0 + \frac{1}{2}2 \cdot 1^2 \right] dt + 2X_t \cdot 1 \cdot dW_t \\ &= 1 \cdot dt + 2W_t dW_t \text{ since } X_t = W_t. \end{aligned}$$

So Y_t is an Ito process with a drift (α) of 1 and a diffusion (σ) of $2W_t$.

2. $dX_t = \alpha dt + \sigma dW_t$ an arithmetic Brownian motion, can also be written

$$X_t = \alpha t + \sigma W_t \text{ (Since } \sigma \text{ is a constant). Consider a process defined by } S_t = S_0 \exp(X_t),$$

with $S_0 > 0$

$$S_t = f(X_t, t), \text{ where the function } f: \mathfrak{R} \times [0, T] \rightarrow \mathfrak{R} \text{ is defined by } f(X_t, t) = S_0 \exp(X_t).$$

$$f_x = S_0 \exp(X_t), \quad f_{xx} = S_0 \exp(X_t), \quad f_t = 0,$$

$$dS_t = \left[S_0 \exp(X_t) \cdot \alpha + \frac{1}{2} S_0 \exp(X_t) \sigma^2 \right] dt + S_0 \exp(X_t) \cdot \sigma \cdot dW_t$$

$dS_t = \left[\left(\alpha + \frac{1}{2} \cdot \sigma^2 \right) S_t \right] dt + \sigma S_t dW_t$, and we see that S is a geometric Brownian motion. X_t is normally distributed so $\ln(S_t) + X_t$ is normal with mean $\ln(S_0) + \alpha t$ and variance $\sigma^2 t$. The log of S has normal increments so S has lognormal increments.

Chapter 4: Constant-Elasticity-of-Variance Model

Modeling the underlying asset price is very important for solving the option-pricing problem. In fact, the price of the option depends essentially on the underlying asset price. One of the most important models is the Black-Scholes model because of its simplicity. In this chapter, we deal with the Black and Scholes model (F.Black, 1973) and the CEV model of (Cox, J.C., and S.A. Ross, 1976).

4.1 Black and Scholes Model

The Black-Scholes formula is widely used in practice. However, it has several shortcomings. In the next subsection, we present the Black-Scholes model and formula.

4.1.1 Black-Scholes Formula

The Black-Scholes model is a mathematical model of a financial market containing certain derivative investment instruments. It explains how volatility can be either estimated from historical data or implied from option prices using the model. It is widely used by options market participants.

The Black-Scholes model also assumes stocks move in a manner referred to as a random walk at any given moment, they are as likely to move up as they are to move down. These assumptions are combined with the principle that options pricing should provide no immediate gain to either seller or buyer. The Black-Scholes formula: (F.Black, 1973)

$$S_{n+1} = S_n e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \varepsilon \sigma \sqrt{\Delta t}} \quad (4.1)$$

In Black-Scholes model is principally a formula that is used to calculate option values. The Black-Scholes formula consists of three parts. The main equation and two formulas for calculating parameters.

$$C(S, T) = S\Phi(d_1) - Ke^{-rT}\Phi(d_2) \quad (4.2)$$

This part of the Black-Scholes formula tells us that the price of a European-style call option with expiration date in time T written on stock S is equal to the price of the stock adjusted for volatility, interest rate, and spread minus present value of the stock delivery price (or a strike price) also adjusted for volatility, interest rate, and spread.

The parameters d_1 and d_2 in the Black-Scholes formula can be calculated the following way:

$$d_1 = \left(\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T \right) / \sigma\sqrt{T}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

The d_1 and d_2 are parameters to the Φ in the first equation. (D.M.Kereps, 1979)

Phi represents a cumulative distribution function of Normal distribution. In layman terms, we calculate the parameters d_1 and d_2 and look up a corresponding tabularized value in a book, and then we plug those values back into the first formula. The Black-Scholes formula for a European-style put option is very similar to the Black-Scholes formula for a call option. It is the following:

$$P(S, T) = Ke^{-rT}\Phi(-d_2) - S\Phi(-d_1) \quad (4.3)$$

This Black-Scholes formula tells us that a value of a put option can be calculated as a present value of the stock delivery price minus the price of the stock, both adjusted for volatility, interest rate, and spread.

4.1.2 Black-Scholes Formula Calculation Example

The Black-Scholes formula is used to calculate the value of an option. We can demonstrate the working of the Black-Scholes formula on an example.

Let us assume that the current price of shares of company XYZ is \$100 and you would like to get an option to purchase one share of XYZ company stock for \$95. The option expires in three months. We also assume that the stock pays no dividends. The standard deviation of the stock's returns is 50% per year, and the risk-free rate is 10% per year, we can calculate the value of the option as follows:

$$d1 = [\ln(\$100/\$95) + (0.10 + 0.25/2) * 0.25] / 0.50 * \sqrt{0.25} = 0.43$$

$$d2 = 0.43 - 0.50 * \sqrt{0.25} = 0.18$$

$$N(0.43) = 0.6664$$

$$N(0.18) = 0.5714$$

Plugging these parameters into the formula, we get:

$$\begin{aligned} C(S,T) &= \$100 * 0.6664 - \$95 * e^{- (0.10 * 0.25)} * 0.5714 \\ &= 66.64 - 52.94 = \$13.70 \end{aligned}$$

You can go to our Black-Scholes formula option value on-line calculator page to run some calculations and verify this result.

This Black Scholes formula is used to evaluate a call option. It can be used as a price sticker's technique, to evaluate it we can count it from scratch similar to the previous action using the (PST) model put or reevaluate the Black – Scholes model – Using the call parity technique to evaluate the put option value. (P.Protter, 1990) . This put – call parity will solved as given in the equation as the following example

$$\begin{aligned} P(S,T) &= C(S,T) + B(T) - S(T) = C(S,T) + X * e^{-rT} - S(T) \\ &= \$13.70 + \$95 * e^{- (0.10 * 0.25)} - \$100 = \$6.35 \end{aligned}$$

4.1.3 Black-Scholes Model Work in the Real World.

The Black-Scholes model is a mathematical model based on the notion that prices of stock follow a stochastic process. This model was developed by Fischer Black and Myron Scholes in 1973 (M.Schroder, 1989). Robert Merton also participated in the model's creation; hence that is why the model is sometimes referred to as the Black-Scholes-Merton model. In addition to that, it is a mathematical model of a financial market containing certain derivative instruments that explains how volatility can be either estimated from historical data or implied from option prices using the model.

4.1.4 The Assumptions behind the Black-Scholes Model

There are several assumptions underlying the Black-Scholes model. One of these assumptions is constant volatility which is a measure of how much a stock can be expected to move in the near-term, is a constant overtime (F.Black, 1973) .

The Black-Scholes model assumes stocks move in a manner referred to as a random walk. Furthermore, interest rates constant and known is another assumption through which the Black-Scholes model uses the risk-free rate to represent this constant and known rate. Moreover, the Black-Scholes model assumes that returns on the underlying stock are normally distributed and it also assumes European-style options which can only be exercised on the expiration date. American-style options can be exercised at any time during the life of the option, making American options more valuable due to their greater flexibility.

4.2 Numerical Methods for Stochastic Differential Equation

This section is based on chapter 6 from (Hirsa, 2013).

Let $(X_t)_{0 \leq t \leq T}$ be a stochastic process that satisfied the following SDE:

$$dX_t = a(X_t, t)dt + b(X_t, t) dW_t,$$

a and b are drift and the diffusion coefficient respectively. The assumption is that μ and σ are defined and measurable.

The Itô Taylor expansion is

$$\begin{aligned} X_t = X_{t_0} + \mu(X_{t_0}) \int_{t_0}^t ds \\ + \sigma(X_{t_0}) \int_{t_0}^t dW(s) + \frac{1}{2} \sigma(X_{t_0}) \sigma'(X_{t_0}) ([W(t) - W(t_0)]^2 - (t - t_0)) \\ + R, \end{aligned} \tag{4.4}$$

where R is the remainder .

When we simulate an SDE we generate samples of the discretized version of SDE at a finite number of points

$$X_{\Delta t}, X_{2\Delta t}, \dots, X_{m\Delta t}, \tag{4.5}$$

where m is the number of time steps and Δt is the time step assuming equidistant

subintervals, $\Delta t = \frac{T-t_0}{m}$. To write it more formally

$$X_{t_1}, X_{t_2}, \dots, X_{t_j}, \dots, X_{t_m},$$

where $t_j = t_0 + j\Delta t = j\Delta t$ for $j = 1, 2, \dots, m$. As Δt approaches zero, our discretized

path will converge toward the theoretical continuous – time path. For the interval $[t_j, t_{j+1}]$

, by choosing

$$t_0 = t_j,$$

$$t = t_{j+1}$$

$$\Delta t = t_{j+1} - t_j$$

$$\Delta W_j = W(t_{j+1}) - W(t_j)$$

We get the following expression for (4.4)

$$X_{t_{j+1}} = X_{t_j} + \mu(X_{t_j})\Delta t + \sigma(X_{t_j})\Delta W_j + \frac{1}{2}\sigma(X_{t_j})\sigma'(X_{t_j})((\Delta W_j)^2 - \Delta t) + R \quad (4.6)$$

There are various schemes for simulating SDEs of this form, and the most common ones are

- Euler scheme
- Milstein scheme
- Range – Kurta scheme

4.2.1 Euler Scheme

The Euler scheme is the simplest discretization scheme a variable for discretizing SDEs.

Keeping the first three terms in equation (4.6) gives us the explicit Euler method

$$\begin{aligned} X_{t_{j+1}} &= X_{t_j} + \mu(X_{t_j}, t_j)\Delta t + \sigma(X_{t_j}, t_j)\Delta W_j \\ &= X_{t_j} + \mu(X_{t_j}, t_j)\Delta t + \sigma(X_{t_j}, t_j)\sqrt{\Delta t}Z_j \end{aligned}$$

4.2.2 Milstein Scheme

The Milstein scheme improves upon the Euler discretization by adding a second diffusion term, expanding the diffusion term to $O(\Delta t)$. This method is obtained by simply keeping all terms of $O(\Delta t)$ in equation (4.6), that is

$$X_{t_{j+1}} = X_{t_j} + \mu(X_{t_j}, t_j)\Delta t + \sigma(X_{t_j}, t_j)\Delta W_j + \frac{1}{2}\sigma(X_{t_j})\sigma'(X_{t_j})((\Delta W_j)^2 - \Delta t)$$

So while the Milstein scheme has a higher order in discretization, it requires knowing that the first derivative of the volatility functions.

4.2.3 Range – Kurta Scheme

While the Milstein scheme improves on the accuracy of the Euler scheme, it requires both knowledge of the first derivative of the volatility function, which may not be available at all or may be expensive to compute. The Range – Kurta scheme allows to avoid using the first derivative of the volatility function, by using the Range – Kurta approximation.

We have the following Range – Kurta scheme

$$\hat{X}_i = X_i + \mu(X_i)\Delta t + \sigma(X_i)\sqrt{\Delta t}$$

$$X_{i+1} = X_i + \mu(X_i)\Delta t + \sigma(X_i)\Delta W_i + \frac{1}{2\sqrt{\Delta t}}[\sigma(\hat{X}_i) - \sigma(X_i)]((\Delta W_i)^2 - \Delta t)$$

4.3 Simulation of the Black-Scholes Model

The Brownian motion model states that the value S of a security follows the stochastic process

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

The Exact solution of Black and Scholes model is:

$$S_t = S_0 e^{[(r - \frac{\sigma^2}{2})t + \sigma W(t)]}$$

Euler-Maruyama method For $n = 0, 1, 2, \dots, N - 1$

$$S_{n+1} = \mu S_n dt + \sigma S_n \Delta W_n$$

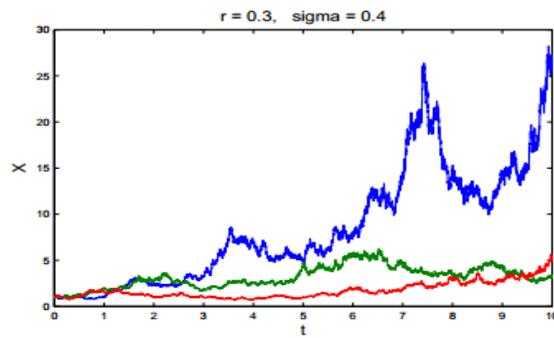


Figure 3: Geometric Brownian motion and Euler method

4.4 Figures of EM Approximation and Exact Solution

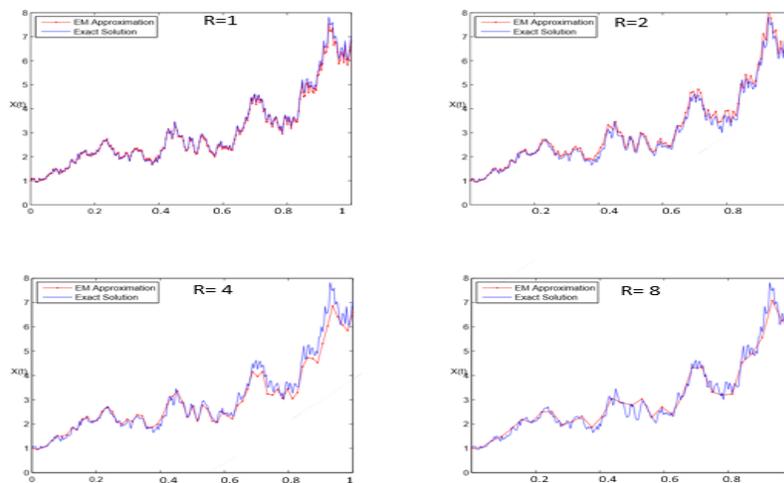


Figure 4: The effect of increasing the step size in the Euler-Maruyama method

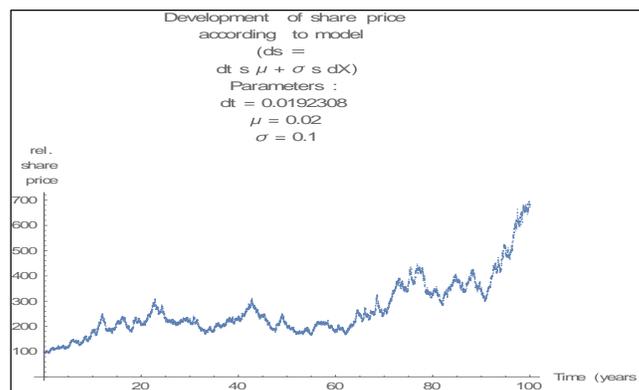


Figure 5: Black-Scholes Model in different prices

4.5 CEV Model

4.5.1 Introduction

As before the CEV model has won an important attention because it allows the price change of stock goes back to be time change to mend the product that the present price change based on the present product prices and not on the chosen differences.

The standard CEV model assumes that share price S_t is a solution to the following SDE:

$$dS_t = rS_t dt + \sigma S_t^{\frac{\beta}{2}} dW_t \quad (4.4)$$

Where r, β, σ are constant and with initial condition $W_0 = 0$. The CEV diffusion process has two volatility related parameters, the volatility coefficient σ , and the elasticity of variance β , which is assumed to be time-invariant. Looking at (4.4), we can observe that the GBM process assumed by Black and Scholes is a special case of the CEV process. In particular, if $\beta = 2$, the process is equivalent to the lognormal diffusion and continuously compounded returns will follow a stationary normal distribution. Cox considers the cases where $\beta < 2$, and (D.C.Emacuel, J.D.MacBeth, 1982) have extended the model to price options for those cases where $\beta > 2$. As a result, the option price over an underlying asset whose price is a solution to the SDE (4.4) is known for any value of β .

4.5.2 The CEV Evolution Process of Share Prices

CEV model has increased considerable attention because it licenses the volatility of stock revenues to be time-varying but still keeps the property that the present volatility depends only on the current stock price and not on any other random variable. There are also some other advantageous structures of the model that has prepared it famous. First, if stock prices follow a CEV process with $\beta < 2$, then we have an inverse relationship between the stock price and the variance of the stock's instantaneous rate of return. Thus, for a CEV

process with $\beta < 2$, the instantaneous variance of stock returns decreases as stock price increases. This is consistent with an empirical observation that the volatility of stock returns has an inverse relation with stock prices. For the Black-Scholes case with $\beta = 2$. For the CEV process with $\beta > 2$, the relationship between the instantaneous variance of stock returns and stock price is positive. This is again not consistent with the empirical findings.

4.5.3 The Exact Solution of CEV Option Pricing Model

When $\beta < 2$, the no dividend-paying CEV call pricing formula is as follows: (Schroder, 1989)

$$C = S \left[\sum_{n=0}^{\infty} g(S' | n+1) G(K' | n+1 + \frac{1}{2-\beta}) \right] - Ke^{-r\tau} \left[\sum_{n=0}^{\infty} g(S' | n+1 + \frac{1}{2-\beta}) G(K' | n+1) \right]$$

When $\beta > 2$, the CEV call pricing formula is as follows:

$$C = S \left[1 - \sum_{n=0}^{\infty} g(S' | n+1 + \frac{1}{2-\beta}) G(K' | n+1) \right] - Ke^{-r\tau} \left[1 - \sum_{n=0}^{\infty} g(S' | n+1) G(K' | n+1 + \frac{1}{2-\beta}) \right]$$

where

$$S' = \left[\frac{2re^{r\tau(2-\beta)}}{\delta^2(2-\beta)(e^{r\tau(2-\beta)} - 1)} \right] S^{2-\beta}$$

$$K' = \left[\frac{2r}{\delta^2(2-\beta)(e^{r\tau(2-\beta)} - 1)} \right] K^{2-\beta}$$

$$g(x | m) = \frac{e^{-x} x^{m-1}}{\Gamma(m)} \text{ is the gamma density function}$$

$$G(x | m) = \int_x^{\infty} g(y | m) dy$$

C is the call price; S, the stock price; τ , the time to maturity; r, the risk-free rate of interest; K, the strike price; and β and δ , the parameters of the formula.

4.5.4 Simulation Method for CEV

In the case of GBM, where the share price S_T could be written as a function of a standard normal variable, using Mathematica we could simulate the time series $\{S_t\}$ directly. In order to simulate the time series $\{S_t\}$, it will be necessary to compute an approximate solution of the continuous time SDE (4.4). This method discretizes the continuous time SDE to:

$$S_{t_i} - S_{t_{i-1}} \approx \mu S_{t_{i-1}} (t_i - t_{i-1}) + \sigma S_{t_{i-1}}^{\frac{\beta}{2}} (W_{t_i} - W_{t_{i-1}}), \quad (4.5)$$

where the interval of interest $[t, T]$ is divided into n subintervals:

$$t = t_0 < t_1 < \dots < t_i < \dots < t_{n-1} < t_n = T$$

It not necessarily of equal length. We can rewrite the Brownian motion increment $W_{t_i} - W_{t_{i-1}}$ from (4.10) in terms of the standard normal variable using the following property of Brownian motion:

$$W_{t_i} - W_{t_{i-1}} \sim N(0, t_i - t_{i-1})$$

The above property implies that:

$$W_{t_i} - W_{t_{i-1}} = \sqrt{t_i - t_{i-1}} Z_i$$

Where Z_i are independent $N(0,1)$ variables, and the relationship between the share prices at time

t_i and the share price at time t_{i-1} can be written as:

$$S_{t_i} \approx: S_{t_{i-1}} + rS_{t_{i-1}}\Delta_i + \sigma S_{t_{i-1}}^{\frac{\beta}{2}} \sqrt{\Delta_i} Z_i \quad (4.6)$$

For $i = 1, 2, \dots, n$, where $\Delta_i = t_i - t_{i-1}$ for any $t_i > t_{i-1}$. To sum up, using this Euler scheme, an individual share price at a particular period can be computed by first calculating the share price from the last period.

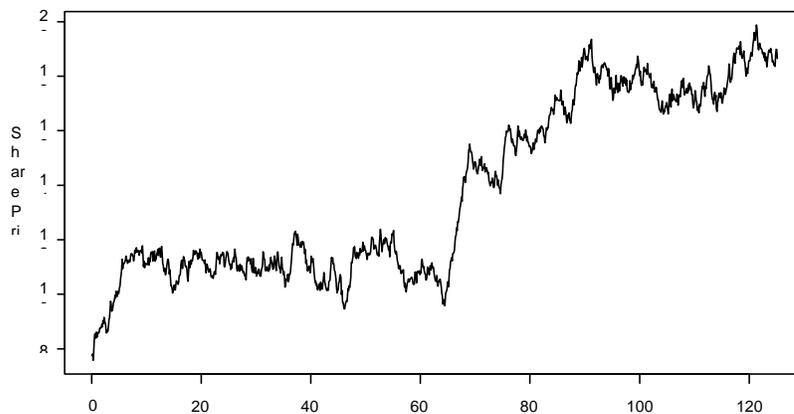


Figure 6: Realization of CEV share prices with CEV parameter $\beta=0.67$ and additional parameters $S_t=\$7.753$, $\Delta\tau=5$ years, $\mu=0.1$, $\sigma=1.2$ and $n=1250$ subintervals

4.5.5 The CEV Option Pricing Model

The formula for the price of a European option over a stock whose share price evolves according to the SDE can be developed using the risk-neutral pricing. The risk-neutral method is appropriate because the CEV option pricing formula is derived independently of investor risk preferences. Recall that the value of a European call option at t is:

$$C_t = E_t[\max(S_T - K)e^{-rt}]$$

where the discounting is performed using the risk-free rate r , and E is an expectation operator taken in a risk-neutral world, in which the underlying stock price evolves according to the following risk-neutral CEV process:

$$dS_t^* = rS_t^*dt + \sigma S_t^{*\frac{\beta}{2}}dW_t^*$$

subject to the initial condition $S_t^* = S_t$

To obtain the pricing formula, we need to determine the resulting distribution of the stock price at the option's payoff date, and derive the form of $E_t[\max(S_T - K)]$.

However this is not as easy as seen in the Black-Scholes case. An excellent detailed description of the procedure can be found in Randal (1998). The pricing formula for a European call option whose underlying asset's price evolves according to the SDE.

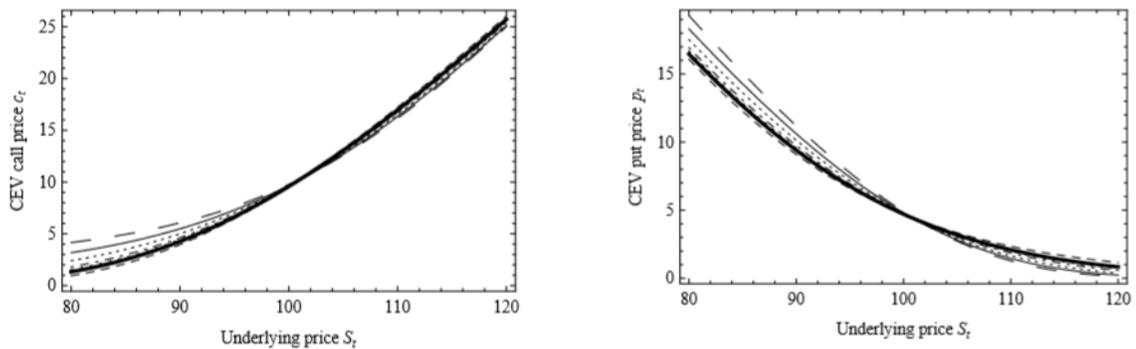


Figure 7: European call and put option prices under CEV processes

Chapter 5: Constant-Elasticity-of-Variance Model during High Volatile Periods

A model that describes perfectly the behavior of a financial asset price does not exist. New model or improvement of the existing uses the stylized facts: A stylized fact is a term used in economics to refer to empirical findings that are so consistent that they are accepted as truth. We suggest a modified CEV model that has an increased volatility.

Is the suggested model an improvement?

Apparently, yes, since the suggested model satisfies the stylized fact: during crisis, the volatility increases.

How to solve the option-pricing problem?

In this work, we use Monte Carlo method to find a numerical solution.

5.1 The CEV High Volatility Model

As in the Black-Scholes and the CEV model, we assume that the market is living in a probability space (Ω, F, P) , W_t is a standard Brownian motion, $F = F_T$ where (F_t) is the filtration generated by the Brownian motion. We add an additional constant parameter γ that designates the increase in the volatility during a financial crisis. The market has two assets. The risk-free asset price is following:

$$dA_t = rA_t dt, \quad t \in [0, T], \quad A_0 = 1$$

The price of the risky asset S_t under the risk-neutral probability is driven by the following SDE:

$$dS_t = rS_t dt + \left(\sigma S_t^{\frac{\beta}{2}} + \gamma \right) dW_t, \quad t \in [0, T], \quad S_0 = x > 0$$

Here r, β, σ and γ are all constant. γ is a new parameter related to the increase in the volatility.

Using the risk-neutral valuation, one can see that the price of the European call option with strike C is

$$C = E[\max(S_T - K, 0)]e^{-rT}$$

The above expected value is under the risk-neutral probability.

5.2 The Numerical Solution for in CEV Model with High Volatility

We assume that the stock price S_t is driven by the stochastic differential equation (SDE)

$$dS_t = rS_t dt + (\sigma S_t^{\beta/2} + \gamma)dW_t, \quad S_0 > 0, \quad (5.8)$$

where W_t is Brownian motion. We simulate S_t over the time interval $[0, T]$, which we assume to be discretized as $0 = t_1 < t_2 < \dots < t_m = T$, where the time increments are equally spaced with width dt : Equally-spaced time increments is primarily used for notational convenience, because it allows us to write $t_i - t_{i-1}$ as simply dt .

A discrete model for change in the price of a stock over a time interval $[0, T]$ is

$$S_{n+1} = S_n + rS_n \Delta t + \left(\sigma (S_n)^{\beta/2} + \gamma \right) \Delta W_t, \quad S_0 = s \quad (5.9)$$

$$\Delta W_t \sim N(0, \sqrt{\Delta t}) \text{ or } \sqrt{\Delta t} N(0, 1), \quad (5.10)$$

where $S_n = S_{t_n}$ is the stock price at time $t_n = n\Delta t, n = 0, 1, \dots, N - 1, \Delta t = T/N$ is the annual growth rate of the stock, and σ is a measure of the stocks annual price volatility. Highly volatile stocks have large values of σ .

5.3 Simulation Monte Carlo Method of the Option Price

We then find numerical solution for using Monte Carlo method. The price S_T can be obtained by applying the Euler scheme techniques for the stochastic differential equation.

After, we compute the payoff= $\max(S_T - K, 0)$.

The next step is to compute the mean of the resulting payoffs. Finally we estimate the price of the option by discounting the mean of the payoff at the risk-free rate.

An ensemble

$$\{S_N^{(l)} = S^l(T), l = 1, \dots, M\}$$

of M stock prices at expiration is generated using the difference equation

$$S_{n+1}^{(l)} = S_n^{(l)} + rS_n^{(l)}\Delta t + \left(\sigma(S_n^{(k)})^{\frac{\beta}{2}} + \gamma \right) \Delta W_t,$$

$$n = 1 \quad S_1^{(l)} = S_0^{(l)} + rS_0^{(l)} + \left(\sigma(S_0^{(l)})^{\frac{\beta}{2}} + \gamma \right) \sqrt{\Delta t} N(0, 1)$$

$$n = 2 \quad S_2^{(l)} = S_1^{(l)} + rS_1^{(l)}\Delta t + \left(\sigma(S_1^{(l)})^{\frac{\beta}{2}} + \gamma \right) \sqrt{\Delta t} N(0, 1)$$

.....

$$n = N \quad S_N^{(l)} = S_{N-1}^{(l)} + rS_{N-1}^{(l)}\Delta t + \left(\sigma(S_{N-1}^{(l)})^{\frac{\beta}{2}} + \gamma \right) \sqrt{\Delta t} N(0, 1)$$

Consider the European call, whose value at expiration time T is $\max\{S_T - K, 0\}$, where

S_t is the price of the underlying stock and K is the strike price.

Now, the price of the option under the CEV model with high volatility model by using Monte Carlo method

$$C(S_0, T) = e^{-rT} E(\max\{S_T - K, 0\}) \approx e^{-rT} \sum_{l=1}^M \frac{(S_T(l) - K)}{M}$$

5.4 Results and Findings

By using Mathematica, it is very easy to create a sequence of random number. With this sequence, the equation (5.9) can then be used to simulate a sample path or trajectory of stock prices, $\{S_0, S_1, S_2, \dots, S_N\}$. For our purpose here, it has been shown as a relatively accurate method of pricing options and very useful for options that depend on paths.

5.4.1 Observation of the High Volatility from the Suggested High Volatility Model

The following figures show clearly that our suggested model takes into account the high volatility periods. Actually, when $\gamma=0$, the range of the asset price is [4:6] (figure 1), while when $\gamma=0.5$ the volatility increases and the range of the asset price is [3:8]. If we take $\gamma=1$, the volatility is increased and the asset price has values between 2 and 9.

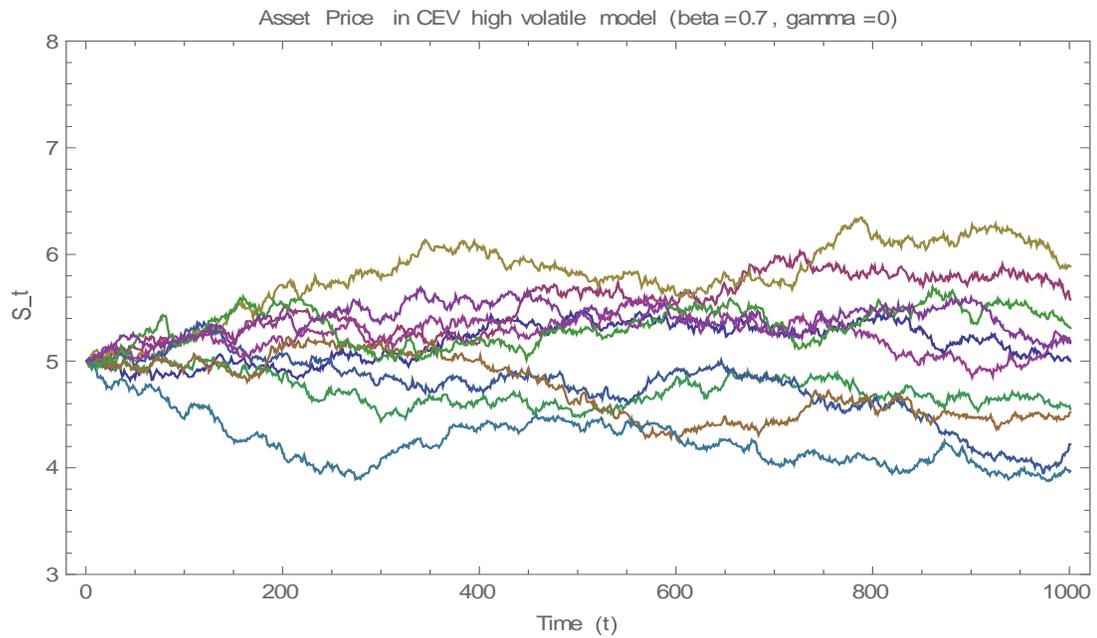


Figure 8: Asset Price in the normal CEV model with $r = 0.04$; $\text{sig} = 0.2$; $\text{bet} = 0.7$; $\text{gam} = 0$

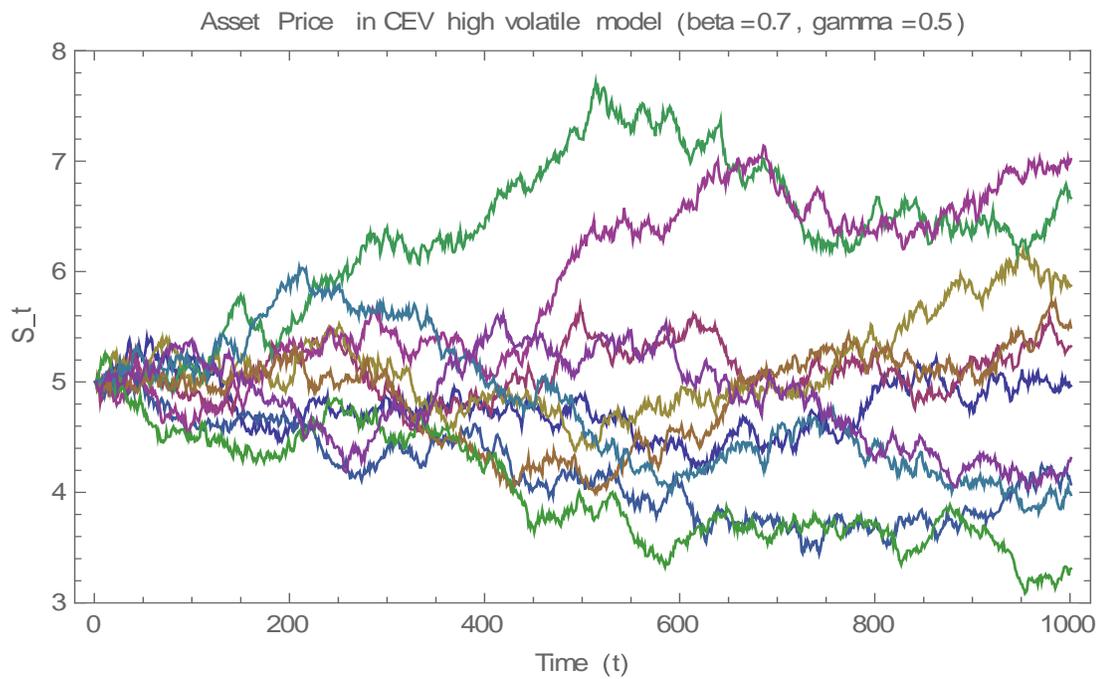


Figure 9: Asset Price in CEV with high volatility model (beta = 0.7, gamma= 0.5)

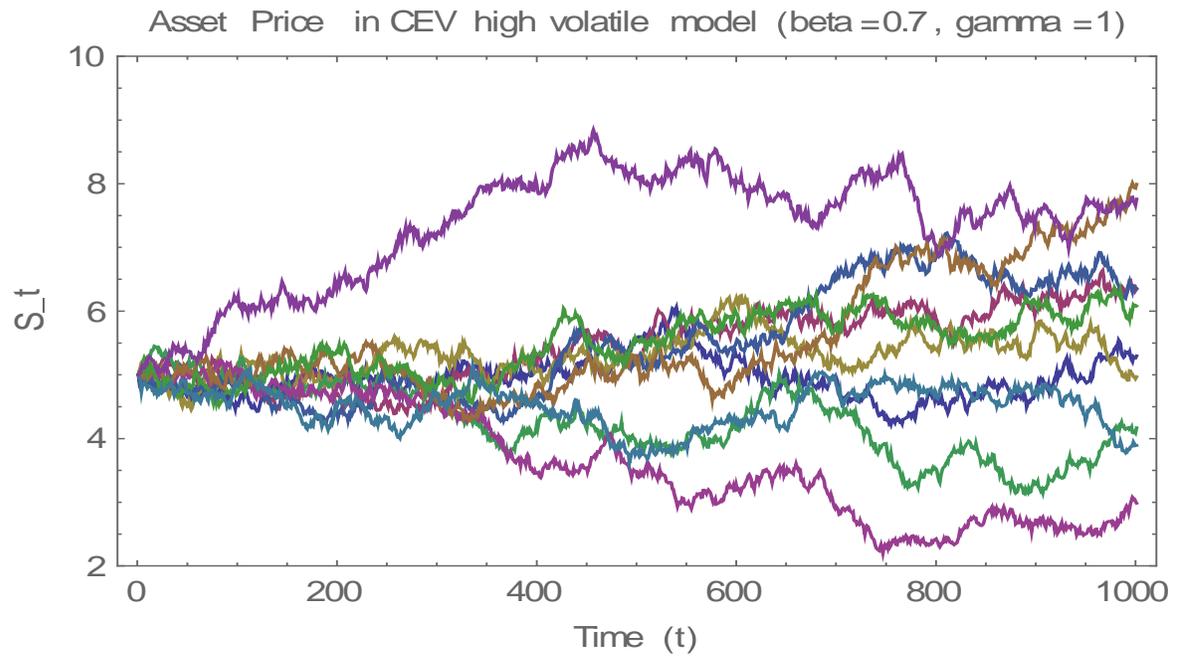


Figure 10: Asset Price in CEV with high volatility model ($\beta = 0.7$, $\gamma = 1$)

5.4.2 Comparison between the Options Price for the CEV and CEV High Volatile Models

The figure below shows that the price of European option under the CEV high volatile model (the line) is almost equal to the price of the CEV standard model (the dots). However, when the strike is higher than the spot, the option price for the high volatile model is more expensive than the option price of the regular CEV model.

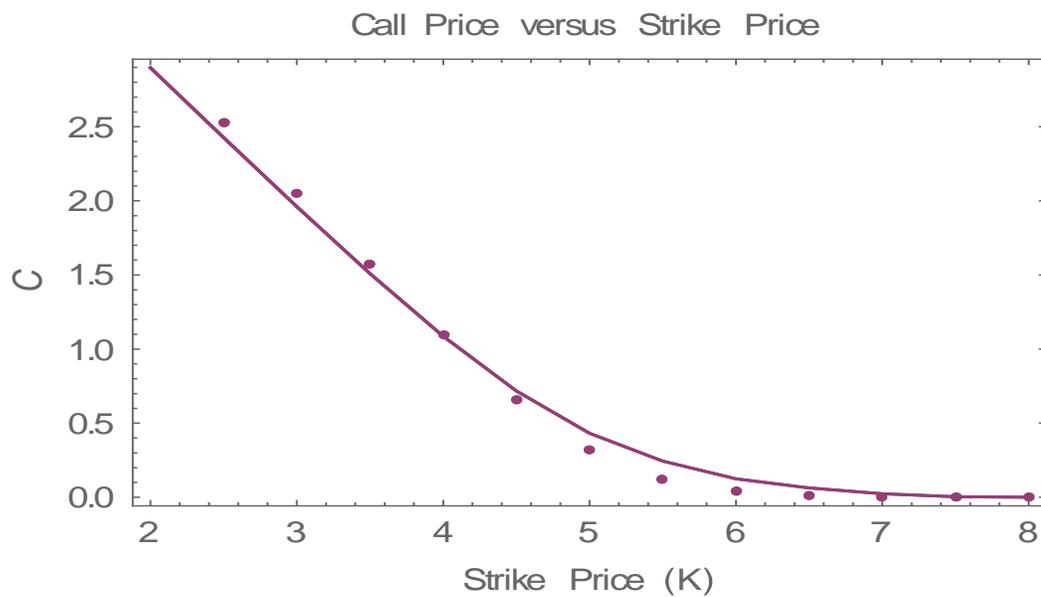


Figure 11: Compare between the option price in CEV model and CEV model volatile model

5.4.3 Observation of the Leverage effect from the suggested high

The suggested model conserves the leverage effect stylized fact. In fact the figure below shows clearly the inverse proportional relation between the underlying asset price (up) and its volatility (down).

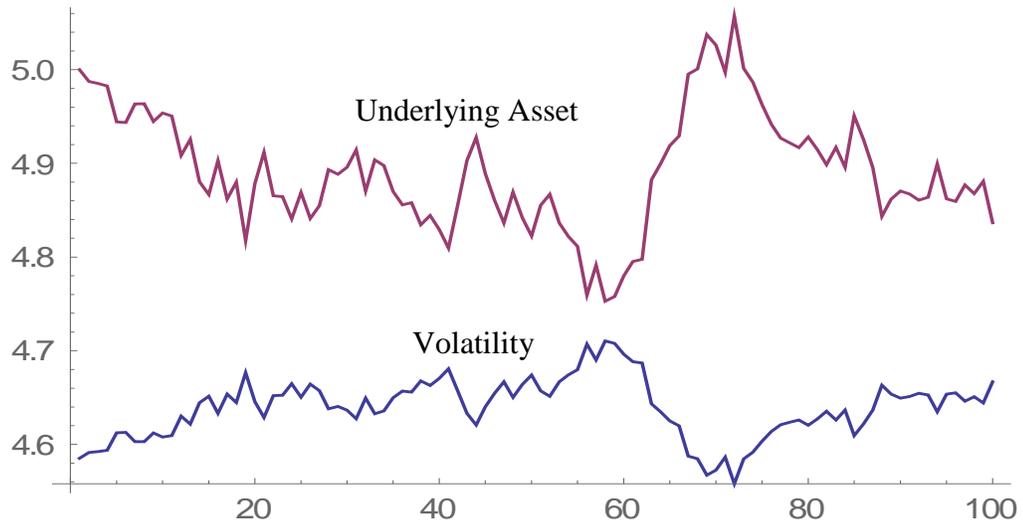


Figure 12: Relation between the underlying asset price (up) and its volatility (down)

Chapter 6: Conclusion

This thesis suggests a new model that generalizes the CEV model. The thesis provides numerical solutions for the probabilistic expression of the European option price under the modified CEV model.

After an introduction to financial derivatives in general and options in particular, the reader can find an elementary introduction to the stochastic calculus and to the options pricing theory. Then we deal with the Black & Scholes and the CEV models. A simulation of the SDE of the underlying asset prices is conducted for the two models. We provide several illustrations for the trajectory of the underlying asset price. They are favorable to the suggested model. In addition, we compare the asset price trajectory and the volatility of the modified model. The comparison shows clearly that the suggested model accounts for the leverage effect.

Moreover, the price of a European option under the modified CEV model is obtained using Monte Carlo methods. Numerical results for the option prices under the CEV and the high volatile models are obtained. They confirm the presence of an increased volatility in the modified CEV model.

The suggested model can be seen as an improved version of the CEV model, since it conserves its advantages (for instance, stochastic volatility and leverage effect). However, as a future direction of research, we need to calibrate the model so that it can be used in practice.

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Appendix

Mathematica code of CEV high volatile model

```

r = 0.04; sig = 0.2; bet = 0.7; gam = 0.5;

T = 1; (* Maturity time*)

NN = 1000; (*number of samples in a path from t=0 to T*)

h = T/NN; (* Time step size *)

KK = 100;(*number of paths to generate*)

(*define strike price *)

strikeprices = Table[2 + i 0.5, {i, 0, 12}];

CallPrices = Table[, {i, 0, 12}];

For[strikepriceindex = 1, strikepriceindex <= Length[strikeprices],

strikeprice = strikeprices[[strikepriceindex]];

SS = {};

For[k = 1, k <= KK, (* begin of path*)

S = {5};

(* generating 1001 independent normally distributed values Z_0,

Z_1, .... Z_1000 *)

U1 = RandomReal[1, 1001]; U2 = RandomReal[1, 1001];

R = -2 Log[U1]; V = 2 Pi U2;

Z = Sqrt[R] Cos[V];

(*For[i=1,i<=NN,temp=S[[i]](1+r h)+(sig S[[i]]^bet+gam)RandomReal[

NormalDistribution[0,h]];*)

For[i = 1, i <= NN,

```

```

temp = S[[i]] (1 + r h) + (sig S[[i]]^bet + gam) Sqrt[h] Z[[i]];
S = Join[S, {temp}];
i++;
SS = Join[SS, {S}];
k++; \[AliasDelimiter]
priceatmaturity = Table[, {index, 1, KK}];
For[index = 1, index <= KK,
priceatmaturity[[index]] = SS[[index]][[Length[S]]]; index++;
CallPrices[[
strikepriceindex]] = (Sum[
Max[priceatmaturity[[index]] - strikeprice, 0], {index, 1,
KK}]/KK) Exp[-r T];
Print["For strikeprice = ", strikeprice,
" the Call Price at Maturity is ",
CallPrices[[strikepriceindex]]];
strikepriceindex++;
];(* end of path*)
ListPlot[
Table[{strikeprices[[in]], CallPrices[[in]]}, {in, 1,
Length[strikeprices]}], Joined -> True]

```

Mathematica code of graph of CEV model with high volatile model

```
ListPlot[{Table[{strikeprices[[in]],CallPricese[[in]]},{in,1,Length[strikeprices]}],  
Table[{strikeprices[[in]],CallPricesm[[in]]},{in,1,Length[strikeprices]}]  
,Frame-> True,FrameLabel->{{C},{ "Strike Price (K)","Call Price versus Strike  
Price"}}
```
