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Direct Iterative Algorithm for Solving Optimal Control Problems Using B-Spline Polynomials

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DIRECT ITERATIVE ALGORITHM FOR SOLVING OPTIMAL CONTROL PROBLEMS USING B-SPLINE POLYNOMIALS

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في هذا البحث، تم اقتراح اسلوب لحل تقريبي لمسائل السيطرة المثلى. هذه الخوارزمية تعتمد على متعدات حدود البي سبلاين مع نظرية المعلمات للحاله. قدمت خاصية مهمه للبي سبلاين ثم استخدمت لاقتراح اسلوب حل لتقليل عدد المجاهيل مع اقتراب سريع للحل. طبقت الطريقة لحل اربعة امثله وتم مقارنة النتائج مع نتائج اخرى..

New technique for achieving an approximate solution to optimal control problems (OCPs) is considered in this paper. The algorithm is based upon B-spline polynomials (BSPs) approximation with state parameterization method. An important property concerning the B-spline functions is first presented then it is utilized to propose a modified restarted technique to reduce the number of unknown parameters with fast convergence. The method is applied through four illustrative examples and is compared with other results.

1. INTRODUCTION

The study of problems in optimal control is very important in our day life. The application of optimal control problems can be studied in many disciplines based on mathematical modeling physics, chemistry, and economy [1-3]. Because of the complexity of most applications, optimal control problems are solved numerically. Various numerical methods have been proposed to solve (OCPs). In [4] Yousef Edrisi studied the solution of OCPs using collocation method using B-spline functions. Authors of [5] presented a numerical solution of OCPs with aid of state parameterization technique. Different numerical algorithms for treating OCPs have been introduced by utilizing the orthogonal functions. The complexity of the OCPs is decreased by reducing it to an algebraic system of equation, for example, B-spline polynomials [6], generalized Laguerre polynomials [7], Chebyshev polynomials [8-10] as well as third kind Chebyshev wavelets functions [11], Boubaker polynomials [12]. Special attention is given to find the approximate solution of OCPs using BEPs. These polynomials have already been utilized for solving OCPs [13] and integral equation [14]. In [15], authors have constructed orthonormal BEPs and applied them to solve integral equations.

The approach in the current paper based on BSPs expansion for solving OCPs . These polynomials introduced by [16-17]. In [17], Mohson, A., applied the operational matrices of BEPs and proposed a numerical solution of fractional optimal control problems while Safaie E. and Farahi M. H. in [16] solved delay fractional OCP with the aid of BEPs. For the historical development of BEPs properties and their applications, the reader can be referred to [18-20].

2. B-SPLINE POLYNOMIALS DEFINITION AND PROPERTIES

2.1. DEFINITION OF BSPS [18]

The general form of B-spline polynomials of degree in n of the interval (0, 1) is defined by:

$$Bs_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i} \quad 0 \le i \le n$$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$

The first four B-spline bases are:

$$Bs_0^0 = 1$$

$$Bs_0^1 = 1 - t \quad Bs_1^1 = t$$

$$Bs_0^2 = (1-t)^2 \quad Bs_1^2 = 2t(1-t) \quad Bs_2^2 = t^2$$

$$Bs_0^3 = (1-t)^3 \quad Bs_1^3 = 3t(1-t)^2 \quad Bs_2^3 = 3t^2(1-t)$$

$$Bs_0^4 = (1-t)^4 \quad Bs_1^4 = 4t(1-t)^3 \quad Bs_2^4 = 6t^2(1-t)^2$$

For mathematical convenience, $Bs_i^n(t) = 0$ if $i < 0$ or $i < n$.

Remark 1:

Any nth degree of B-spline polynomial can be expressed in terms of the power basis. Using the Bernstein-spline polynomial

$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i} \quad 0 \le i \le n$

Then the binomial theorem, one can get

$$B_i^n(t) = \binom{n}{i} t^i \sum_{s=0}^{n-i} (-1)^s \binom{n-i}{s} t^s$$

$$= \sum_{s=i}^{n} (-1)^{s-i} \binom{n}{i} \binom{n-i}{s-i} t^s$$

$$= \sum_{s=i}^{n} (-1)^{s-i} \binom{n}{s} \binom{s}{i} t^s$$

This means that

$$\binom{n}{i} t^i (1-t)^{n-i} = \sum_{s=i}^{n} (-1)^{s-i} \binom{n}{s} \binom{s}{i} t^s$$

The derivative of the n^{th} degree BSPs are polynomials of degree $n-1$ and are given by

$$DBs_i^n(t) = n(Bs_{i-1}^{n-1}(t) - Bs_i^{n-1}(t)) \quad (1)$$

where $D \equiv \frac{d}{dt}$

it is important in numerical formulation of the problem using the basis for $n \ge 1$ with the following useful degree elevation property

$$Bs_{i-1}^{n-1}(t) = \frac{1}{2}[(n-i)Bs_i^n(t) + (i+1)Bs_{i+1}^n(t)] \quad (2)$$

The values of BSFs at the end points are

$Bs_i^n(0) = \begin{cases} 1, & i = 0 \\ 0, & i = 1, 2, \dots, n \end{cases}$ and

$Bs_i^n(1) = \begin{cases} 0 & i = 0, 1, \dots, n-1 \\ 1 & i = n \end{cases}$

A square integrable functions $f(t)$ in $(0,1)$ can be expressed in terms of the BSPs basis

$f(t) = \sum_{c=0}^{n} c_i Bs_i^n(t) = c^T Q(t)$

where $c^T = [c_0, c_1, \dots, c_n]$ and $Q(t) = [Bs_0^n(t),\ Bs_1^n(t), \dots, Bs_n^n(t)]^T$

2.2. HGMH NEW PROPERTY OF B-SPLINE POLYNOMIALS FOR CONVERTING THE POWER BASIS TO B-SPLINE BASIS

The expression of converting the bases between the power basis and B-spline functions is presented in this subsection. Given power basis$\{t^i\}_{i=0}^{s}$, this can be written in the B-spline as follows

$$t^n = [Bs_{n-1}^{n-1}(t) - \frac{1}{n} Bs_{n-1}^n(t)] \quad (3)$$

for $n = 2, 3, \dots$.

Proof:

In order to prove Eq. 3, the mathematical induction is used.

In order to establish the validity of formula in Eq.3, the following steps are needed.

- To prove that Eq. 3 is true for $n = 2$

$$t^2 = Bs_1^1(t) - \frac{1}{2} Bs_1^2(t) = t - \frac{1}{2} 2t(1-t) = t^2$$

This has been shown in Eq. 3

- For fixed k, assume Eq. 3 is true

$$t^k = Bs_{k-1}^{k-1}(t) - \frac{1}{k} Bs_{k-1}^k(t) \quad (4)$$

Then prove that Eq. 3 is true for $n = k+1$ that is we want to prove that

$$t^{k+1} = Bs_k^k(t) - \frac{1}{k+1} Bs_k^{k+1}(t) \quad (5)$$

Multiply Eq. 4 by t, one can obtain

$$t\, t^k = t\left(Bs_{k-1}^{k-1}(t) - \frac{1}{k} Bs_{k-1}^k(t)\right)$$

or $\; t^{k+1} = t\, Bs_{k-1}^{k-1}(t) - \frac{t}{k} Bs_{k-1}^k(t)$

By using $t\, Bs_k^k(t) = \frac{k+1}{k+1} B_{k+1,k+1}$, the result can be obtained.

Note that the above formula Eq. 3 can be generalized for $t \in [a, b]$

$$t^n = \frac{1}{(b-a)^n}(Bs_n^n(t) - \frac{1}{n} Bs_n^{n+1}(t))$$

where

$$Bs_{in}(t) = \frac{1}{(b-a)^n} \binom{n}{i} (x-a)^i (b-x)^{n-1} \quad (6)$$

3. OUTLINE OF THE METHOD

3.1. THE PROBLEM STATEMENT

Find the optimal control $u(t)$ which minimizes the cost function

$$J = \int_a^b F\big(t, x(t), u(t)\big) dt \quad (7)$$

subject to $u(t) = f(t, x(t), \dot{x}(t)$

(8)

where $x(.): [a, b] \to R$ is the state variable,

$u(.): [a, b] \to R$ is the control variable

and f: is a real valued continuously differentiable function.

The boundary conditions are $x(a) = x_a$ and $x(b) = x_b$

(9)

where x_a and b are states given in R.

3.2. SOLUTION SCHEME

First we start with the initial approximation

$$x_1(t) = (a_0 Bs_0^1(t) + a_1 Bs_1^1(t)) \quad (10)$$

Using the initial and final conditions to obtain

$$x_a = a_0(1-a) + a_0 a \text{ and } x_b = a_0(1-b) + a_1 b \quad (11)$$

Eq. 11 will give the values of the unknown a_0 and a_1 as below

$a_0 = \frac{bx_a - ax_b}{b-a}$, $a_1 = \frac{x_a(b-1) - x_b\,(1-a)}{b-a}$

After substituting these values into Eq. 10, yields

$x_1(t) = \frac{bx_a - ax_b}{b-a} + (x_b - x_a)t$

The optimal control u(t) can be obtained from Eq. 8 to get

$$u_1(t) = f(t, x_1(t), \dot{x}_1(t)) \tag{12}$$

The functional J can be evaluated using Eq. 7

$J_1 = \int_0^1 F(t, x_1(t), u_1(t)dt$

Now the second approximation is calculated as below

$$x_2(t) = x_1(t) + a_2[-\frac{1}{2}Bs_1^2(t)]$$

$u_2(t) = F(t, x_2(t), \dot{x}_2(t))$

$J_2 = \int_0^1 F(t, x_2(t), u_2(t))$

By continuing the procedure, the n^th approximated solution for $x(t)$, $u(t)$ will be as follows

$$x_{n+1}(t) = x_n(t) + a_{n+1}[Bs_n^n(t) - \frac{1}{n+1}Bs_n^{n+1}(t) - Bs_{n-1}^{n-1}(t) + \frac{1}{n}Bs_{n-1}^n(t)]$$

$u_{n+1}(t) = F(t, x_{n+1}(t), \dot{x}_{n+1}(t))$

$J_{n+1} = \int_a^b F(t, x_{n+1}(t), u_{n+1}(t)dt$

4. APPLICATION EXAMPLES

The following examples are considered to illustrate the efficiency of the proposed algorithm

Example 1

This example clarifies the following concepts:

Find the optimal state and optimal control based on minimizing the performance index

$J = \int_0^1 \left(x(t) - \frac{1}{2}u(t)^2\right)dt$, $0 \le t \le 1$

subject to $u(t) = \dot{x}(t) + x(t)$ with the condition $x(0) = 0$, $x(1) = \frac{1}{2}(1 - \frac{1}{e})^2$

where $J_{exact} = 0.08404562020$

In this example the initial approximation is $x_1(t) = \frac{1}{2}\left(1 - \frac{1}{e}\right)^2 t$

The approximate state variables for n=2, 3 and 4 using B-spline polynomial can be expressed as below:

$x_2(t) = 0.6091\,t - 0.4091t^2$

$x_3(t) = 0.6091\,t - 0.4208t^2 + 0.0117t^3$

$x_4(t) = 0.6091\,t - 0.4208t^2 + 0.0091t^3 + 0.0026t^4$

The approximate state variables for n=2, 3 and 4 using B-spline polynomial can be expressed as below:

$u_2(t) = 0.6091 - 0.2091\,t - 0.4091t^2$

$u_3(t) = 0.6091 - 0.2325\,t - 0.3857t^2 + 0.0117t^3$

$u_4(t) = 0.6091 - 0.2325\,t - 0.3935t^2 + 0.0195t^3 + 0.0026t^4$

The approximate results are obtained by the proposed algorithm based on B-spline polynomial with n=1, 2, 3. The results are compared with results obtained in **[21]**. Table 1 illustrates the total information of the optimal values for the functional J with different iterations and one can observe that our results have almost better accuracy.

Table 1. Results of the functional J

I.	O. M.	E	M.in [21]	E
1	0.08401526011	3.047×10^{-5}	0.0533262210	3.0×10^{-2}
2	0.08402489318	2.073×10^{-5}	0.0840152600	3.0×10^{-2}
3	0.08402519637	2.042×10^{-5}	0.8402496180	2.0×10^{-2}

[1] I. – Iteration; O.M. – Our Method; E. - Error; M. – Method

The primacy of present algorithm compared with method in [21] is clear in this example because by the same number of iteration n, the present algorithm error is lower. The obtained results are plotted in Figure 1 against the actual solution

$x(t) = 1 - 0.5e^{t-1} - 0.8160603e^{-t}$and

$u(t) = 1 - e^{t-1}$

Example 2

Consider the non-linear control system which consists of minimizing

$$\int_0^1 u^2(t)dt$$

subject to $u(t) = \dot{x}(t) - x^2(t)sint$, $x(0) = 0$, $x(1) = 0.5$

In this example the initial approximation is $x_1(t) = 0.5t$

The approximate state variables for $n = 2, 3$ and 4 using B-spline polynomial can be expressed as below:

$x_2(t) = 0.4622\,t + 0.0378t^2$

$x_3(t) = 0.4622\,t + 0.0215t^2 + 0.0136t^3$

$x_4(t) = 0.4622\,t + 0.0215t^2 + 2.1159t^3 + 2.0996t^4$

The approximate state variables for $n = 2, 3$ and 4 using B-spline polynomial can be expressed as below:

$u_2(t) = 0.4622 + 0.756\,t - \sin t[(0.4622t + 0.765t^2)^2]$

$u_3(t) = 0.4622 - 0.043\,t + 0.0489t^2 - sint(0.4622t + 0.0215t^2 + 0.0163t^3)$

$$u_4(t) = 0.4622 - 0.043\,t + 6.3477t^2 - 8.3984t^3 - sint(0.4622t + 0.0215t^2 + 2.1153t^3 - 2.0996t^4)$$

The approximate performance index is $J = 0.2005$. the obtained results and the actual solution are plotted in Figure 2.

Example 3

The proposed method in this example is applied to the following problem

$$J= \frac{1}{2}\int_0^1 (3x(t)^2 + u(t)^2)dt$$

subject to $u(t) = \dot{x}(t) + x(t), x(0) = 0, x(1) = 2$

In this example the initial approximation is $x_1(t) = 2t$

The approximate state variables for n=2, 3 and 4 using B-spline polynomial can be expressed as below:

$$x_2(t) = 0.5714\,t + 1.4286t^2$$

$$x_3(t) = 0.5714\,t + 1.0397t^2 + 0.3889t^3$$

$$x_4(t) = 0.5714\,t + 1.0397t^2 + 01798t^3 + 0.2091t^4$$

The approximate state variables for n=2, 3 and 4 using B-spline polynomial can be expressed as below:

$$u_2(t) = 0.5714 + 3.4286t + 1.4286t^2$$

$$u_3(t) = 0.5714 + 2.6508\,t + 2.2063t^2 + 0.3889t^3$$

$$u_4(t) = 0.5714 + 2.6508\,t + 1.5912t^2 + 1.0161t^3 + 0.2091t^4$$

The comparison among the B-spline algorithm with different iterations beside method in [21] are listed in Table 2 . The exact value for the cost is J=6.1586. The obtained results and the actual solution are plotted in Figure 3.

Table 2: The values of cost functional J in Example 3

I.	O. M.	E	M.in [21]	E
1	6.190476192	0.03187624	6.195	0.0319
2	6.177513228	0.01891323	6.1775	0.0189
3	6.174827155	0.01622723	6.1753	0.0167

[1] I. – Iteration; O.M. – Our Method; E. - Error; M. – Method

Example 4

The proposed method in this example is applied to the following problem

$$J= \int_0^1 (x(t)^2 + u(t)^2)dt$$

subject to $u = \dot{x}$, $x(0) = 0$, $x(1) = 0.5$

In this example the initial approximation is $x_1(t) = 0.5\,t$

The approximate state variables for n=2, 3 and 4 using B-spline polynomial can be expressed as below:

$$x_2(t) = 0.3863\,t + 0.1136t^2$$

$$x_3(t) = 0.3863\,t + 0.0844t^2 + 0.0292t^3$$

$$x_4(t) = 0.3863\,t + 0.0844t^2 + 0.0242t^3 + 0.0485t^4$$

The approximate state variables for n=2, 3 and 4 using B-spline polynomial can be expressed as below:

$$u_2(t) = 0.3864 + 0.2273t$$

$$u_3(t) = 0.3864 + 0.1689\,t + 0.0875t^2$$

$$u_4(t) = 0.3864 + 0.1689\,t + 00497t^2 + 0.0504t^3$$

Table 3 illustrates the results based on B-spline algorithm for this example in comparison with results obtained by [21]. The optimal values for performance index J are also compared with the exact solution while the exact state and control solution as well as the actual value for J are Note that, the actual solution of this problem is

$x(t) = \frac{e(e^t - e^{-t})}{2(e^2-1)}$, $u(t) = \frac{e(e^t + e^{-t})}{2(e^2-1)}$ and $J_{exact} = 0.3023$

The obtained results and the actual solution are plotted in Figure 4.

Table 3 The values of cost functional J in Example 4

I.	O. M.	E	M.in [21]	E
1	0.3286	3.37910×10^{-4}	0.3333	5.0×10^{-3}
2	0.3262	2.18140×10^{-4}	0.3286	3.4×10^{-3}
3	0.3226	2.03089×10^{-4}	0.3285	2.1×10^{-4}

5. CONCLUSION

Some modification is proposed by introducing accelerating iterative algorithm for solving optimal control problem directly based on B-spline functions with only unknown coefficient must be evaluated in each approximation. A new resulted modification solution was constructed which based upon an interesting property of B-spline functions. The examples illustrated the reliability of the devoted B-spline algorithm presented in this paper.

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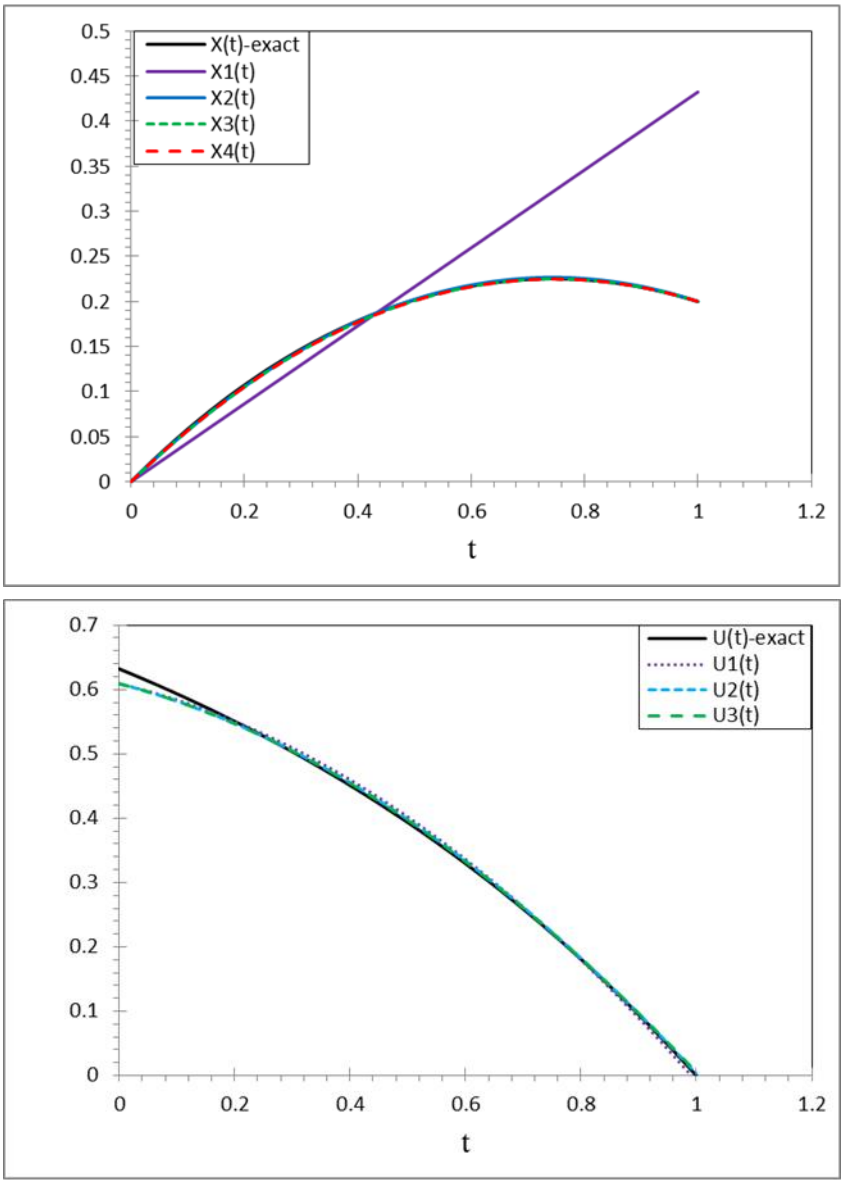


Fig. 1 Solution of Example 1. The solution of the five iterative compared with the analytical solution

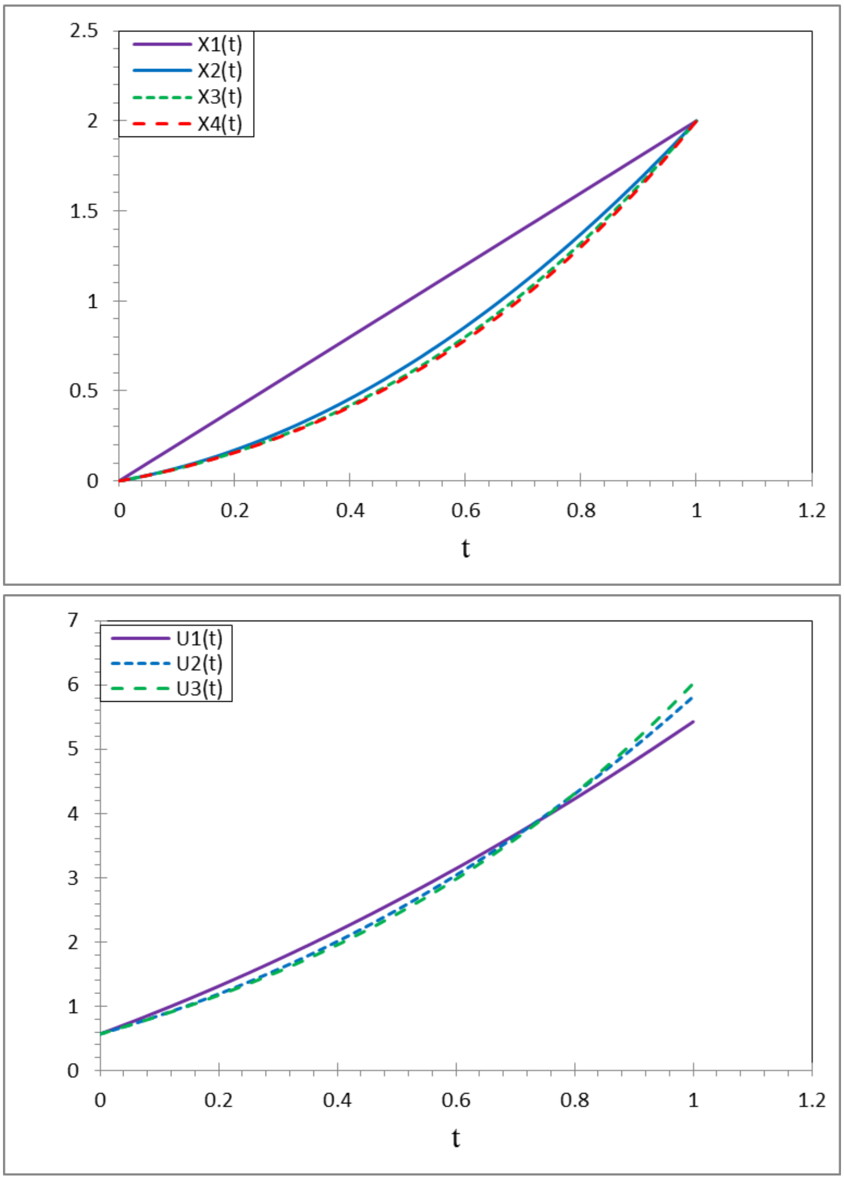
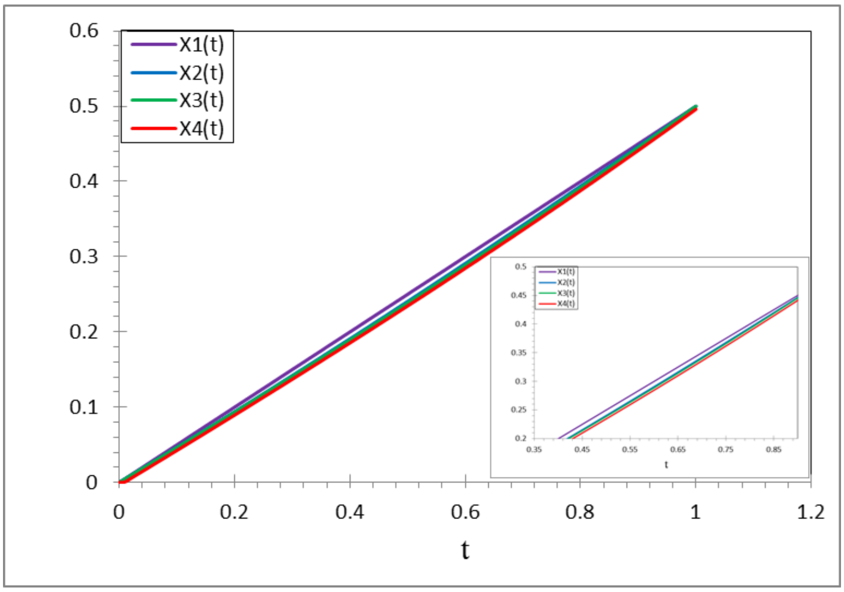


Fig. 2 Solution of Example 2. The solution of the first iterative compared with the exact solution.



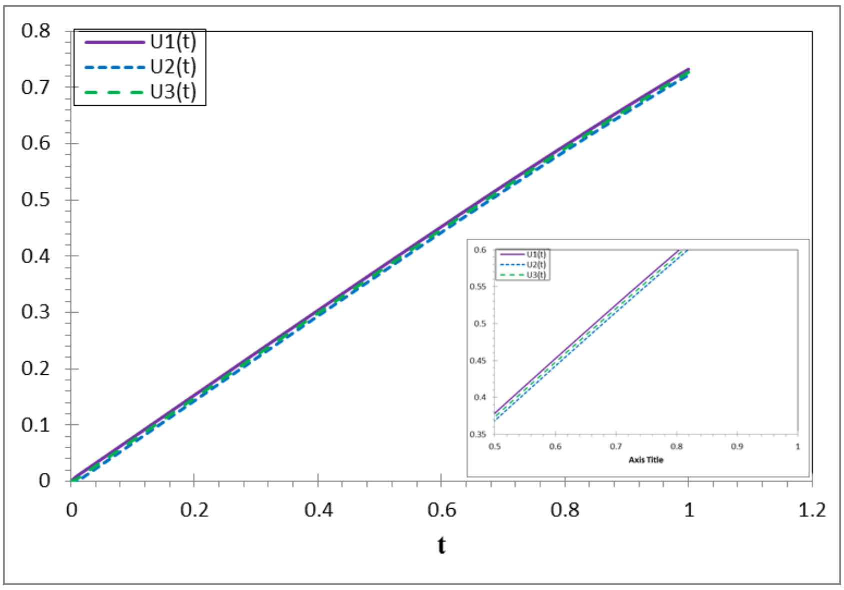


Fig. 3 Solution of Example 3. The solution of the first iterative compared with the exact analytical solution.

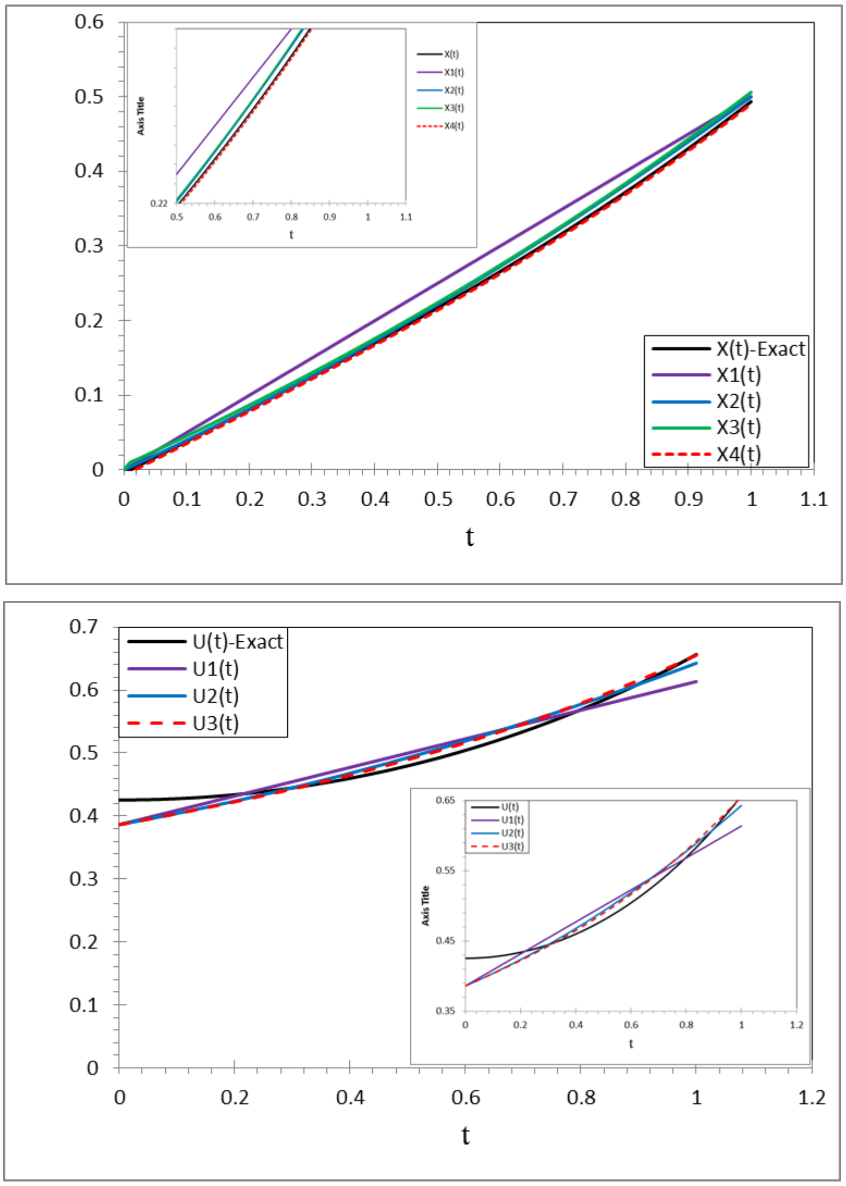


Fig. 4 Solution of Example 4. The solution of the first iterative compared with the exact solution.