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SHIFTED THIRD KIND CHEBYSHEV OPERATIONAL MATRIX TO SOLVE BVPS OVER INFINITE INTERVAL

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Abstract

The main purpose of this research is to solve boundary value problems (BVPs) with an infinite number of boundary conditions. By reducing the infinite interval to finite interval that is large and approximating the variable using finite difference method, the resulting boundary value problem is reduced to linear system of algebraic equations with unknown shifted third kind chebyshev coefficients. The applications are demonstrated via test examples.

Keywords: Shifted third Chebyshev, Operational Matrix, Boundary Value Problem, Infinite interval.

1. Introduction:

Boundary Value Problems on finite intervals appear often in applied mathematics and physics. More examples and collection of works on the existence of solution of boundary value problems on infinite interval for differential difference and integral equations may be found in monographs [1,2,3,4]. For some works and various techniques dealing with such boundary value problems see [5,6,7,8,9,10] and the references therein.

Consider the Tow Point BVPs on the form

\[ \dot{y} = f(x, y, \dot{y}) \quad (a, b) \quad \ldots(1) \]

\[ y(a) = A \quad , \quad y(\infty) = B \]

Where \( f(x, y, \dot{y}) \) are continuous functions before computing the solution, we plummet the infinite interval to finite, then we introduce a collocation method with shifted third chebyshev polynomials for solving (1). 

2. Some Properties of Third Chebyshev Polynomials and Their Shifted Ones:

[11,12,13]

The chebyshev polynomial \( V_r(x) \) of the third kind has trigonometric definitions involving
the half angle $\theta/2$ (where $x=\cos \theta$). It is the polynomials of degree $r$ in $x$ defined by:

$$V_r(x) = \frac{\cos(r+1)\theta}{\cos(2^r\theta)} \text{ where } x=\cos \theta \ldots (2)$$

the recurrence relation of $V_r(x)$ with initial condition for $r=1$

$$V_{r+1}(x) = 2xV_r(x) - V_{r-1}(x) \ldots (3)$$

with initial conditions $V_0(x) = 1$ , $V_1(x) = 2x - 1$ , $r=1,2,\ldots$.

Generally, we define shifted third kind chebyshev polynomials $V_r^*(x)$ appropriate to any given finite range $a \leq x \leq b$ of $x$ by making the interval correspond to the interval $-1 \leq x \leq 1$ of a new variable $s$ under the linear transformation:

$$S = \frac{2(x-a)}{b-a} - 1 \ldots (4)$$

The third kind polynomials appropriate to $a \leq x \leq b$ are thus given by $V_r^*(s)$, where $s$ is given in eq.(4),using this in conjunction with (3) yields

$$V_0^*(x) = 1$$

$$V_1^*(x) = 2 \left[\frac{2(x-a)}{b-a} - 1\right] - 1$$

$$V_2^*(x) = 4 \left[\frac{2(x-a)}{b-a} - 1\right]^2 - 2 \left[\frac{2(x-a)}{b-a} - 1\right] + 1$$

$$V_3^*(x) = 8\left[\frac{2(x-a)}{b-a} - 1\right]^3 \ldots (5)$$

with the recursive formula given as:

$$V_{r+1}^*(x) = 2 \left[\frac{2(x-a)}{b-a} - 1\right] V_r^*(x) - V_{r-1}^*(x) \ldots (5)$$

let $\Phi(x) = [V_0^*(x),V_1^*(x),\ldots,V_n^*(x)]^T$,then the operational matrix of the first derivative of shifted third kind chebyshev polynomial (SC3OMD) may be defined by:

$$\frac{d\Phi(x)}{dx} = E \Phi(x)$$

\ldots (6)

where $E=(e_{ij})_{0 \leq i,j \leq N}$ is the square operational matrix of order $(N+1)$ whose nonzero elements are given explicitly by:

$$e_{ij} = \begin{cases} \frac{2}{b-a} (i+j+1) & i > j, (i+j)\text{odd} \\ \frac{2}{b-a} (i-j) & i > j, (i+j)\text{even} \end{cases}$$

For example if $N=6$, then the operational matrix $E$ is given explicitly by:

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 & 0 & 0 \\ 4 & 4 & 2 & 0 & 0 & 0 \\ 6 & 4 & 8 & 2 & 0 & 0 \\ 6 & 8 & 12 & 8 & 2 & 0 \end{bmatrix} \ldots (7)$$

It can be easily shown that for any $n \in \mathbb{Z}^+$

$$\frac{d^n\Phi(x)}{dx^n} = E^n \Phi(x) \ldots (8)$$

$n=0,1,2,\ldots$

3. SC3OMD Collocation Method

Let us consider BVPs (1), first stage we replaced the boundary condition at infinite with the same conditions at a finite value $b$.

[12]

$$y(\infty) \rightarrow B \text{, } b \rightarrow \infty, \text{ then } \ y(b) = B$$

$$b^n = a + (N+1)h \ldots (9)$$

let $\varepsilon$ be an a small arbitrary value, then we use the finite difference method as let

$$f_n = f_n(x), \quad g_n = g_n(x)$$

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} = f_n y_n + g_n$$

$$y_{n+1} = (2 + h^2 f_n) y_n - y_{n-1} + h^2 g_n \ldots (10)$$

by substitute $n=1,2,3,\ldots,N+1$ in eq. (9) we get

$$y_2 = (2 + h^2 f_1) y_1 - y_0 + h^2 g_1$$

and $y_0 = A \text{, } y_{N+1} = B$
Consider BVPs $\ddot{y} = 9y - 9$ with infinite boundary conditions $y(0) = 2, \quad y(\infty) = 1$

exact solution $y(x) = e^{-3x+1}$

we take $h=0.5$ and $\epsilon = 10^{-18}$

\[
\frac{-y_{n+1} + 2y_{n} - y_{n-1}}{h^2} = 9y_n - 9
\]

$y_{n+1} = 4.25y_n - y_{n-1} - 2.25 \quad \text{...(13)}$

Substituted $n=1,2,3, \ldots$ in (13) to obtain

$y_2 = 4.25y_1 - y_0 - 2.25$ by substituted $y_3 = 0$

to obtain $y_3 = 1.19047619$

but $|1.19047619 - 1| > \epsilon$

so continue to $n=7$ the second condition becomes as follows

$b \frac{n+1}{2} = a + (n + 1)h \quad \text{yields} \quad b^{10} = 2 + 8(0.5) \quad \text{yields} \quad b^{10} = 4$

The boundary condition becomes $y(0)=2, \quad y(4)=1$

Then by approximate $y(x)$ as

$y_N(x) = \sum_{i=0}^{N} a_i V_i(x)$

Table 1 and Figure 1 represent comparison between the exact and approximated which on least square error

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Table 1
Problem 2

\[
\ddot{y} = 2y - e^{-x}, \quad y(0) = 1, \ y(\infty) = 0,
\]

exact solution: \( y(x) = e^{-x} \)

we take \( \epsilon = 10^{-18} \) by solving in the same ways on examples we obtain \( n=19, \ b=10 \) so the boundary condition becomes \( y(0)=1, \ y(10)=0 \)

and by assuming the approximate solution

\[
y_N(x) = \sum_{i=0}^{n} a_i V_i'(x)
\]

Table 2 and Figure 2 represent comparisons between the exact and approximated which on least square error.

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Table 2
Conclusions:

The aim of present work is develop an efficient and accurate method for solving boundary value problem over infinite interval first the infinite interval changed by finite difference method to finite interval, then the shifted third kind chebyshev operational matrix of drivitive together with collocation method are used to reduce the problem into system of nonlinear equation in unknown approximate coefficient. Illustrative examples are include to demon state the validity and applicability of the technique.

Reference


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