Spring 4-2015

EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A CLASS OF NON-LINEAR BOUNDARY VALUE PROBLEMS OF FRACTIONAL ORDER

Arwa Abdulla Omar Salem Ba Abdulla

United Arab Emirates University

College of Science

Department of Mathematical Sciences

EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A CLASS OF NON-LINEAR BOUNDARY VALUE PROBLEMS OF FRACTIONAL ORDER

Arwa Abdulla Omar Salem Ba Abdulla

This thesis is submitted in partial fulfilment of the requirements for the degree of Master of Science in Mathematics

Under the Supervision of Dr. Mohammed Al-Refai

April 2015

Declaration of Original Work

I, Arwa Abdulla Omar Salem Ba Abdulla, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this master thesis, entitled *"Existence and Uniqueness of Solutions for a Class of Non-Linear Boundary Value Problems of Fractional Order"*, hereby, solemnly declare that this thesis is an original research work that has been done and prepared by me under the supervision of Dr. Mohammed Al-Refai, in the College of Science at UAEU. This work has not been previously formed as the basis for the award of any academic degree, diploma or a similar title at this or any other university. The materials borrowed from other sources and included in my thesis have been properly cited and acknowledged.

Student's Signature ______________________ Date ______________

Approval of the Master Thesis

This Master Thesis is approved by the following Examining Commitee Members:

1) Advisor (Committee Chair): Mohammed Al-Refai

Title: Associate Professor

Department of Mathematical Sciences

College of Science

Signature ______________________ Date ______________

2) Member: Ahmed Al-Rawashdeh

Title: Associate Professor

Department of Mathematical Sciences

College of Science

Signature ______________________ Date ______________

3) Member (External Examiner): Suheil Khoury

Title: Professor

Department of Mathematics and Statistics

Institution: American University of Sharjah

Signature ______________________ Date ______________

This Master Thesis is accepted by:

Dean of the College of Science: Prof. Frederick Chi-Ching Leung

Signature ______________________________ Date ____________________

Dean of the College of Graduate Studies: Prof. Nagi Wakim

Signature ______________________________ Date ____________________

Copy __________ of __________

Abstract

In this thesis, we extend the maximum principle and the method of upper and lower solutions to study a class of nonlinear fractional boundary value problems with the Caputo fractional derivative $1 < \delta < 2$. We first transform the problem to an equivalent system of equations, including integer and fractional derivatives. We then implement the method of upper and lower solutions to establish existence and uniqueness of the resulting system. At the end, some examples are presented to illustrate the validity of our results.

Keywords: Fractional differential equations, Boundary value problems, Maximum principle, Lower and upper solutions, Caputo fractional derivative.

Title and Abstract (in Arabic)

وجود ووحدانية الحلول لنوع من المعادلات الحدية غير الخطية ذات الدرجة الكسرية

الملخص

في هذه الرسالة، قمنا بتعميم مبدأ الحد الأقصى و طريقة الحل باستخدام الحد الأعلى و الأدنى لدراسة نوع من معادلات القيم الحدية غير الخطية ذات الرتبة الكسرية من نوع كابوتو و ذات المشتقة الكسرية دلتا بحيث دلتا بين ١ و ٢. في البداية، قمنا بتحويل المسألة إلى جملة معادلات مكافئة تتضمن مشتقة ذات درجة صحيحة وأخرى ذات درجة كسرية. ثم قمنا بعد ذلك بتطبيق طريقة الحل باستخدام الحد الأعلى والأدنى لإثبات وجود ووحدانية الحل لجملة المعادلات الناتجة. في النهاية، تم تقديم بعض الأمثلة التوضيحية لتدعيم أهمية النتائج و قابليتها للتطبيق.

Acknowledgements

I would like to express my deep gratitude to my advisor Dr. Mohammed Al-Refai, whose expertise, understanding, and patience, added considerably to my graduate experience. Furthermore, I would like to thank Dr. Ahmed Al-Rawashdeh for the valuable comments and remarks. Also, I would like to thank Prof. Suheil Khoury for spending time reading this thesis and letting my defense be an enjoyable moment.

I must also acknowledge Prof. Fathi Allan, the coordinator of this program, for his assistance in many aspects. In addition, I thank Dr. Mohamed Salim, the chair of the Department of Mathematical Sciences in the United Arab Emirates University, and my colleagues at the department for their support and cooperation.

Last but not the least important, I owe more than thanks to my family members for their support and encouragement throughout my life.

Dedication

To my beloved parents and family

Table of Contents

Title i
Declaration of Original Work ii
Copyright iii
Approval of the Master Thesis iv
Abstract vi
Title and Abstract (in Arabic) vii
Acknowledgments viii
Dedication ix
Table of Contents x
List of Figures xi
Chapter 1: Fractional Calculus: Introduction 1
1.1 Motivation 1
1.2 Basic Definitions and Theorems 1
1.2.1 Special Functions 1
1.2.2 The Riemann-Liouville Integral 4
1.2.3 The Fractional Derivatives 5
1.2.4 Main Theorems 10
Chapter 2: Boundary Value Problems of Fractional Order 14
2.1 Introduction 14
2.2 Definitions and Preliminary Results 15
2.3 The Linear System of Fractional Differential Equations 17
2.4 Monotone Sequences of Lower and Upper Solutions 21
2.5 Existence and Uniqueness of Solutions 28
2.6 Illustrated Examples 34
2.7 Conclusion 37
Bibliography 38

List of Figures

1.1 The graph of the Gamma function in the real domain. 2

Chapter 1: Fractional Calculus: Introduction

1.1 Motivation

Differential equations with fractional order are generalization of ordinary differential equations to non-integer order. In recent years, a great interest was devoted to study fractional differential equations, because of their appearance in various applications in Engineering and Physical Sciences [for more details, the reader is referred to [1, 2]]. Therefore, numerical and analytical techniques have been developed to deal with fractional differential equations. The maximum principle and the method of lower and upper solutions are well established for differential equations of elliptic, parabolic and hyperbolic types [3, 4]. Recently, there are several studies devoted to extend, if possible, these results for fractional differential equations [5, 6, 7, 8, 9, 10]. It is noted that the extension is not a straightforward process, due to the difficulties in the definition and the rules of fractional derivative. Therefore, the theory of fractional differential equations is not established yet and there are still many open problems in this area. Unlike, the integer derivative, there are several definitions of fractional derivatives which are not equivalent in general. However, the most popular ones are the Caputo and Riemann-Liouville fractional derivatives.

1.2 Basic Definitions and Theorems

In this section, we present basic definitions in fractional calculus and some important theorems that will be used in this thesis.

1.2.1 Special Functions

In the following, we present the definitions and some properties of the Gamma function and the Mittag-Leffler function. The Gamma function is a generalization of the factorial function and it appears in the definition of fractional derivatives, while the

Mittag-Leffler function is a generalization of the exponential function, and it appears in the solution of some fractional differential equations.

Definition 1.2.1. The Gamma function is defined by

$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt,$

for all $x \in \mathbb{R}$, provided the integral exists.

Here are some common properties of the Gamma function:

1. $\Gamma(x+1) = x\Gamma(x)$,
2. $\Gamma(n) = (n-1)!$, where $n \in \mathbb{N}$.

Figure 1.1 depicts the Gamma function. One can see that the Gamma function approaches infinity or negative infinity at non-positive integers.

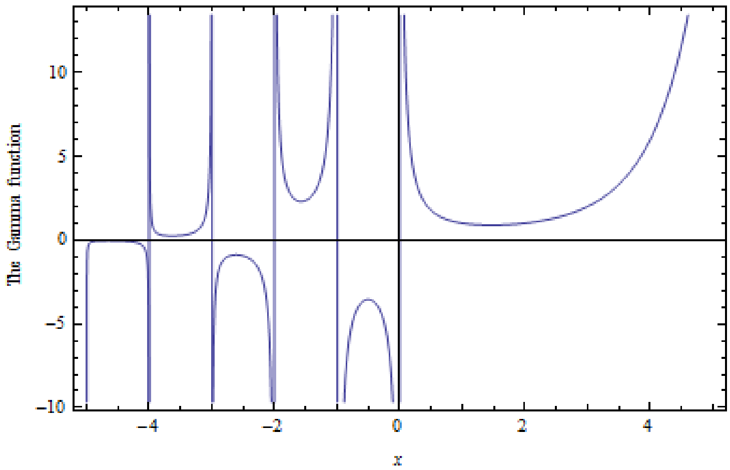

Figure 1.1: The graph of the Gamma function in the real domain.

Definition 1.2.2. The Mittag-Leffler function of one parameter is defined by the power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \ \alpha \in \mathbb{R}^+, \ z \in \mathbb{C}. \tag{1.1}$$

This function was introduced first by Mittag-Leffler [11]. Later on, Agarwal [12] introduced a generalization of the Mittag-Leffler function of one parameter to the two parameters Mittag-Leffler function, which is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \ \alpha, \ \beta \in \mathbb{R}^+, \, z \in \mathbb{C}. \tag{1.2}$$

It follows from the definition of $E_{\alpha,\beta}(z)$ in (1.2) that
$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$, and in general

$$E_{1,m}(z) = \frac{1}{z^{m-1}} \left[e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!} \right], \, m \geq 2. \tag{1.3}$$

The following properties hold for the Mittag-Leffler function of two parameters:

Lemma 1.2.1. $E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \dfrac{1}{\Gamma(\beta)}$.

Proof. We have

$$\begin{aligned} E_{\alpha,\beta}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \\ &= \sum_{k=-1}^{\infty} \frac{z^{k+1}}{\Gamma(\alpha k+\alpha+\beta)} = \frac{1}{\Gamma(\beta)} + \sum_{k=0}^{\infty} \frac{z.z^k}{\Gamma(\alpha k+(\alpha+\beta))} \\ &= \frac{1}{\Gamma(\beta)} + z\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+(\alpha+\beta))} \\ &= \frac{1}{\Gamma(\beta)} + zE_{\alpha,\alpha+\beta}(z). \end{aligned}$$

□

Lemma 1.2.2. $E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z) + \alpha z \dfrac{d}{dz}(E_{\alpha,\beta+1}(z))$.

Proof. We have

$$\begin{aligned} \beta E_{\alpha,\beta+1}(z) + \alpha z\frac{d}{dz}(E_{\alpha,\beta+1}) &= \beta \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta+1)} + \alpha z \frac{d}{dz} \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta+1)} \\ &= \beta \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta+1)} + \alpha z \sum_{k=1}^{\infty} \frac{k\, z^{k-1}}{\Gamma(\alpha k+\beta+1)} \\ &= \beta \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta+1)} + \alpha \sum_{k=0}^{\infty} \frac{k\, z^k}{\Gamma(\alpha k+\beta+1)} \\ &= \sum_{k=0}^{\infty} \frac{z^k(\alpha k+\beta)}{\Gamma(\alpha k+\beta+1)} = \sum_{k=0}^{\infty} \frac{z^k(\alpha k+\beta)}{\Gamma(\alpha k+\beta)(\alpha k+\beta)} \end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} = E_{\alpha,\beta}(z).$$

1.2.2 The Riemann-Liouville Integral

We start with the Cauchy's formula for the n-fold repeated integrals [see [1], p.64],

$$\begin{aligned} I^n f(t) &= \int_a^t \int_a^{s_1} \dots \int_a^{s_{n-1}} f(s_n)\, ds_n \dots ds_2\, ds_1 \\ &= \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s)\, ds. \end{aligned}$$

This formula can be generalized to any positive real number α, using the fact that $(n-1)! = \Gamma(n)$, to obtain the left Riemann-Liouville fractional integral.

Definition 1.2.3. The left Riemann-Liouville fractional integral of order $\alpha \geq 0$, of a function $f \in C[0,1]$ is defined by

$$I^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, & \text{if } \alpha > 0\ . \\ f(t), & \text{if } \alpha = 0\ . \end{cases}$$

The following properties hold true for the Reimann-Liouville fractional integral:

1. The linearity property: $I^\alpha(c_1 f(t) + c_2 g(t)) = c_1 I^\alpha f(t) + c_2 I^\alpha g(t)$, $\alpha \geq 0$, c_1, $c_2 \in \mathbb{C}$.
2. If $f(t)$ is continuous for $t \geq 0$, then $I^\alpha(I^\beta f(t)) = I^\beta(I^\alpha f(t)) = I^{\alpha+\beta}(f(t))$, where $\alpha, \beta \in \mathbb{R}^+$.
3. $I^\alpha(t^\beta) = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\beta+\alpha}$, where α, $\beta \in \mathbb{R}^+$.

For the proof and more properties of the Riemann-Liouville integral the reader is referred to [1, 13].

1.2.3 The Fractional Derivatives

Knowing the definition of the fractional integral enables us to define the fractional derivative for any positive real number. There are several definitions of the fractional derivative. However, the most popular ones are the Riemann-Liouville fractional derivative and the Caputo fractional derivative.

Definition 1.2.4. Let $\alpha \in \mathbb{R}^+$, and $n = [\alpha]+1$, the Riemann-Liouville fractional derivative of order α of function $f(t) \in C^n[0,1]$ is defined by:

$$D_R^{\alpha} f(t) = \frac{d^n}{dt^n}(I^{n-\alpha} f(t)) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s)\, ds, & \text{if } n-1 < \alpha < n \in \mathbb{N}, \\ f^{(n)}(t), & \text{if } \alpha = n \in \mathbb{N}. \end{cases}$$

The following properties hold true for Riemann-Liouville fractional derivative:

1. Linearity property: Let $n-1 < \alpha < n$, $n \in \mathbb{N}, \alpha \geq 0$, $c_1, c_2 \in \mathbb{C}$ and $D_R^{\alpha} f(t)$ and $D_R^{\alpha} g(t)$ exist, then
   $D_R^{\alpha}(c_1 f(t) + c_2 g(t)) = c_1 D_R^{\alpha} f(t) + c_2 D_R^{\alpha} g(t)$.
2. $D_R^{\alpha} C = \dfrac{C}{\Gamma(1-\alpha)} t^{-\alpha}$, for $0 < \alpha < 1$ and C is constant.
3. $D_R^{\alpha}(t^{\beta}) = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}$, $n-1 < \alpha < n$, $\beta > -1$, $\beta \in \mathbb{R}$.

For the proof the reader is referred to [[1], p.72].

The following equation holds true for the Riemann-Liouville fractional derivative.

Lemma 1.2.3. Let $\alpha \in \mathbb{R}^+$ and $m,n \in \mathbb{N}$ such that $n-1 < \alpha < n$ and $f(t) \in C^{n+m}[0,1]$. Then $D^m(D_R^{\alpha} f(t)) = D_R^{m+\alpha} f(t)$.

Proof. For $n-1<\alpha<n$, we have $n+m-1<\alpha+m<n+m$. Therefore

$$\begin{aligned} D_R^{\alpha+m} f(t) &= \frac{1}{\Gamma(n+m-\alpha-m)} \frac{d^{n+m}}{dt^{n+m}} \int_0^t (t-s)^{n+m-\alpha-m-1} f(s)\,ds \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^{n+m}}{dt^{n+m}} \int_0^t (t-s)^{n-\alpha-1} f(s)\,ds. \end{aligned} \tag{1.4}$$

$$\begin{aligned} \text{Also, } D^m(D_R^\alpha f(t)) &= D^m\left(\frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s)\,ds\right) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^{n+m}}{dt^{n+m}} \int_0^t (t-s)^{n-\alpha-1} f(s)\,ds. \end{aligned} \tag{1.5}$$

Compare equations (1.4) and (1.5) to obtain the result. □

Definition 1.2.5. The Caputo fractional derivative of order $\alpha>0$, $n-1<\alpha<n$, $n\in\mathbb{N}$ of a function $f\in C^n[0,1]$ is defined by

$$D_C^\alpha f(t) = I^{n-\alpha}\left(\frac{d^n}{dt^n} f(t)\right) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,ds, & \text{if } n-1<\alpha<n\in\mathbb{N}, \\ f^{(n)}(t), & \text{if } \alpha=n\in\mathbb{N}. \end{cases}$$

The following properties hold true for the Caputo fractional derivative:

1. Linearity property: Let $n-1<\alpha<n, n\in\mathbb{N}, \alpha\geq 0$, $c_1, c_2\in\mathbb{C}$, and $D_C^\alpha f(t)$ and $D_C^\alpha g(t)$ exist, then
   $D_C^\alpha(c_1 f(t)+c_2 g(t)) = c_1 D_C^\alpha f(t) + c_2 D_C^\alpha g(t)$.

2. $D_C^\alpha K = 0, \alpha>0$, where K is a constant.

3. $D_C^\alpha(I^\alpha f(t)) = f(t)$ for $\alpha\in\mathbb{R}^+$, $f(t)\in C^n[0,1]$ and $n=[\alpha]+1$.

4. $I^\alpha(D_C^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k$ for $\alpha\in\mathbb{R}^+$, $f(t)\in C^n[0,1]$ and $n=[\alpha]+1$.

For the proof the reader is referred to [[13], p.95-96].

Lemma 1.2.4. Let $n-1<\alpha<n, n\in\mathbb{N}, \alpha\in\mathbb{R}^+$ and $f(t)\in C^{n+1}[a,b]$ such that $D_C^\alpha f(t)$ exists, then the following hold true

$$\lim_{\alpha\to n} D_C^\alpha f(t) = f^{(n)}(t), \tag{1.6}$$

$$\lim_{\alpha\to n-1} D_C^\alpha f(t) = f^{(n-1)}(t) - f^{(n-1)}(0). \tag{1.7}$$

Proof. By using integration by parts, we get

$$\begin{aligned}
D_C^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t f^{(n)}(s)(t-s)^{n-\alpha-1}\, ds \\
&= \frac{1}{\Gamma(n-\alpha)} \left(\frac{-f^{(n)}(s)}{n-\alpha}(t-s)^{n-\alpha}|_0^t + \int_0^t \frac{f^{(n+1)}(s)}{n-\alpha}(t-s)^{n-\alpha}\, ds \right) \\
&= \frac{1}{\Gamma(n-\alpha+1)} \left(f^{(n)}(0)t^{n-\alpha} + \int_0^t f^{(n+1)}(s)(t-s)^{n-\alpha}\, ds \right).
\end{aligned}$$

Taking the limit $\alpha \to n$ and $\alpha \to n-1$, respectively, and using the fact that $f^{(n+1)}(s)(t-s)^{n-\alpha}$ is continuous, we have

$$\begin{aligned}
\lim_{\alpha \to n} D_C^{\alpha} f(t) &= f^{(n)}(0) + \int_0^t f^{(n+1)}(s)\, ds = f^{(n)}(0) + f^{(n)}(t) - f^{(n)}(0) \\
&= f^{(n)}(t), \\
\text{and} \quad \lim_{\alpha \to n-1} D_C^{\alpha} f(t) &= \left(f^{(n)}(0)t + f^{(n)}(s)(t-s)|_{s=0}^t \right) + \int_0^t f^{(n)}(s)\, ds \\
&= f^{(n)}(0)t - f^{(n)}(0)t + \int_0^t f^{(n)}(s)\, ds \\
&= f^{(n-1)}(s)|_{s=0}^t \\
&= f^{(n-1)}(t) - f^{(n-1)}(0).
\end{aligned}$$

□

Lemma 1.2.5. Suppose that $n-1 < \alpha < n, m, n \in \mathbb{N}, \alpha \in \mathbb{R}^+$ and $f(t) \in C^{n+m}[0,1]$, then $D_C^{\alpha}(D^m f(t)) = D_C^{\alpha+m} f(t)$.

Proof. From $n-1 < \alpha < n$, we have $n+m-1 < \alpha+m < n+m$, therefore

$$\begin{aligned}
D_C^{\alpha+m} f(t) &= \frac{1}{\Gamma(n+m-\alpha-m)} \int_0^t (t-s)^{n+m-\alpha-m-1} f^{(n+m)}(s)\, ds \\
&= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n+m)}(s)\, ds. \qquad (1.8) \\
\text{Also,} \quad D_C^{\alpha}(D^m f(t)) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} (D^m f)^{(n)}(s)\, ds \\
&= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n+m)}(s)\, ds. \qquad (1.9)
\end{aligned}$$

Compare equation (1.8) and (1.9) to get the result. $\square$

The following lemma presents the well-known relationship between the Caputo fractional derivative and the Riemann-Liouville fractional derivative.

Lemma 1.2.6. If $f \in C^n[0,1]$, then $D_C^\alpha f(t) = D_R^\alpha [\, f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)\,]$, where $D_R^\alpha t^k = \dfrac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} t^{k-\alpha}$.

Proof. Applying the Taylor series expansion about $t_0 = 0$, yields;

$$f(t) = \sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)} f^{(k)}(0) + R_{n-1}(t),$$

where

$$\begin{aligned} R_{n-1}(t) &= \int_0^t \frac{f^{(n)}(s)(t-s)^{(n-1)}}{(n-1)!} ds \\ &= \frac{1}{\Gamma(n)} \int_0^t f^{(n)}(s)(t-s)^{n-1} ds \\ &= I^n f^{(n)}(t), \end{aligned}$$

[see [14], p.217].

Applying the properties of the Riemann-Liouville and Caputo fractional derivatives, we have

$$\begin{aligned} D_R^\alpha f(t) &= D_R^\alpha \left(\sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)} f^{(k)}(0) + R_{n-1}(t) \right) \\ &= \sum_{k=0}^{n-1} \frac{D_R^\alpha t^k}{\Gamma(k+1)} f^{(k)}(0) + D_R^\alpha (R_{n-1}(t)) \\ &= \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0) + I^{n-\alpha} f^{(n)}(t) \\ &= \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0) + D_C^\alpha f(t). \end{aligned}$$

$\square$

Lemma 1.2.7. If $\alpha \in \mathbb{R}^+$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$, then

$$D_R^{\alpha}(e^{\lambda t}) = t^{-\alpha} E_{1,1-\alpha}(\lambda t).$$

Proof. We have

$$\begin{aligned} e^{\lambda\, t} &= \sum_{k=0}^{\infty} \frac{\lambda^k\, t^k}{\Gamma(k+1)} \\ \text{and } D_R^{\alpha}\,(e^{\lambda\, t}) &= D_R^{\alpha} \left(\sum_{k=0}^{\infty} \frac{\lambda^k\, t^k}{\Gamma(k+1)} \right) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(k+1)}\, D_R^{\alpha}\,(t^k) \\ &= \sum_{k=1}^{\infty} \frac{\lambda^k\, \Gamma(k+1)}{\Gamma(k+1)\, \Gamma(k+1-\alpha)}\, t^{k-\alpha} \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k\, t^{k-\alpha}}{\Gamma(k+1-\alpha)} \\ &= t^{-\alpha} \sum_{k=0}^{\infty} \frac{\lambda^k\, t^k}{\Gamma(k+1-\alpha)} = t^{-\alpha} \sum_{k=0}^{\infty} \frac{(\lambda\, t)^k}{\Gamma(k+1-\alpha)} \\ &= t^{-\alpha}\, E_{1,1-\alpha}\,(\lambda\, t). \end{aligned}$$

□

Lemma 1.2.8. If $\alpha \in \mathbb{R}^+$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$, then
$D_C^{\alpha}(e^{\lambda t}) = \sum_{k=0}^{\infty} \frac{\lambda^{k+n} t^{k+n-\alpha}}{\Gamma(k+1+n-\alpha)} = \lambda^n t^{n-\alpha} E_{1,n-\alpha+1}(\lambda t).$

Proof. We have

$D_C^{\alpha} f(t) = D_R^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(0).$
From Lemma 1.2.7 we get

$$\begin{aligned} D_C^{\alpha}(e^{\lambda t}) &= D_R^{\alpha} e^{\lambda t} - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} \left(e^{\lambda t}\right)^{(k)} (0) \\ &= t^{-\alpha} E_{1,1-\alpha}(\lambda t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} \lambda^k, \text{ where} (e^{\lambda t})^{(k)}(t) = \lambda^k e^{\lambda t}. \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k t^{k-\alpha}}{\Gamma(k+1-\alpha)} - \sum_{k=0}^{n-1} \frac{\lambda^k t^{k-\alpha}}{\Gamma(k+1-\alpha)} \\ &= \sum_{k=n}^{\infty} \frac{\lambda^k t^{k-\alpha}}{\Gamma(k+1-\alpha)} \end{aligned}$$

$$
\begin{aligned}
&= \lambda^n t^{n-\alpha} \sum_{k=0}^{\infty} \frac{\lambda^k t^k}{\Gamma(k+(n+1-\alpha))} \\
&= \lambda^n t^{n-\alpha} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{\Gamma(k+(n+1-\alpha))} \\
&= \lambda^n t^{n-\alpha} E_{1,n+1-\alpha}(\lambda t).
\end{aligned}
$$

□

1.2.4 Main Theorems

In this section, we present several results that will be used in this thesis. We start with some recent results concerning the fractional derivatives at extreme points.

Theorem 1.2.1. *[15]. Let $f \in C^1[0,1]$ attain its absolute minimum at $t_0 \in (0,1]$. Then $D_C^\alpha f(t_0) \leq \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)}[f(t_0)-f(0)] \leq 0$, for all $0<\alpha<1$.*

Theorem 1.2.2. *[16]. If $x(t) \in C^m[0,1]$ and $m-1<\alpha<m \in Z^+$, then $D_C^\alpha x(t)|_{t=0} = 0$.*

The next lemma is a special case of the previous theorem, and it is essential in the proof of the positivity result in Section 2.2.

Lemma 1.2.9. If $f(t) \in C^1[0,1]$, then $D^\alpha f(t)|_{t=0} = 0, 0<\alpha<1$.

Proof. We have $D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t f'(s)(t-s)^{-\alpha}\, ds$. Since $f \in C^1[0,1]$, then $\int_0^t f'(s)\, ds$ is integrable, with $\int_0^t f'(s)\, ds = c(t)$, where $c(t) = f(t)-f(0)$. And let $M = \max_{0\leq t\leq 1} |f'(t)|$. Also, the improper integral $\int_0^t (t-s)^{-\alpha}\, ds$ exists because $\lim_{R\to t} \int_0^R (t-s)^{-\alpha}\, ds = \frac{t^{1-\alpha}}{1-\alpha}$, and $1-\alpha>0$. Since the product of two integrable functions is integrable, we have
$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t f'(s)(t-s)^{-\alpha}\, ds$ exists for all $t \in [0,1]$, and there holds

$$
\begin{aligned}
|D^\alpha f(t)| &= \frac{1}{\Gamma(1-\alpha)} \left| \int_0^t f'(s)(t-s)^{-\alpha}\, ds \right| \\
&\leq \frac{1}{\Gamma(1-\alpha)} \int_0^t |f'(s)|\, ds \int_0^t |(t-s)^{-\alpha}|\, ds \\
&= \frac{M}{\Gamma(1-\alpha)} \frac{t^{1-\alpha}}{1-\alpha},\ 0<\alpha<1.
\end{aligned}
$$

Therefore, $|D^{\alpha} f(0)| \leq 0$, implies that $D^{\alpha} f(t)|_{t=0} = 0$. $\square$

Theorem 1.2.3 (Dominated Convergence Theorem)**.** *[[14], p.304] Let $\{f_n\}$ be a sequence of Riemann integrable on I, and let $f(x) = \lim_{k\to\infty} f_k(x)$ almost everywhere on I. If there exist integrable functions α, ω such that $\alpha(x) \leq f_k(x) \leq \omega(x)$ for almost every $x \in I$, then f is Riemann integrable and $\int_I f = \lim_{k\to\infty} \int_I f_k$.*

Theorem 1.2.4 (Uniform Continuity Theorem)**.** *[[14], p.138]. Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.*

Theorem 1.2.5 (Weighted Mean Value Theorem for Integrals)**.** *[17]. Suppose $f \in C[a,b]$, the Riemann integral of g exists on $[a,b]$, and $g(x)$ does not change sign on $[a,b]$. Then there exists a number c in (a,b) with $\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$.*

Definition 1.2.6 (Metric Space)**.** Let X be a non-empty set. A metric on X is a function $d : X \times X \to \mathbb{R}$ such that for $x, y, z \in X$, we have

(i) $d(x,y) \geq 0$,

(ii) $d(x,y) = 0$ if and only if $x = y$,

(iii) $d(x,y) = d(y,x)$,

(iv) $d(x,y) \leq d(x,z) + d(z,y)$.

The pair (X,d) is called a metric space.

The following are examples of well-known metric spaces.

Example 1.2.1. On $\mathbb{R}^n = \{(x_1, x_2, x_3, \ldots, x_n), \forall x_i \in \mathbb{R}\}$, the Euclidean metric d is defined by

$$d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Example 1.2.2. Let $C[a,b]$ denote the set of all continuous real-valued functions on $[a,b]$. Define d on $C[a,b] \times C[a,b]$ by $d(x,y) = \max_{t \in [a,b]} |x(t) - y(t)|$. Then d is a metric on $C[a,b]$.

Definition 1.2.7 (Cauchy sequence, completeness)**.** A sequence $\{x_n\}$ in a metric space $X = (X,d)$ is said to be Cauchy if for every $\varepsilon > 0$, there is an $N = N(\varepsilon)$ such that $d(x_m, x_n) < \varepsilon,$ for every $m,n > N$. The space X is said to be complete if every Cauchy sequence in X converges (that is, has a limit which is an element of X).

Theorem 1.2.6. *The space $C[a,b]$ with the above metric is a complete metric space. For the proof the reader is referred to [[18], p.36].*

Example 1.2.3. The space $C[a,b]$ with the metric $d(f,g) = \int_a^b f(x)g(x)\,dx$, where $f(x)$ and $g(x) \in C[a,b]$ is non-complete metric space.

Theorem 1.2.7. $\mathbb{R}$ *with its usual metric is a complete metric space.*

Definition 1.2.8 (Normed Space)**.** Let X be a vector space over a field of scalars F. A norm on X is a function $||.|| : X \to \mathbb{R}$ such that

(i) $||x|| \geq 0, \forall x \in X,$

(ii) $||x|| = 0$ if and only if $x = 0,$

(iii) $||\alpha x|| = |\,\alpha\,|\;||x||\,; \forall \alpha \in F,$

(iv) $||x+y|| \leq ||x|| + ||y||\,, \forall x,y \in X.$

Remark 1.2.1. A norm on X defines a metric d on X given by $d(x,y) = ||x-y||\,.$

Definition 1.2.9 (Contraction)**.** Let $X = (X,d)$ be a metric space. A mapping $T : X \to X$ is called a contraction on X if there is a positive real numbers $\alpha < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \alpha\, d(x, y).$$

Theorem 1.2.8 (Banach Fixed Point Theorem)**.** *[[18], p.300]. Consider a metric space $X = (X, d)$, where $X \neq \phi$, suppose that X is complete and let $T : X \to X$ be a contraction on X. Then T has precisely one fixed point.*

Definition 1.2.10 (Partial Order Set)**.** [[19], p.142]. Let S be a set. A partial order relation $\leqslant$ on S is antisymmetric, transitive and reflexive. The pair $(S, \leqslant)$ is called a partially ordered set.

Example 1.2.4. $\mathbb{R}$ with the usual ordering is a partially ordered set.

Example 1.2.5. Let $\mathscr{F}(X)$ denote the set of real-valued functions on X. Define the order $\leqslant$ on $\mathscr{F}$ by $f \leqslant g \Longleftrightarrow f(x) \leqslant g(x)$; $\forall x \in X$. Then $(\mathscr{F}(X), \leqslant)$ is a partially ordered set.

Example 1.2.6. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. We consider the order $\leqslant$ on $\mathbb{R}^n$ defined by $x \leqslant y$ if and only if $x_i \leqslant y_i$, $\forall i = 1, 2, \ldots, n$. Then $(\mathbb{R}^n, \leqslant)$ is partially ordered set. This ordering is called the simplicial ordering of $\mathbb{R}^n$.

Definition 1.2.11 (Comparable)**.** [[19], p.144]. Let $(S, \leqslant)$ be a partially ordered set. Two members x and y of S are said to be comparable if either $x \leq y$ or $y \leq x$.

Theorem 1.2.9 (Mean value theorem for function of several variables)**.** *[20]. Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ be differentiable and the segment $[a, b]$ joining a to b be contained in U. Then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = \bigtriangledown f(c).(b - a).$$

Chapter 2: Boundary Value Problems of Fractional Order

2.1 Introduction

We consider a class of fractional boundary value problems of the form

$$D^{\delta}y + f(t,y,y') = 0,\ 0 < t < 1,\ 1 < \delta < 2, \tag{2.1}$$

with boundary conditions $\quad y(0) = a,\ y'(1) = b, \qquad (2.2)$

where f is continuous with respect to t on $[0,1]$ and smooth with respect to y and y', and the fractional derivative is considered in the Caputo's sense. Several existence and uniqueness results for various classes of fractional differential equations have been established using the method of lower and upper solutions and fixed points theorems. Problem (2.1) with $f = f(t,y)$ and homogeneous boundary conditions $u(0) = u(1) = 0$, and D^{δ} is the standard Riemann-Liouville fractional derivative was discussed by Bai and Lü [21]. They used some fixed point theorems in a cone to establish the existence and multiplicity of positive solutions. Problem (2.1) with $f = f(t,y)$ and the boundary conditions $y(0) = a$, $y(1) = b$, and D^{δ} is the Caputo fractional derivative was studied by Al-Refai and Hajji [9], where some existence and uniqueness results were established using the monotone iterative sequences of upper and lower solutions. In addition, problem (2.1) with $f(t,y) = f_0(t,y) + f_1(t,y) + f_2(t,y)$ was studied by Hu, Liu, and Xie [22] using quasi-lower and quasi-upper solutions and monotone iterative technique. To the best of our knowledge, the method of monotone iterative sequences of lower and upper solutions has not been implemented for problem (2.1) - (2.2), where the nonlinear term $f = f(t,y,y')$ depends on the variables y and y'. In order to apply the method of lower and upper solutions, we need some information about the fractional derivative of a function at its extreme points, which are difficult for the fractional derivative $1 < \delta < 2$. While some estimates were obtained by Al-Refai in [15], these estimates require more information about the function, unlike the case where $0 < \delta < 1$. Therefore, we will transform the problem (2.1)-(2.2) to a system of two equations and then apply the method of lower and upper solutions to the new system. A similar technique has been used by Syam and Al-Refai [23] for higher order fractional boundary value problems of the form $D^{\delta}y(x)+f(x,y,y'')=0,\ 0<x<1,\ 3<\delta<4$, with the boundary conditions $y(0)=a_1,\ y(1)=b_1,\ y''(0)-\mu_1 y'''(0)=a_2,\ y''(1)+\mu_2 y'''(1)=b_2$, where f is continuous with respect to t on $[0,1]$ and $y\in C^4[0,1],\ a_1,\ a_2,\ b_1,\ b_2\in\mathbb{R},\ \mu_1,\ \mu_2\geq 0$, and D^{δ} is the Caputo fractional derivative. They have established an existence result by using the method of lower and upper solutions. Moreover, the existence of the solutions to the problem (2.1) for $f=f(t,y)$ and $t\in[0,T],\ \alpha\in(0,1]$, and the boundary condition $y(0)+\mu\int_0^T y(s)\,ds=y(T)$, where D^{α} is the Caputo fractional derivative has been studied by Benchohra and Ouaar [24], using the Banach contraction principle and Schauder's fixed point theorem.

In the following, we transform problem (2.1) - (2.2) to a system of differential equations, consisting of a fractional derivative and an integer derivative. Let $y_1=y$, and $y_2=Dy$. Using the fact that $D^{\delta}y=D^{\delta-1}(Dy)$ for $1<\delta<2$, the system (2.1) - (2.2) is reduced to

$$Dy_1-y_2=0,\ 0<t<1, \tag{2.3}$$

$$D^{\alpha}y_2+f(t,y_1,y_2)=0,\ 0<t<1,\ 0<\alpha<1, \tag{2.4}$$

$$\text{with}\quad y_1(0)=a,\ y_2(1)=b, \tag{2.5}$$

where $\alpha=\delta-1$. For the above system we initially require $y_1,y_2\in C^1[0,1]$ and f is continuous with respect to the variable t and smooth with respect to the variables y_1 and y_2.

2.2 Definitions and Preliminary Results

Now, we have the following definition of lower and upper solutions for the system (2.3) - (2.5).

Definition 2.2.1 (Lower and Upper Solutions)**.** A pair of functions $(v_1, v_2) \in C^1[0,1] \times C^1[0,1]$ is called a pair of lower solutions of the problem (2.3) - (2.5), if they satisfy the following inequalities

$$Dv_1 - v_2 \leq 0,\ 0 < t < 1, \tag{2.6}$$

$$\text{and} \quad D^{\alpha} v_2 + f(t, v_1, v_2) \leq 0,\ 0 < t < 1,\ 0 < \alpha < 1, \tag{2.7}$$

$$\text{with} \quad v_1(0) \leq a,\ v_2(1) \leq b. \tag{2.8}$$

Analogously, a pair of functions $(w_1, w_2) \in C^1[0,1] \times C^1[0,1]$ is called a pair of upper solutions of the problem (2.3) - (2.5), if they satisfy the reversed inequalities. In addition, if $v_1(t) \leq w_1(t)$ and $v_2(t) \leq w_2(t), \forall t \in [0,1]$, we say that (v_1, v_2) and (w_1, w_2) are ordered pairs of lower and upper solutions.

The following new positivity result is essential in this thesis.

Lemma 2.2.1 (Positivity Result)**.** Let $\omega(t)$ be in $C^1[0,1]$ that satisfies the fractional inequality

$$D^{\alpha}\omega(t) + \mu(t)\omega(t) \geq 0,\ 0 < t < 1,\ 0 < \alpha < 1, \tag{2.9}$$

where $\mu(t) \geq 0$ and $\mu(0) \neq 0$. Then $\omega(t) \geq 0$, $\forall t \in [0,1]$.

Proof. Assume that $\omega(t) < 0$ for some $t \in [0,1]$. Since $\omega(t)$ is continuous on $[0,1]$, then $\omega(t)$ attains an absolute minimum value at $t_0 \in [0,1]$ with $\omega(t_0) < 0$. If $t_0 \in (0,1]$, then by Theorem 1.2.1, we have

$$\Gamma(1-\alpha) D^{\alpha}\omega(t_0) \leq t_0^{-\alpha}[\omega(t_0) - \omega(0)] < 0.$$

Since $\Gamma(1-\alpha) > 0$, for $0 < \alpha < 1$, we have $D^{\alpha}\omega(t_0) < 0$, and hence

$$D^{\alpha}\omega(t_0) + \mu(t_0)\omega(t_0) < 0,$$

which contradicts (2.9). If $t_0 = 0$, then by Lemma 1.2.9, $D^{\alpha}\omega(0) = 0$, and as $\mu(0) \neq 0$, we get

$$D^{\alpha}\omega(0) + \mu(0)\omega(0) < 0,$$

which contradicts (2.9). Thus the assumption made at the beginning of the proof is not valid and the statement of the lemma is proved. □

2.3 The Linear System of Fractional Differential Equations

In this section, we study the existence and uniqueness of solutions to the system of linear boundary value problems of the form

$$\begin{cases} Dy_1(t) = g(t),\ 0 < t < 1, \\ D^{\alpha}y_2(t) + \mu y_2(t) = f(t),\ 0 < t < 1,\ 0 < \alpha < 1, \\ y_1(0) = a,\ y_2(1) = b, \end{cases} \tag{2.10}$$

where μ is a positive constant and D^{α} is the Caputo fractional derivative. These results will be used later on to establish the existence and uniqueness of monotone iterative sequences of the nonlinear system (2.3) - (2.5).

Lemma 2.3.1. Let $f(t)$ and $g(t)$ be in $C[0,1]$. Then a pair of $(y_1(t), y_2(t)) \in C^1[0,1] \times C^1[0,1]$ is a solution to the system (2.10) if and only if it is a solution to the system of integral equations:

$$y_1(t) = a + \int_0^t g(s)\,ds, \tag{2.11}$$

$$y_2(t) = b + \int_0^1 G(t,s)\,[\mu y_2(s) - f(s)]\,ds, \tag{2.12}$$

where

$$G(t,s) = \begin{cases} \dfrac{(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, \; 0 \le s < t \le 1, \\ \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, \; 0 \le t \le s < 1. \end{cases} \tag{2.13}$$

Proof. As $y_1 \in C^1[0,1]$, it is clear that y_1 which satisfies Eq.(2.11) will also satisfy the first equation in the system (2.10) with $y_1(0) = a$.
Applying the fractional integral operator I^α to the second equation in the system (2.10), we get

$$y_2(t) - y_2(0) + \mu I^\alpha y_2(t) = I^\alpha f(t),$$

which can be written as

$$y_2(t) = y_2(0) - \frac{\mu}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y_2(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds.$$

Now, at $t = 1$, we have

$$b = y_2(1) = y_2(0) - \frac{\mu}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y_2(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)\,ds.$$

Thus,

$$y_2(0) = b + \frac{\mu}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y_2(s)\,ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)\,ds,$$

and

$$y_2(t) = b + \frac{\mu}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y_2(s)\,ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)\,ds$$

$$
\begin{aligned}
&-\frac{\mu}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y_2(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,ds. \\
&= b + \frac{1}{\Gamma(\alpha)}\int_0^1 (\mu y_2(s) - f(s))\,(1-s)^{\alpha-1}\,ds - \frac{1}{\Gamma(\alpha)}\int_0^t (\mu y_2(s) - f(s))\,(t-s)^{\alpha-1}\,ds.
\end{aligned}
\tag{2.14}
$$

The last equation can be written as

$$
y_2(t) = b + \int_0^1 G(t,s)\,[\mu y_2(s) - f(s)]\,ds,
$$

where $G(t,s)$ is defined in (2.13).

Conversely, let $y_2(t) \in C^1[0,1]$ satisfy Eq.(2.12), then y_2 satisfies Eq.(2.14) which can be written as

$$
y_2(t) = b + \frac{1}{\Gamma(\alpha)}\int_0^1 (\mu y_2(s) - f(s))(1-s)^{\alpha-1}\,ds - I^{\alpha}(\mu y_2(t) - f(t)).
$$

Applying the fractional derivative operator D^{α} yields

$$
\begin{aligned}
D^{\alpha} y_2(t) = D^{\alpha}&\left(b + \frac{\mu}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} y_2(s)\,ds - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s)\,ds\right) \\
&- D^{\alpha} I^{\alpha}(\mu y_2(t) - f(t)) \\
&= -\mu y_2(t) + f(t).
\end{aligned}
$$

Thus, $D^{\alpha} y_2(t) + \mu y_2(t) = f(t)$.

At $t = 1$, we have

$$
\begin{aligned}
y_2(1) = b &+ \frac{\mu}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha} y_2(s)\,ds - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s)\,ds \\
&- \frac{\mu}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} y_2(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s)\,ds
\end{aligned}
$$

$= b,$ which completes the proof of the Theorem. □

In the following theorem, we establish the existence and uniqueness result of the system (2.10), using the Banach Fixed Point Theorem.

Theorem 2.3.1. *Suppose that $f(t) \in C[0,1]$ and the constant μ satisfies*

$$0 < \frac{2\mu}{\Gamma(\alpha+1)} < 1, \tag{2.15}$$

then the system (2.10) has exactly one solution given by the Eq's.(2.11) - (2.12).

Proof. In Lemma 2.3.1, we proved that the system (2.10) is equivalent to the system (2.11)-(2.12). Therefore we will prove the existence and uniqueness of solutions for the system (2.11)-(2.12). The existence and uniqueness of a solution to Eq.(2.11) is guaranteed as $g(t) \in C[0,1]$.

To prove the existence and uniqueness of solution to Eq.(2.12), we will use the Banach Fixed Point Theorem. we define the operator $b + \int_0^1 G(t,s)\,[\mu x(s) - f(s)]\,ds$. For each $x \in C^1[0,1]$, we prove $b + \int_0^1 G(t,s)\,[\mu x(s) - f(s)]\,ds \in C[0,1]$. We have

$$\begin{aligned} b + &\frac{\mu}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} x(s)\,ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)\,ds \\ &- \frac{\mu}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds. \end{aligned}$$

Since $(1-s)^{\alpha-1}$ and $(t-s)^{\alpha-1}$, $0 < \alpha < 1$ are integrable, $x(s) \in C^1[0,1]$ and $f(s) \in C[0,1]$, then $\int_0^1 (1-s)^{\alpha-1} f(s)\,ds$ and $\int_0^1 (1-s)^{\alpha-1} x(s)\,ds$ exist.

Applying the Weighted Mean Value Theorem for integrals to $\int_0^t (t-s)^{\alpha-1} x(s)\,ds$ and $\int_0^t (t-s)^{\alpha-1} f(s)\,ds$, yields

$$\begin{aligned} b + &\frac{\mu}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} x(s)\,ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)\,ds - \frac{\mu x(\eta_1)}{\Gamma(\alpha+1)} t^\alpha \\ &+ \frac{f(\eta_2)}{\Gamma(\alpha+1)} t^\alpha, \end{aligned}$$

for some $\eta_1,\ \eta_2 \in (0,t)$ and $0 < \alpha < 1$. Thus $b + \int_0^1 G(t,s)\,[\mu x(s) - f(s)]\,ds \in C[0,1]$.

Now, let $T : C[0,1] \to C[0,1]$ with $Tx = b + \int_0^1 G(t,s)\,[\mu x(s) - f(s)]\,ds$. To show that the system (2.10) has unique solution, we apply the Banach Fixed Point Theorem to T. Let $x_1(t)$ and $x_2(t)$ be in $C[0,1]$, then we have

$$\begin{aligned}
\|Tx_1 - Tx_2\| &= \left\|\mu \int_0^1 G(t,s)(x_1 - x_2)\,ds\right\| \\
&\leq \|x_1 - x_2\|\,\mu \left\|\int_0^1 G(t,s)\,ds\right\| \\
&\leq \|x_1 - x_2\|\,\mu \max_{0\leq t\leq 1} \int_0^1 |G(t,s)|\,ds \\
&= \|x_1 - x_2\|\,\mu \max_{0\leq t,s\leq 1} |-\int_0^t \frac{(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds + \int_t^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds| \\
&= \|x_1 - x_2\| \frac{\mu}{\Gamma(\alpha+1)} \max_{0\leq t\leq 1} |-1 + t^{\alpha} + 2(1-t)^{\alpha}| \\
&= \|x_1 - x_2\| \frac{2\mu}{\Gamma(\alpha+1)}.
\end{aligned}$$

Since $\dfrac{2\mu}{\Gamma(\alpha+1)} < 1$, we have T is a contraction and by Banach Fixed Point Theorem, the equation $Tx = x$ has a unique solution on $C[0,1]$. □

2.4 Monotone Sequences of Lower and Upper Solutions

In this section, we construct monotone iterative sequences of lower and upper solutions to the system (2.3) - (2.5). Then we use these sequences to establish an existence and uniqueness result.

Given ordered pairs $V = (v_1^{(0)}, v_2^{(0)})$ and $W = (w_1^{(0)}, w_2^{(0)})$ of lower and upper solutions, respectively, to the problem (2.3) - (2.5), we define the set

$[V,W] = \{(h_1,h_2) \in C^1[0,1] \times C^1[0,1] : v_1^{(0)} \leq h_1 \leq w_1^{(0)}, v_2^{(0)} \leq h_2 \leq w_2^{(0)}\}$.

We assume that the nonlinear term $f(t,y_1,y_2)$ satisfies the following conditions on $[V,W]$:

(A1) The function $f(t,h_1,h_2)$ is decreasing with respect to h_1, that is $\dfrac{\partial f}{\partial h_1}(t,h_1,h_2) \leq 0$ for all $(h_1,h_2) \in [V,W]$, and $t \in [0,1]$.

(A2) There exists a positive constant c, such that $\dfrac{\partial f}{\partial h_2}(t,h_1,h_2) \leq c$, for all $(h_1,h_2) \in [V,W]$, and $t \in [0,1]$.

The following theorem describes the monotone iterative sequences of lower and upper pairs of solutions.

Theorem 2.4.1. *Assume that the conditions (A1) and (A2) are satisfied and consider the iterative sequence* $U^{(k)} = (y_1^{(k)}, y_2^{(k)})$, $k \geq 0$ *which is defined by*

$$Dy_1^{(k)}(t) = y_2^{(k-1)}(t),\ 0 < t < 1 \tag{2.16}$$

$$D^\alpha y_2^{(k)}(t) + cy_2^{(k)}(t) = cy_2^{(k-1)}(t) - f(t, y_1^{(k-1)}, y_2^{(k-1)}),\ 0 < t < 1,\ 0 < \alpha < 1, \tag{2.17}$$

$$\text{with} \quad y_1^{(k)}(0) = a_k,\ y_2^{(k)}(1) = b_k. \tag{2.18}$$

We have

1. *If* $U^{(0)} = V = (v_1^{(0)}, v_2^{(0)})$ *and* a_k, b_k *are increasing sequences with* $a_k \leq a$, $b_k \leq b$, *then* $U^{(k)} = (y_1^{(k)}, y_2^{(k)}) = (v_1^{(k)}, v_2^{(k)}) = V^{(k)}$ *is an increasing sequence of lower pairs of solutions to the problem (2.3)-(2.5).*

2. *If* $U^{(0)} = W = (w_1^{(0)}, w_2^{(0)})$ *and* a_k, b_k *are decreasing sequences with* $a_k \geq a$, $b_k \geq b$, *then* $U^{(k)} = (y_1^{(k)}, y_2^{(k)}) = (w_1^{(k)}, w_2^{(k)}) = W^{(k)}$ *is a decreasing sequence of upper pairs of solutions to the problem (2.3) - (2.5).*

   Moreover,

3. $v_1^{(k)} \leq w_1^{(k)}$ *and* $v_2^{(k)} \leq w_2^{(k)}$, $\forall k \geq 0$.

Proof.

1. First, we use mathematical induction to show that $U^{(k)} = (v_1^{(k)}, v_2^{(k)})$ is an increasing sequence. For $k = 1$, we have

$$Dv_1^{(1)}(t) = v_2^{(0)}(t),\ 0 < t < 1 \tag{2.19}$$

$$D^\alpha v_2^{(1)}(t) + cv_2^{(1)} = cv_2^{(0)} - f(t, v_1^{(0)}, v_2^{(0)}),\ 0 < t < 1,\ 0 < \alpha < 1, \tag{2.20}$$

$$\text{with} \quad v_1^{(1)}(0) = a_1,\ v_2^{(1)}(1) = b_1. \tag{2.21}$$

Since $V = (v_1^{(0)}, v_2^{(0)})$ is a pair of lower solution, we have

$$Dv_1^{(0)} - v_2^{(0)} \leq 0,\ 0 < t < 1 \tag{2.22}$$

$$D^{\alpha} v_2^{(0)} + f(t, v_1^{(0)}, v_2^{(0)}) \leq 0,\ 0 < t < 1,\ 0 < \alpha < 1, \tag{2.23}$$

$$\text{and} \quad v_1^{(0)}(0) = a_0 \leq a, v_2^{(0)}(1) = b_0 \leq b. \tag{2.24}$$

Let $z_1 = v_1^{(1)} - v_1^{(0)}$ and by substituting Eq.(2.19) in Eq.(2.22), we have

$$0 \geq Dv_1^{(0)} - Dv_1^{(1)} = -D(v_1^{(1)} - v_1^{(0)}) = -Dz_1.$$

Thus $Dz_1 \geq 0$, with $z_1(0) = a_1 - a_0 \geq 0$. Since $Dz_1 \geq 0$, this means z_1 is non-decreasing which together with $z_1(0) \geq 0$ imply that $z_1 \geq 0$, and hence $v_1^{(1)} \geq v_1^{(0)}$. To prove that $v_2^{(1)} \geq v_2^{(0)}$, let $z_2 = v_2^{(1)} - v_2^{(0)}$ and by substituting Eq.(2.20) in Eq.(2.23), we have

$$\begin{aligned} 0 &\geq D^{\alpha} v_2^{(0)} - D^{\alpha} v_2^{(1)} - cv_2^{(1)} + cv_2^{(0)} \\ &= -D^{\alpha}(v_2^{(1)} - v_2^{(0)}) - c(v_2^{(1)} - v_2^{(0)}) \\ &= -D^{\alpha} z_2 - cz_2. \end{aligned}$$

Therefore $D^{\alpha} z_2 + cz_2 \geq 0$. By applying the positivity lemma, we have that $z_2 \geq 0$, and hence $v_2^{(1)} \geq v_2^{(0)}$. Now, assume that $v_1^{(k)} \geq v_1^{(k-1)}$ and $v_2^{(k)} \geq v_2^{(k-1)}$, for $k = 0, 1, 2, \ldots, n$.

From Equations (2.19) and (2.20), we have

$$Dv_1^{(n)} = v_2^{(n-1)},\ 0 < t < 1 \tag{2.25}$$

$$Dv_1^{(n+1)} = v_2^{(n)},\ 0 < t < 1 \tag{2.26}$$

$$D^{\alpha}v_2^{(n)} + cv_2^{(n)} = cv_2^{(n-1)} - f(t, v_1^{(n-1)}, v_2^{(n-1)}),\ 0 < \alpha < 1 \tag{2.27}$$

$$\text{and} \quad D^{\alpha}v_2^{(n+1)} + cv_2^{(n+1)} = cv_2^{(n)} - f(t, v_1^{(n)}, v_2^{(n)}),\ 0 < \alpha < 1. \tag{2.28}$$

Subtracting Eq.(2.25) from Eq.(2.26), and Eq.(2.28) from Eq.(2.27), we have

$$\begin{aligned} &D(v_1^{(n+1)} - v_1^{(n)}) = v_2^{(n)} - v_2^{(n-1)}, \\ \text{and} \quad &D^{\alpha}(v_2^{(n+1)} - v_2^{(n)}) + c(v_2^{(n+1)} - v_2^{(n)}) \\ &= c(v_2^{(n)} - v_2^{(n-1)}) + f(t, v_1^{(n-1)}, v_2^{(n-1)}) - f(t, v_1^{(n)}, v_2^{(n)}). \end{aligned}$$

Let $z_1 = v_1^{(n+1)} - v_1^{(n)}$ and using the induction hypothesis, we have that $Dz_1 = v_2^{(n)} - v_2^{(n-1)} \geq 0$, with $z_1(0) \geq 0$, which proves that $v_1^{(n+1)} \geq v_1^{(n)}$.
Let $z_2 = v_2^{(n+1)} - v_2^{(n)}$ and applying the induction hypothesis, the conditions (A1) and (A2) and the Mean Value Theorem, we have

$$\begin{aligned} D^{\alpha}z_2 + cz_2 &= c(v_2^{(n)} - v_2^{(n-1)}) + (v_1^{(n-1)} - v_1^{(n)})\frac{\partial f}{\partial y_1}(\rho_1) \\ &\quad + (v_2^{(n-1)} - v_2^{(n)})\frac{\partial f}{\partial y_2}(\rho_2) \\ &= (v_2^{(n-1)} - v_2^{(n)})(\frac{\partial f}{\partial y_2}(\rho_2) - c) + (v_1^{(n-1)} - v_1^{(n)})\frac{\partial f}{\partial y_1}(\rho_1) \geq 0. \end{aligned}$$

where $\rho_1 = \mu v_1^{(n-1)} + (1-\mu)v_1^{(n)}, \rho_2 = \nu v_2^{(n-1)} + (1-\nu)v_2^{(n)}$ with $0 \leq \mu, \nu \leq 1$.
Again, by the positivity lemma, $z_2 \geq 0$ and hence $v_2^{(n+1)} \geq v_2^{(n)}$.
Second, we prove that $(v_1^{(k)}, v_2^{(k)})$, for all $k \geq 0$ is a pair of lower solutions.
Since the sequence $\{v_2^{(k)}\}$ is increasing and $Dv_1^{(k)} = v_2^{(k-1)}$, we have $Dv_1^{(k)} - v_2^{(k)} = v_2^{(k-1)} - v_2^{(k)} \leq 0$, which together with $v_1^{(k)}(0) = a_k \leq a$, prove that $v_1^{(k)}$ is a lower solution. From Eq.(2.17), we have

$$\begin{aligned} D^{\alpha}v_2^{(k)} + cv_2^{(k)} &= cv_2^{(k-1)} - f(t, v_1^{(k-1)}, v_2^{(k-1)}) \\ D^{\alpha}v_2^{(k)} &= -cv_2^{(k)} + cv_2^{(k-1)} - f(t, v_1^{(k-1)}, v_2^{(k-1)}) \\ &= -c(v_2^{(k)} - v_2^{(k-1)}) - f(t, v_1^{(k-1)}, v_2^{(k-1)}). \end{aligned}$$

By adding $f(t, v_1^{(k)}, v_2^{(k)})$, applying the Mean Value Theorem and using the fact that the sequences $\{v_1^{(k)}\}$ and $\{v_2^{(k)}\}$ are increasing, we have

$$\begin{aligned}
D^\alpha v_2^{(k)} + f(t, v_1^{(k)}, v_2^{(k)}) &= -c(v_2^{(k)} - v_2^{(k-1)}) + f(t, v_1^{(k)}, v_2^{(k)}) - f(t, v_1^{(k-1)}, v_2^{(k-1)}) \\
&= -c(v_2^{(k)} - v_2^{(k-1)}) + \frac{\partial f}{\partial y_1}(\rho_1)(v_1^{(k)} - v_1^{(k-1)}) + \frac{\partial f}{\partial y_2}(\rho_2)(v_2^{(k)} - v_2^{(k-1)}) \\
&= (-c + \frac{\partial f}{\partial y_2}(\rho_2))(v_2^{(k)} - v_2^{(k-1)}) + \frac{\partial f}{\partial y_1}(\rho_1)(v_1^{(k)} - v_1^{(k-1)}),
\end{aligned}$$

where $\rho_1 = \zeta_1 v_1^{(k)} + (1-\zeta_1)v_1^{(k-1)}$, $\rho_2 = \zeta_2 v_2^{(k)} + (1-\zeta_2)v_2^{(k-1)}$, and $0 \le \zeta_1, \zeta_2 \le 1$.

Applying the conditions (A1) and (A2), we have $D^\alpha v_2^{(k)} + f(t, v_1^{(k)}, v_2^{(k)}) \le 0$, which together with $v_2^{(k)}(1) = b_k \le b$, prove that $v_2^{(k)}$ is a lower solution.

2. Similar to the proof of (1). First, we apply induction arguments to prove that the two sequences $\{w_1^{(k)}\}$ and $\{w_2^{(k)}\}$ are decreasing. Then, we use these results to show that $(w_1^{(k)}, w_2^{(k)})$ is a pair of upper solutions for each $k \ge 0$.

3. Since $V = (v_1^{(0)}, v_2^{(0)})$ and $W = (w_1^{(0)}, w_2^{(0)})$ are ordered pairs of lower and upper solutions, we have $v_1^{(0)} \le w_1^{(0)}$ and $v_2^{(0)} \le w_2^{(0)}$. Hence the result is true for $n = 0$.
Assume that $v_1^{(k)} \le w_1^{(k)}$ and $v_2^{(k)} \le w_2^{(k)}$, for all $k = 0, 1, 2, \ldots, n$.
We have $Dv_1^{(n+1)} = v_2^{(n)}$ and $Dw_1^{(n+1)} = w_2^{(n)}$. Thus

$$Dw_1^{(n+1)} - Dv_1^{(n+1)} = w_2^{(n)} - v_2^{(n)} \ge 0.$$

Let $z_1 = w_1^{(n+1)} - v_1^{(n+1)}$, thus $Dz_1 \ge 0$, and with $w_1^{(n+1)}(0) \ge v_1^{(n+1)}(0)$ imply $z_1 \ge 0$, and hence $w_1^{(n+1)} \ge v_1^{(n+1)}$. Similarly, we have

$$D^\alpha v_2^{(n+1)} + c v_2^{(n+1)} = c v_2^{(n)} - f(t, v_1^{(n)}, v_2^{(n)}), \tag{2.29}$$

$$\text{and} \quad D^\alpha w_2^{(n+1)} + c w_2^{(n+1)} = c w_2^{(n)} - f(t, w_1^{(n)}, w_2^{(n)}). \tag{2.30}$$

Subtract Eq.(2.29) from Eq.(2.30), we have

$$D^{\alpha}(w_2^{(n+1)} - v_2^{(n+1)}) + c(w_2^{(n+1)} - v_2^{(n+1)})$$
$$= c(w_2^{(n)} - v_2^{(n)}) + f(t, v_1^{(n)}, v_2^{(n)}) - f(t, w_1^{(n)}, w_2^{(n)}).$$

Let $z_2 = w_2^{(n+1)} - v_2^{(n+1)}$. Then z_2 satisfies
$D^{\alpha}z_2 + cz_2 = c(w_2^{(n)} - v_2^{(n)}) + f(t, v_1^{(n)}, v_2^{(n)}) - f(t, w_1^{(n)}, w_2^{(n)})$.
Applying the Mean Value Theorem yields

$$\begin{aligned} D^{\alpha}z_2 + cz_2 &= c(w_2^{(n)} - v_2^{(n)}) + \frac{\partial f}{\partial y_1}(\rho_1)(v_1^{(n)} - w_1^{(n)}) + \frac{\partial f}{\partial y_2}(\rho_2)(v_2^{(n)} - w_2^{(n)}) \\ &= (v_2^{(n)} - w_2^{(n)})(\frac{\partial f}{\partial y_2}(\rho_2) - c) + \frac{\partial f}{\partial y_1}(\rho_1)(v_1^{(n)} - w_1^{(n)}), \end{aligned}$$

for some $\rho_1 = \zeta_1 v_1^{(n)} + (1-\zeta_1)w_1^{(n)}$, $\rho_2 = \zeta_2 v_2^{(n)} - (1-\zeta_2)w_2^{(n)}$ and
$0 \le \zeta_1, \zeta_2 \le 1$.
By the induction hypothesis $w_1^{(n)} \ge v_1^{(n)}$ and $w_2^{(n)} \ge v_2^{(n)}$ and the conditions (A1) and (A2), we have $D^{\alpha}z_2 + cz_2 \ge 0$, which proves that $z_2 \ge 0$. Therefore, $w_2^{(n+1)} \ge v_2^{(n+1)}$, and the proof is completed. □

Remark 2.4.1. The existence and uniqueness of solutions to the sequence defined in (2.16) - (2.18) is guaranteed by Theorem 2.3.1.

Now, we state the convergence results of the two sequences of ordered pairs of lower and upper solutions described in Theorem 2.4.1.

Theorem 2.4.2. *Assume that the conditions (A1) and (A2) are satisfied, and consider the two iterative sequences $V^{(k)} = (v_1^{(k)}, v_2^{(k)})$ and $W^{(k)} = (w_1^{(k)}, w_2^{(k)})$, obtained from (2.16) - (2.18), with $U^{(0)} = V = (v_1^{(0)}, v_2^{(0)})$ and $U^{(0)} = W = (w_1^{(0)}, w_2^{(0)})$, respectively. Then*

(1) The two sequences converge pointwise to $V^* = (v_1^*, v_2^*)$ and $W^* = (w_1^*, w_2^*)$, respectively with $v_1^* \le w_1^*$ and $v_2^* \le w_2^*$. Moreover,

(2) For any solution $Y = (y_1, y_2) \in [V, W]$ of (2.3) - (2.5), we have $v_1^* \le y_1 \le w_1^*$ and $v_2^* \le y_2 \le w_2^*$.

Proof. (1) The two sequences $v_1^{(k)}$ and $v_2^{(k)}$ are increasing and bounded above by $w_1^{(0)}$ and $w_2^{(0)}$, respectively. Hence, they converge pointwise to v_1^* and v_2^*, respectively. By applying similar arguments, the two sequences $w_1^{(k)}$ and $w_2^{(k)}$ are decreasing and bounded below by $v_1^{(0)}$ and $v_2^{(0)}$, respectively. Hence, they converge pointwise to w_1^* and w_2^*, respectively.
Since $v_1^{(k)} \le w_1^{(k)}$ and $v_2^{(k)} \le w_2^{(k)}$, $\forall k \ge 0$, then $v_1^* \le w_1^*$ and $v_2^* \le w_2^*$.
(2) It is enough to show that $v_1^{(k)} \le y_1 \le w_1^{(k)}$ and $v_2^{(k)} \le y_2 \le w_2^{(k)}$, $\forall k \ge 0$. We use mathematical induction to show that $v_1^{(k)} \le y_1$ and $v_2^{(k)} \le y_2$, $\forall k \ge 0$. Similar arguments can be used to prove that $y_1 \le w_1^{(k)}$ and $y_2 \le w_2^{(k)}$, $\forall k \ge 0$.
Since $Y = (y_1, y_2) \in [V, W]$, then the result is true for $k = 0$.
Assume that $v_1^{(k)} \le y_1$ and $v_2^{(k)} \le y_2$, $\forall k = 0, 1, \dots, n$. Then we have
$Dv_1^{(n+1)} = v_2^{(n)}$ and $Dy_1 - y_2 = 0$. Therefore, there holds
$-Dv_1^{(n+1)} + Dy_1 - y_2 = -v_2^{(n)}$, or
$D(y_1 - v_1^{(n+1)}) = y_2 - v_2^{(n)}$.
By induction hypothesis, we have $D(y_1 - v_1^{(n+1)}) \ge 0$, which together with $y_1(0) \ge v_1^{(n+1)}(0)$, prove that $y_1 - v_1^{(n+1)} \ge 0$, and $y_1 \ge v_1^{(n+1)}$.
By subtracting Eq.'s (2.4) and (2.28), we get

$$D^\alpha(y_2 - v_2^{(n+1)}) - cv_2^{(n+1)} = -cv_2^{(n)} + f(t, v_1^{(n)}, v_2^{(n)}) - f(t, y_1, y_2).$$

Adding $c(y_2 - v_2^{(n+1)})$ for both sides, we get

$$\begin{aligned}
D^\alpha(y_2 - v_2^{(n+1)}) + c(y_2 - v_2^{(n+1)}) - cv_2^{(n+1)} &= -cv_2^{(n)} + c(y_2 - v_2^{(n+1)}) \\
&\quad + f(t, v_1^{(n)}, v_2^{(n)}) - f(t, y_1, y_2) \\
&= cv_2^{(n+1)} - cv_2^{(n)} + c(y_2 - v_2^{(n+1)}) + f(t, v_1^{(n)}, v_2^{(n)}) - f(t, y_1, y_2) \\
&= cv_2^{(n+1)} - cv_2^{(n)} + cy_2 - cv_2^{(n+1)} + f(t, v_1^{(n)}, v_2^{(n)}) - f(t, y_1, y_2)
\end{aligned}$$

$$= -c(v_2^{(n)} - y_2) + f(t, v_1^{(n)}, v_2^{(n)}) - f(t, y_1, y_2).$$

By applying the Mean Value Theorem, we have

$$\begin{aligned} D^\alpha(y_2 - v_2^{(n+1)}) + c(y_2 - v_2^{(n+1)}) &= -c(v_2^{(n)} - y_2) + \frac{\partial f}{\partial y_1}(\rho_1)(v_1^{(n)} - y_1) \\ &+ \frac{\partial f}{\partial y_2}(\rho_2)(v_2^{(n)} - y_2) \\ &= (v_2^{(n)} - y_2)(\frac{\partial f}{\partial y_2}(\rho_2) - c) + \frac{\partial f}{\partial y_1}(\rho_1)(v_1^{(n)} - y_1) \geq 0, \end{aligned}$$

for some $\rho_1 = \mu v_1^{(n)} + (1 - \mu y_1), \rho_2 = \beta v_2^{(n)} + (1 - \beta y_2)$ and $0 \leq \mu, \beta \leq 1$.
Thus $D^\alpha(y_2 - v_2^{(n+1)}) + c(y_2 - v_2^{(n+1)}) \geq 0$,
which proves that $y_2 \geq v_2^{(n+1)}$ by the positivity lemma. By applying similar argument, one can show that $y_1 \leq w_1^*$ and $y_2 \leq w_2^*$. □

2.5 Existence and Uniqueness of Solutions

In order to establish the existence and uniqueness of solutions to problem (2.3) - (2.5), we start with the following lemma.

Lemma 2.5.1. A pair of functions $(y_1(t), y_2(t)) \in C^1[0,1] \times C^1[0,1]$ is a solution to the problem (2.3) - (2.5) if and only if it is a solution to the system of integral equations

$$y_1(t) = a + \int_0^t y_2(s)\,ds, \; 0 < t < 1 \tag{2.31}$$

and $$y_2(t) = \eta - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_1(s), y_2(s))\,ds, \; 0 < t < 1, \; 0 < \alpha < 1, \tag{2.32}$$

where $\eta = b + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, y_1(s), y_2(s))\,ds$.

Proof. Let $(y_1(t), y_2(t)) \in C^1[0,1] \times C^1[0,1]$ be a solution to the boundary value problem (2.3) - (2.5). Applying the integral operator I for Eq.(2.3), we have

$$y_1(t) = y_1(0) + Iy_2(t) = a + \int_0^t y_2(s)\,ds,\ 0 < t < 1,$$

which proves the result in the Eq.(2.31). Applying the Riemann-Liouville fractional integral I^α for Eq.(2.4), we have

$$y_2(t) = y_2(0) - I^\alpha f(t, y_1(t), y_2(t)),\ 0 < t < 1,\ 0 < \alpha < 1. \tag{2.33}$$

Since, $y_2(1) = y_2(0) - I^\alpha f(t, y_1(t), y_2(t))|_{t=1}$, we have

$$\begin{aligned} y_2(0) &= y_2(1) + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, y_1(s), y_2(s))\,ds \\ &= b + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, y_1(s), y_2(s))\,ds. \end{aligned}$$

By substituting the last Eq. in Eq.(2.33) the result is obtained.

Conversely, let $y_1(t) \in C^1[0,1]$ satisfy Eq.(2.31). Applying the derivative operator D yields $Dy_1(t) = y_2(t)$.

Similarly, let $y_2(t) \in C^1[0,1]$ satisfy Eq.(2.32). Applying the Caputo fractional derivative operator D^α, and using the fact that $D^\alpha \eta = 0$, for any constant η and $D^\alpha I^\alpha f(t) = f(t)$, we have $D^\alpha y_2(t) + f(t, y_1, y_2) = 0$. For the boundary conditions we have,

$$y_1(0) = a + \int_0^0 y_2(s)\,ds = a,$$

and
$$\begin{aligned} y_2(1) &= \eta - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, y_1(s), y_2(s)\,ds \\ &= b + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, y_1(s), y_2(s)\,ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, y_1(s), y_2(s)\,ds \\ &= b, \end{aligned}$$
which completes the proof. □

The next theorem proves the existence of solutions of problem (2.31) - (2.32).

Theorem 2.5.1 (Existence Result)**.** *Let $V^* = (v_1^*, v_2^*)$ and $W^* = (w_1^*, w_2^*)$ be the limits of the two sequences $V^{(k)} = (v_1^{(k)}, v_2^{(k)})$ and $W^{(k)} = (w_1^{(k)}, w_2^{(k)})$ defined by (2.16) - (2.18) with $V^{(0)} = (v_1^{(0)}, v_2^{(0)})$ and $W^{(0)} = (w_1^{(0)}, w_2^{(0)})$, respectively. Assume that $\lim_{k\to\infty} a_k = a$ and $\lim_{k\to\infty} b_k = b$. Then V^* and W^* are solutions to (2.31) - (2.32).*

Proof. We have that

$$Dv_1^{(k)} = v_2^{(k-1)}, \tag{2.34}$$

$$\text{and} \quad D^\alpha v_2^{(k)} + cv_2^{(k)} = cv_2^{(k-1)} - f(t, v_1^{(k-1)}, v_2^{(k-1)}). \tag{2.35}$$

Applying the integral operator I for Eq.(2.34), we have
$v_1^{(k)} - v_1^{(k)}(0) = I(v_2^{(k-1)})$, where $v_1^{(k)}(0) = a_k$. Taking the limit and using the fact that $v_1^{(k)}$ converges pointwise to v_1^*, we have $v_1^* = a + \lim_{k\to\infty} I(v_2^{(k-1)})$.
Since $v_2^{(k)}$ converges pointwise to v_2^*, is bounded and Riemann integrable, then by Theorem 1.2.3, we have

$$v_1^* = a + I(v_2^*) = a + \int_0^t v_2^* ds, \; 0 < t < 1, \tag{2.36}$$

which proves that v_2^* is a solution to Eq.(2.31).
Similarly, applying the fractional integral operator I^α for the Eq.(2.35), we have
$I^\alpha(D^\alpha v_2^{(k)}) + cI^\alpha(v_2^{(k)}) = cI^\alpha(v_2^{(k-1)}) - I^\alpha(f(t, v_1^{(k-1)}, v_2^{(k-1)}))$,
or
$v_2^{(k)} - v_2^{(k)}(0) + cI^\alpha(v_2^{(k)}) = cI^\alpha(v_2^{(k-1)}) - I^\alpha(f(t, v_1^{(k-1)}, v_2^{(k-1)}))$.
Taking the limit and using the facts that $v_1^{(k)}$ and $v_2^{(k)}$ converge pointwise to v_1^* and v_2^*, respectively, they are bounded and Riemann integrable, and f is continuous, we have

$$v_2^* - v_2^*(0) + cI^\alpha(v_2^*) = cI^\alpha(v_2^*) - I^\alpha(f(t, v_1^*, v_2^*)). \tag{2.37}$$

$$\text{Thus,} \quad v_2^*(t) = v_2^*(0) - I^\alpha(f(t, v_1^*, v_2^*)). \tag{2.38}$$

Now, at $t = 1$ we have $v_2^*(1) = v_2^*(0) - I^\alpha f(t, v_1^*, v_2^*)|_{t=1}$ and then

$$v_2^*(0) = b + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, v_1^*(s), v_2^*(s))\, ds. \tag{2.39}$$

Substitute Eq.(2.39) in Eq.(2.38) to obtain the result. By similar arguments, one can show that (w_1^*, w_2^*) is also a solution to the problem (2.31) - (2.32). $\square$

Remark 2.5.1. Since in general, we don't guarantee that $V^*, W^* \in C^1[0,1] \times C^1[0,1]$, V^* and W^* are called weak solutions of the problem (2.3) - (2.5). In the literature we refer to V^* and W^* by the minimal and maximal solutions, respectively.

Before establishing the uniqueness result, we have the following definition of comparable solutions.

Definition 2.5.1 (Comparable Solutions)**.** Assume that $(u_1, u_2) \neq (v_1, v_2)$ are two solutions of the problem (2.3) - (2.5). We say that (u_1, u_2) and (v_1, v_2) are comparable solutions, if either $(u_1, u_2) \leq (v_1, v_2)$ or $(v_1, v_2) \leq (u_1, u_2)$.

Theorem 2.5.2. *Let $(y_1(t), y_2(t)) \in C^1[0,1] \times C^1[0,1]$ and $(x_1(t), x_2(t)) \in C^1[0,1] \times C^1[0,1]$ be comparable solutions of the problem (2.3) - (2.5), such that the conditions (A1) and (A2) are satisfied and there exists $c_1 < 0$ such that $\dfrac{\partial f}{\partial y_2}(t, y_1, y_2) \leq c_1 < 0$. Then $(y_1, y_2) = (x_1, x_2)$, for all $t \in [0,1]$.*

Proof. Since (y_1, y_2) and (x_1, x_2) are solutions of problem (2.3)-(2.5), we have

$$Dy_1 - y_2 = Dx_1 - x_2 = 0, \tag{2.40}$$

$$\text{and} \quad D^\alpha y_2 + f(t, y_1, y_2) = D^\alpha x_2 + f(t, x_1, x_2) = 0, \tag{2.41}$$

$$\text{with} \quad y_1(0) = x_1(0) = a,\ y_2(1) = x_2(1) = b. \tag{2.42}$$

Equations (2.40) and (2.41) can be written as:

$$D(x_1-y_1)=x_2-y_2,\ 0<t<1,$$
$$D^{\alpha}(x_2-y_2)+f(t,x_1,x_2)-f(t,y_1,y_2)=0,\ 0<t<1,\ 0<\alpha<1.$$

As (y_1,y_2) are (x_1,x_2) are comparable solutions we assume without loss of generality that $y_1\le x_1$ and $y_2\le x_2$. Let $z_1=x_1-y_1$, and $z_2=x_2-y_2$. Applying the Mean Value Theorem for the last equation we obtain

$$Dz_1=z_2,\ 0<t<1, \tag{2.43}$$
$$D^{\alpha}z_2+\frac{\partial f}{\partial y_1}(\rho_1)z_1+\frac{\partial f}{\partial y_2}(\rho_2)z_2=0,\ 0<t<1,\ 0<\alpha<1, \tag{2.44}$$

with $z_1(0)=0$ and $z_2(1)=0$, where $\rho_1=\mu x_1+(1-\mu)y_1,\rho_2=\nu x_2+(1-\nu)y_2$ and $0\le\mu,\nu\le1$.

By the continuity of $z_1(t)$ and $z_2(t)$ for $t\in[0,1]$, we have

$$D^{\alpha}z_2(0)+\frac{\partial f}{\partial y_1}(\rho_1)z_1(0)+\frac{\partial f}{\partial y_2}(\rho_2)z_2(0)=0.$$

As $z_2\in C^1[0,1]$ by Lemma 1.2.9, $D^{\alpha}z_2(0)=0$, and since $z_1(0)=0$, we have $\frac{\partial f}{\partial y_2}(\rho_2)z_2(0)=0$. Since $\frac{\partial f}{\partial y_2}(\rho_2)\neq0$, we have $z_2(0)=0$. Since $\frac{\partial f}{\partial y_1}(t,y_1,y_2)\le0$, and $z_1\ge0$, the Eq.(2.44) leads to

$$D^{\alpha}z_2+\frac{\partial f}{\partial y_2}(\rho_2)z_2=-\frac{\partial f}{\partial y_1}(\rho_1)z_1\ge0. \tag{2.45}$$

Since $\frac{\partial f}{\partial y_2}(t,y_1,y_2)\le c_1<0$, and $z_2\ge0$, we have $\frac{\partial f}{\partial y_2}(\rho_2)z_2\le c_1z_2\le0$, therefore Inequality (2.45) leads to

$$0\le D^{\alpha}z_2+\frac{\partial f}{\partial y_2}(\rho_2)z_2\le D^{\alpha}z_2+c_1z_2. \tag{2.46}$$

Applying the fractional integral operator I^{α} to the last inequality, we have

$0 \leq I^{\alpha}D^{\alpha}z_2 + c_1 I^{\alpha}z_2 = z_2(t) - z_2(0) + c_1 I^{\alpha}z_2.$

Since $z_2(0) = 0$, we have

$$0 \leq z_2(t) + c_1 I^{\alpha}z_2(t),\ \forall t \in [0,1]. \tag{2.47}$$

In the following, we prove that $z_2(t) = 0,\ \forall t \in [0,1]$. Assume by contradiction that $z_2(t) \neq 0$ in $[0,1]$. Since $z_2(1) = 0$, we have at $t = 1$,

$$0 \leq z_2(1) + c_1 I^{\alpha}z_2(1) = c_1 I^{\alpha}z_2(1) = c_1 \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} z_2(s)\, ds. \tag{2.48}$$

The function z_2 satisfies the following: $z_2(0) = z_2(1) = 0$, $z_2 \geq 0$. As $z_2 \neq 0$ on $[0,1]$, there exists $t_0 \in (0,1)$ such that $z_2(t_0) > 0$. By the continuity of $z_2(t)$ there exists a neighborhood of t_0, $N_{\delta}(t_0) = (t_0 - \delta, t_0 + \delta)$, such that $z_2(t) > 0,\ \forall t \in N_{\delta}(t_0)$. Therefore

$$\begin{aligned} I^{\alpha}z_2(1) = \frac{1}{\Gamma(\alpha)} \Big[& \int_0^{t_0-\delta} (1-s)^{\alpha-1} z_2(s)\, ds + \int_{t_0-\delta}^{t_0+\delta} (1-s)^{\alpha-1} z_2(s)\, ds \\ & + \int_{t_0+\delta}^{1} (1-s)^{\alpha-1} z_2(s)\, ds \Big]. \end{aligned}$$

The first and last integrals are non-negative since both $(1-s)^{\alpha-1}$ and z_2 are non-negative. Applying the Weighted Mean Value Theorem for Integrals for the second integral, we have

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_{t_0-\delta}^{t_0+\delta} (1-s)^{\alpha-1} z_2\, ds &= \frac{z_2(\mu)}{\Gamma(\alpha)} \int_{t_0-\delta}^{t_0+\delta} (1-s)^{\alpha-1}\, ds \\ &= \frac{z_2(\mu)}{\Gamma(\alpha+1)} \left((1 - t_0 + \delta)^{\alpha} - (1 - t_0 - \delta)^{\alpha} \right) > 0, \end{aligned}$$

for some $\mu \in (t_0 - \delta, t_0 + \delta)$. Therefore $I^{\alpha}z_2(1) > 0$, which together with $c_1 < 0$, lead to $c_1 I^{\alpha}z_2(1) < 0$, a result that contradicts Eq.(2.48). Hence the assumption made is not correct and therefore $z_2(t) = 0,\ \forall t \in [0,1]$. Substituting the last result in Eq.(2.43) yields $Dz_1 = 0$, which together with $z_1(0) = 0$, lead to $z_1 = 0,\ \forall t \in [0,1]$. Thus, $x_1 = y_1$

and $x_2 = y_2$ and the result of the theorem is proved. $\square$

Theorem 2.5.3 (Existence and Uniqueness Result). *Let* $V^* = (v_1^*, v_2^*) \in C^1[0,1] \times C^1[0,1]$ *and* $W^* = (w_1^*, w_2^*) \in C^1[0,1] \times C^1[0,1]$ *be as in Theorem 2.5.1 and assume that they satisfy the conditions in Theorem 2.5.2 with* $\dfrac{\partial f}{\partial y_2}(t, y_1, y_2) \le c_1 < 0$, *for some* $c_1 < 0$. *Then* $v_1^* = w_1^*$ *and* $v_2^* = w_2^*$ *and the problem (2.3) - (2.5) has a unique solution on* $[V, W]$.

Proof. Since V^*, $W^* \in C^1[0,1] \times C^1[0,1]$ and satisfy $v_1^* \le w_1^*$ and $v_2^* \le w_2^*$, then (v_1^*, v_2^*) and (w_1^*, w_2^*) are comparable solution for the problem (2.3) - (2.5). As $\dfrac{\partial f}{\partial y_2}(t, y_1, y_2) \le c_1 < 0$, for some $c_1 < 0$, by Theorem 2.5.2, we have $v_1^* = w_1^*$ and $v_2^* = w_2^*$. $\square$

2.6 Illustrated Examples

In this section, we apply the analysis described in the previous sections for two examples to illustrate the validity of our result.

Example 2.6.1. Consider the linear fractional boundary value problem

$$D^{\frac{5}{3}} y(t) = \frac{1}{4} \sqrt[3]{t}\, y(t) - \frac{1}{4} y'(t),\ 0 < t < 1, \tag{2.49}$$

$$\text{with} \quad y(0) = 1,\ y'(1) = 0. \tag{2.50}$$

We first transform the problem to the following system

$$D y_1(t) - y_2(t) = 0,\ 0 < t < 1, \tag{2.51}$$

$$D^{\frac{2}{3}} y_2(t) - \frac{1}{4} \sqrt[3]{t}\, y_1(t) + \frac{1}{4} y_2(t) = 0,\ 0 < t < 1, \tag{2.52}$$

$$\text{with} \quad y_1(0) = 1,\ y_2(1) = 0, \tag{2.53}$$

where $y_1(t) = y(t)$ and $y_2(t) = y'(t)$. In the following we show that $V^{(0)} = (v_1^{(0)}, v_2^{(0)}) = (1,0)$ and $W^{(0)} = (w_1^{(0)}, w_2^{(0)}) = (t+1,t)$ are ordered pairs of lower and upper solutions to the system (2.51) - (2.53). It is clear that $V^{(0)}$ satisfies the definition of the lower solutions given in Eq's.(2.6) - (2.8). We now show that $W^{(0)}$ is an upper solution. We have

$$D(t+1) - t = 1 - t \geq 0,\ 0 < t < 1,$$

$$\text{and}\quad D^{\frac{2}{3}}\,t - \frac{1}{4}t^{\frac{4}{3}} - \frac{1}{4}t^{\frac{1}{3}} + \frac{1}{4}t = \frac{1}{\Gamma(\frac{4}{3})}t^{\frac{1}{3}} - \frac{1}{4}t^{\frac{4}{3}} - \frac{1}{4}t^{\frac{1}{3}} + \frac{1}{4}t$$
$$= t^{\frac{1}{3}}\Big(\frac{1}{\Gamma(\frac{4}{3})} - \frac{1}{4}t - \frac{1}{4} + \frac{1}{4}t^{\frac{2}{3}}\Big) \geq 0,\ \text{for } 0 < t < 1,$$

which together with $w_1^{(0)}(0) = 1$, $w_2^{(0)}(1) = 1$ prove that $W^{(0)} = (t+1,t)$ is an upper solution for the system (2.51) - (2.53). In the last equation we use the fact that $\frac{1}{\Gamma(\frac{4}{3})} > 1$ and $-\frac{1}{4}t - \frac{1}{4} \geq -\frac{1}{2}$, for $0 \leq t \leq 1$. Since $v_1^{(0)} = 1 \leq 1+t = w_1^{(0)}$ and $v_2^{(0)} = 0 \leq t = w_2^{(0)}$, $\forall t \in [0,1]$, we have $V^{(0)}$ and $W^{(0)}$ are ordered pairs of lower and upper solutions. Now, from Eq.(2.52), we have $f(t,y_1,y_2) = -\frac{1}{4}\sqrt[3]{t}\,y_1(t) + \frac{1}{4}y_2(t)$ satisfying $\frac{\partial f}{\partial y_1}(t,y_1,y_2) = -\frac{1}{4}\sqrt[3]{t}$ and $\frac{\partial f}{\partial y_2}(t,y_1,y_2) = \frac{1}{4}$, hence we can choose $c = \frac{1}{4}$ and the result in Theorem 2.5.1 guarantee the existence of solution to the problem.

Example 2.6.2. Consider the non-linear fractional boundary value problem

$$D^{\frac{3}{2}}\,y(t) - y^5(t) - \frac{1}{8}y'(t) = 0,\ 0 < t < 1, \tag{2.54}$$

$$\text{with}\quad y(0) = 0,\ y'(1) = 1. \tag{2.55}$$

We transform the problem to the following system

$$Dy_1(t) - y_2(t) = 0,\ 0 < t < 1, \tag{2.56}$$

$$D^{\frac{1}{2}}y_2(t) - y_1^5(t) - \frac{1}{8}y_2(t) = 0,\ 0 < t < 1, \tag{2.57}$$

$$\text{with} \quad y_1(0) = 0,\ y_2(1) = 1, \tag{2.58}$$

where $y_1(t) = y(t)$ and $y_2(t) = y'(t)$. In the following we show that $V^{(0)} = (v_1^{(0)}, v_2^{(0)}) = (0,0)$ and $W^{(0)} = (w_1^{(0)}, w_2^{(0)}) = (t^2, t)$ are ordered pairs of lower and upper solutions to the system (2.56) - (2.58). It is clear that $V^{(0)}$ satisfies the definition of the lower solutions given in Eq's.(2.6) - (2.8). We now show that $W^{(0)}$ is an upper solution. We have

$$Dt^2 - t = 2t - t = t \geq 0,\ 0 < t < 1,$$

$$\text{and} \quad D^{\frac{1}{2}}t - t^{10} - \frac{1}{8}t = \frac{2}{\sqrt{\pi}}t^{\frac{1}{2}} - t^{10} - \frac{1}{8}t = t^{\frac{1}{2}}\Big(\frac{2}{\sqrt{\pi}} - t^{\frac{19}{2}} - \frac{1}{8}t^{\frac{1}{2}}\Big) \geq 0,\ 0 < t < 1,$$

which together with $w_1^{(0)}(0) = 0$, $w_2^{(0)}(1) = 1$ prove that $W^{(0)} = (t^2, t)$ is an upper solution for the system (2.56) - (2.58). In the last equation we use the fact that $\frac{2}{\sqrt{\pi}} > \frac{9}{8}$. Since $v_1^{(0)} = 0 \leq w_1^{(0)} = t^2$ and $v_2^{(0)} = 0 \leq w_2^{(0)} = t$, $\forall\, t \in [0,1]$, we have $V^{(0)}$ and $W^{(0)}$ are ordered pairs of lower and upper solutions. Now, from Eq.(2.57), we have $f(t, y_1, y_2) = -y_1^5(t) - \frac{1}{8}y_2(t)$ satisfying $\dfrac{\partial f}{\partial y_1}(t, y_1, y_2) = -5y_1^4$ and $\dfrac{\partial f}{\partial y_2}(t, y_1, y_2) = -\frac{1}{8}$, hence we can choose $c = \dfrac{1}{4}$ and the result in Theorem 2.5.3 guarantee the existence of unique solution to the problem in $[W^{(0)}, V^{(0)}]$.

2.7 Conclusion

In this thesis, a class of boundary value problems of fractional order $1 < \delta < 2$ has been discussed, where the fractional derivative is of Caputo's type. To establish an existence and uniqueness results using the method of lower and upper solutions, we transform the problem to an equivalent system of differential equations including the fractional and integer derivatives. To establish an existence result, we generate a decreasing sequence of upper solutions that converges to a maximal solution of the system, as well as, an increasing sequence of lower solutions that converges to a minimal solution of the system. Under the condition $\frac{\partial f}{\partial y_2}(t, y_1, y_2) \leq c_1 < 0$, we guarantee that the maximal and minimal solutions coincide, and hence a uniqueness result is established. We apply the Banach Fixed Point Theorem to show that these sequences are well-defined and have unique solutions provided that $0 < \frac{2c}{\Gamma(\alpha+1)} < 1$. The presented examples illustrate the validity of our result. Because of the non-sufficient information about the fractional derivative $1 < \delta < 2$ of a function at its extreme points, the current results cannot be obtained without transforming the original problem to a system of fractional derivatives of less order. The problem with general boundary conditions of Robin type is of interests, and we leave it for a future work.

Bibliography

[1] Podlubny, I. (1993). Fractional Differential Equations, Mathematics in Science and Engineering. San Diego, Calif, USA: Academic Press.

[2] Ortigueira, M.D. (2011). Fractional Calculus for Scientists and Engineers. New York: Springer.

[3] Protter. M. H., & Weinberger. H. (1984). Maximum Principles in Differential Equations. New York: Springer-Verlag.

[4] Pao. C. V. (1992). Nonlinear Parabolic and Elliptic Equations. New York: Plenum Press.

[5] Furati, K., & Kirane, M. (2008). Necessary conditions for the existence of global solutions to systems of fractional differential equations. *Fractional Calculus and Applied Analysis*, 2008(11), 281-298.

[6] Lakshmikantham, V., & Vatsala, A.S. (2008). General uniqueness and monotone iterative technique for fractional differential equations. *Applied Mathematics Letters*, 2008(21), 828-834.

[7] Luchko, Y. (2009). Maximum principle for the generalized time-fractional diffusion equation. *Journal of Mathematical Analysis and Applications*, 2009(351), 218-223.

[8] Agarwal, R.P., Benchohra, N. & Hamani, S. (2010). A survey on existing results for boundary value problems of nonlinear fractional differential equations and inclusions. *Acta Appl. Math*, 2010(109), 973-1033.

[9] Al-Refai, M. & Hajji, M. (2011). Monotone iterative sequences for nonlinear boundary value problems of fractional order. *Nonlinear Analysis*, 74(2011), 3531–3539.

[10] Al-Refai, M. (2012). Basic Results on Nonlinear Eigenvalue Problems with Fractional Order. *Electronic Journal of Differential Equations*, 2012(191), 1-12.

[11] Mittag-Leffler, G.M. (1903). Surla la nouvelle function $E_{\alpha}(x)$. *C.R. Acad. Scie. Paris.*, 137, 554-558.

[12] Agarwal, R.P. (1953). Apropos d'une note de M. Pierre Humbert. *C.R. Seances Acad. Sci.*, 236(21), 2031-2032.

[13] Kilbas, A., Srivastava, H., & Trujillo, J. (2006). Theory and Applications of Fractional Differential Equations. (p.69-79). First edition. Amesterdam: North-Holland Mathematics Studies.

[14] Bartle, R., & Sherbert, D. (2000). Introduction To Real Analysis. Third edition. New York: John Willy & Sons, Inc.

[15] Al-Refai, M. (2012). On the fractional derivatives at extreme points. *Electronic Journal of Qualitative Theory of Differential Equations*, 2012(55), 1-5.

[16] Changpin, L., & Weihua, D. (2007). Remarks on fractional derivatives. Applied Mathematics and Computation, 187(2), 777-784.

[17] Burden, R., & Faires, J. (2001). Numerical Analysis. (p.9). Seventh edition. USA: Brooks/Cole.

[18] Keryszig, E. (1978). Introductory Functional Analysis with Applications. (p.300). First edition. USA: John Willy & Sons, Inc.

[19] Fletcher, P., & Patty, C. (1996). Foundations of higher mathematics. Third edition. USA: Brooks/Cole.

[20] Hubbard, J. H., & Hubbard, B. B. (2001). Vector Calculus, Linear Algebra, and Differential Forms: A Unified Approach. (p.120). Second edition. USA: Prentice Hall.

[21] Bai, Z. & Lu, H. (2005). Positive solutions for boundary value problem of nonlinear fractional differential equation. *Journal of Mathematical Analysis and Applications*, 2005(311,2), 495–505.

[22] Hu, C., Liu, B. & Xie, S. (2013). Monotone Iterative Solutions for Nonlinear Boundary Value Problems of Fractional Differential Equation. *Abstract and Applied Analysis*, 2013, 1-8.

[23] Syam, M., & Al-Refai, M. (2013). Positive solutions and monotone iterative sequences for a class of higher order boundary value problems of fractional order. *Journal of Fractional Calculus and Applications*, 4(1), 147-159.

[24] Benchohra, M., & Ouaar, F. (2010). Existence results for nonlinear fractional differential equations with integral boundary conditions. *Bulletin of mathematical analysis and applications*, 2(4), 7-15.